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# FUNCTIONALS OF MARKOV PROCESSES AND SUPERPROCESSES

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It is well known that a contraction multiplicative functional $\alpha_t$, $t \geqq 0$ on some Markov process with transition $P_t$, $t \geqq 0$, yields another Markov process whose semigroup $Q_t(x, A) = E_x(\alpha_t, X_t \in A)$ is subordinate to $P_t$, $t \geqq 0$. The second process results from the original one by adding a killing operation at a rate of $-d\alpha_t/\alpha_t$. This paper deals with expansion multiplicative functionals (satisfying $\alpha_t \geqq 1$ and $E_x(\alpha_t) < \infty$). It is proved that such functionals yield a Markov process with creation and annihilation of mass. Relations to the original process are established. Finally the results are generalized to, so-called, conditionally monotone functionals.

**1. Introduction.** Let $X = (\Omega, F, F_t, X_t, \theta_t, P_x)$ be a Markov process with transition function $P_t$, $t \geqq 0$. Then it is well known that for every multiplicative functional (MF), $\alpha_t$, $t \geqq 0$, of $X$ satisfying $0 \leqq \alpha_t \leqq 1$ there exists another Markov process subordinated to $X$, in the sense that its transition function $Q_t$, $t \geqq 0$, satisfies $Q_t \leqq P_t$. $Q_t$ is given in terms of $\alpha_t$ by the equation

$$Q_t(x, A) = E_x(\alpha_t, X_t \in A)\,. \tag{1.1}$$

The relation between the two processes is that the later process is generated from the original one by adding a killing operator. In other words what $\alpha_t$ does, is that for every path $\omega \in \Omega$ it assigns a time $0 \leqq \tau(\omega) \leqq \infty$ after which the particle does not continue its motion in the state space $E$ but is killed. Or in a different terminology, it is transferred to a new state added to $E$—the annihilation state.

A natural question arises. What if $\alpha_t$ is a MF which is not a contraction. Then the relation (1.1) yields a semigroup of kernels which are not sub Markov but rather super Markov. The question then arises—can we still find a process having $Q_t$, $t \geqq 0$, as its quasi-transition function? The fact that $\alpha_t \leqq 1$ yields a killing operator suggests that the opposite should happen for $\alpha_t \geqq 1$. Namely we can expect processes with creation of particles. Such processes were already defined and treated in recent years by several authors under the name "Markov processes with creation and annihilation" (MPCA).

In this paper we prove the relation between MF $\alpha_t \geqq 1$ and MPCA (Section 3). We will also prove in Section 4 that for an existence of an MPCA having quasi transition function $Q_t$, $t \geqq 0$, satisfying (1.1) one actually does not need the condition of multiplicativity of $\alpha_t$ but rather a weaker condition of conditional

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monotonicity. In Section 5 we will consider some examples of functionals $\alpha_t \geqq 1$ and the resulting MPCA. A well-known special case is where the original process is Brownian motion and $\alpha_t = \exp \int_0^t u(x_s)\, ds$ where $u$ is a bounded continuous function. This can be shown to be equivalent to perturbation of the infinitesimal generator of Brownian motion by $u \times I$ and as has been proved by Helms [4], it results in an MPCA.

**2. Definitions and preliminaries.** Let $E$ be a locally compact space with a countable basis. Let $\mathscr{E}$ be its $\sigma$-algebra of Borel subsets. Let $\Delta$, $\nabla$ be two points not in $E$. Adjoin $\nabla$ to $E$ as an isolated point or as the point of infinity according as $E$ is compact or not, let $\mathscr{E}_\nabla$ be the resulting $\sigma$-algebra. Adjoin $\Delta$ to $E_\nabla$ as an isolated point and let $\mathscr{E}_{\nabla,\Delta}$ be the resulting $\sigma$-algebra. We will use the notations of [1] throughout this work. Let $X = (\Omega, F, F_t, X_t, \theta_t, P_x)$ be a Markov process with state space $(E_\nabla, \mathscr{E}_\nabla)$ having transition function $P_t$, $t \geqq 0$, where $P_t(x, A) = P_x(X_t \in A)$, $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$.

DEFINITION 2.1. A family $\{\alpha_t : 0 \leqq t < \infty\}$ of random variables on $(\Omega, F)$ is called a functional of the process if $\alpha_t$ is $F_t$ measurable for $t \geqq 0$. A functional is called an expansion functional if it satisfies $\alpha_t \geqq 1$ and $E_x(\alpha_t) < \infty$, for $t \geqq 0$.

DEFINITION 2.2. (a) A functional $\alpha_t$ of the process is called a multiplicative functional (MF), if $\alpha_{s+t} = \alpha_t(\alpha_s \circ \theta_t)$ a.e. $P_x$, $x \in E$ for $s, t \geqq 0$.

(b) $\alpha_t$ is called conditionally monotone if $\alpha_{s+t} \geqq E_x(\alpha_s \mid X_{s+t})$, a.e. $P_x$, $x \in \mathscr{E}_\nabla$.

An MF $\alpha_t$ satisfying $\alpha_t \geqq 1$ is obviously conditionally monotone. Indeed $\alpha_{s+t} = \alpha_t(\alpha_s \circ \theta_t) \geqq \alpha_t$.

DEFINITION 2.3. A function $Q_t(x, A)$, $t \geqq 0$, $X \in E_\nabla$, $A \in \mathscr{E}_\nabla$, is called a quasi-transition function (QTF) if $Q_t(\cdot, A)$ is $\mathscr{E}_\nabla$-measurable, $Q_t(x, \cdot)$ is a measure on $\mathscr{E}_\nabla$, $Q_0(x, E_\nabla - x) = 0$ and $Q_{s+t}(x, A) = Q_s Q_t(x, A)$.

DEFINITION 2.4. Let $P_t$, $Q_t$, $t \geqq 0$ be two QTF's. $P_t$ is subordinate to $Q_t$ if $P_t(x, A) \leqq Q_t(x, A)$ for each $t \geqq 0$, $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$. $Q_t$ is said to dominate $P_t$.

For any functional $\alpha_t \geqq 1$ of the process $X$, define $Q_t(x, A) = E_x(\alpha_t\, x_t \in A)$. Then $Q_t$, $t \geqq 0$, is a QTF dominating $P_t$, $t \geqq 0$. $Q_t$, $t \geqq 0$ is called the QTF generated by $\alpha_t$, $t \geqq 0$.

The last definition is that of an MPCA. The exact definition was given in [4], [5]. Mainly, an MPCA having state space $(E_{\nabla\Delta}, \mathscr{E}_{\nabla\Delta})$ and transition function $P_t$, $t \geqq 0$, is a process $\{\xi_t, \mathscr{F}_t : t \geqq 0\}$ defined on a $\sigma$-finite measure space $(\Omega, \mathscr{F}, \mathscr{P}_x)$, $x \in E_\nabla$, satisfying the following two conditions.

(i) Each $\omega \in \Omega$ is a function from $[0, \infty)$ to $E_{\nabla,\Delta}$ satisfying $\omega(s) = \Delta$ implies $\omega(t) = \Delta$ for $t \leqq s$ and $\omega(s) = \nabla$ implies $\omega(t) = \nabla$ for $t \geqq s$.

(ii) For each $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$ the following Markov property holds:

$$\mathscr{P}_x(\xi_{s+t} \in A \mid \mathscr{F}_t \cap \{\xi_t \in E_\nabla\}) = P_s(\xi_t, A) \qquad \text{a.e. } \mathscr{P}_x \text{ on } \{\xi_t \in E_\nabla\}$$

where $\xi_t : \Omega \to E_{\nabla,\Delta}$ is defined by $\xi_t(\omega) = \omega(t)$ and where $\mathscr{F}_t = \sigma(\xi_s : s \leqq t)$

(generated $\sigma$-algebra). For some regularity conditions see [3]. For any MPCA we can define the associated family of marginal distributions $Q_t$, $t \geqq 0$, by $Q_t(x, A) = \mathscr{P}_x(\xi_t \in A)$, $t \geqq 0$, $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$. Define two random variables $\alpha(\omega) = \inf\{t : \xi_t(\omega) \in E_\nabla\}$, $\beta(\omega) = \inf(t : \xi_t(\omega) = \nabla\}$ called the starting time and annihilation (killing) time, respectively.

**3. Expansion multiplicative functionals.** In this section we establish the relations between an MF of a Markov process with transition function $P_t$, $t \geqq 0$, and an MPCA having the same transition function. We will show that under some mild conditions, for every MF $\alpha_t \geqq 1$ on the process there exists an MPCA whose marginal distribution is a quasi transition function generated by $\alpha_t$. For the reverse implication, we need some extra conditions of absolute continuity and uniform integrability of the quasi transition function. These conditions turn out to be necessary and sufficient. Notice the analogy of these results to Meyer's results for contraction MF [7]. Using the multiplicativity of $\alpha_t$, we can prove,

LEMMA 3.1. *Let $X = (\Omega, F, F_t, X_t, \theta_t, P_x)$ be a Markov process having transition function $P_t$, $t \geqq 0$. Let $\alpha_t$, $t \geqq 0$, be an expansion* MF *on $X$. Then $Q_t(x, A) = E_x(\alpha_t, X_t \in A)$ is a quasi transition function dominating $P_t$, $t \geqq 0$.*

THEOREM 3.2. *Let $X = (\Omega, F, F_t, X_t, \theta_t, P_x)$ be a Markov process having transition function $P_t$, $t \geqq 0$. Let $\alpha_t$ be a right continuous expansion* MF. *Then there exists an* MPCA *$Y = (\hat{\Omega}, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ with transition function $P_t$, $t \geqq 0$, whose marginal distribution is the quasi transition function generated by $\alpha_t$.*

PROOF. Let $\hat{\Omega} = \Omega \times [0, \infty]$ with points $\hat{\omega} = (\omega, \lambda)$, $\omega \in \Omega$, $0 \leqq \lambda \leqq \infty$. Let $\mathscr{F} = F \times \mathscr{B}$ be a $\sigma$-algebra of subsets of $\hat{\Omega}$, where $\mathscr{B}$ is the $\sigma$-algebra of Borel sets of $[0, \infty]$. Let $\pi_1$, $\pi_2$, be the natural projections of $\hat{\Omega}$ on $\Omega$ and on $[0, \infty]$ resp. Define a process $\xi_t$ on $(\hat{\Omega}, \mathscr{F})$ by $\xi_t(\hat{\omega}) = X_t(\omega)$ for $t \geqq \lambda$ and $\xi_t(\hat{\omega}) = \Delta$ for $t < \lambda$. Let $\beta(\hat{\omega}) = \beta(\omega)$, $\alpha(\hat{\omega}) = \inf\{t : \xi_t(\hat{\omega}) \in E_\nabla)$. Define a shift operator $\theta_t(\hat{\omega}) = (\theta_t(\omega), \max(\lambda - t, 0))$. Let $\mathscr{F}_t$, $t \geqq 0$, be the sub $\sigma$-algebra of $\mathscr{F}$ generated by the sets $[0, s] \times A$ for $s \leqq t$, $A \in F_s$ and $\Omega \times (s, \infty)$, $s \leqq t$. $\mathscr{F}_t$ is an increasing sequence of $\sigma$-algebras. Also $\xi_t$ is $\mathscr{F}_t$ measurable, since for $A \in \mathscr{E}_\nabla$ $\{\xi_t(\hat{\omega}) \in A\} = [0, t] \times \{X_t(\omega) \in A\} \in \mathscr{F}_t$ and $\{\xi_t(\hat{\omega}) = \Delta\} = (t, \infty] \times \Omega \in \mathscr{F}_t$. $\mathscr{F}_t$, $t \geqq 0$, are also right continuous.

Let $\Omega'$ be the subset of $\Omega$ consisting of all $\omega \in \Omega$ for which $\alpha_t(\omega) < \infty$, $t \geqq 0$, $t \to \alpha_t(\omega)$ is right continuous and non-decreasing and whence $\alpha_0 = 1$. Clearly $P_x(\Omega') = 1$, $x \in E_\nabla$. Thus we can assume $\Omega = \Omega'$. Define for each $\omega \in \Omega$ a measure $\mu_\omega$ on $[0, \infty]$ via $\mu_\omega[0, t] = \alpha_t(\omega)$, $\mu_\omega(\infty) = 0$ and extend it uniquely to all Borel subsets. $\mu_\omega$ is a $\sigma$-finite measure and $\mu_\bullet(A)$ is clearly measurable for $A \in \mathscr{B}$.

Let $\hat{\Lambda} \in \mathscr{F} = F \times \mathscr{B}$ let $\hat{\Lambda}_\omega = \{\lambda : (\omega, \lambda) \in \hat{\Lambda}\}$. Then as a section of $\hat{\Lambda}$, $\hat{\Lambda}_\omega$ is $\mathscr{B}$-measurable. Also $\omega \to \mu_\omega(\hat{\Lambda}_\omega)$ is $\mathscr{F}$-measurable. Define

$$\mathscr{P}_x(\hat{\Lambda}) = E_x(\mu_\omega(\hat{\Lambda}_\omega)) = E_x(\mu_\omega\{\lambda : (\omega, \lambda) \in \hat{\Lambda}\}) . \tag{3.1}$$

$\mathscr{P}_x$ is a $\sigma$-finite measure on $(\hat{\Omega}, F)$. To show that $(\hat{\Omega}, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ is the required MPCA, let $A \in \mathscr{E}_\nabla$ then clearly $\{\xi_t \in A\} = \{X_t \in A\} \times [0, t]$, thus $\mathscr{P}_x(\xi_t \in A) = E_x(\alpha_t, X_t \in A)$. Denote this by $Q_t(x, A)$. As for the Markov property, let $B \in \mathscr{E}_\nabla$, $s, t \geqq 0$, $x \in E_\nabla$, we must show

$$\mathscr{P}_x(\xi_{s+t} \in B, \hat{\Lambda}) = \mathscr{E}_x(P_{\xi_t}(\xi(s) \in B), \hat{\Lambda}) \tag{3.2}$$

for all $\hat{\Lambda} \in \mathscr{F}_t \cap \Omega_t$ where $\Omega_t = \Omega \cap [0, t]$.

It is enough to prove (3.2) for sets of the forms $\hat{\Lambda} = \{\xi_{t_i} \in A_i, i = 1, \cdots, n\}$ where $t_n = t$, $A_i \in \mathscr{E}_{\nabla,\Delta}$ since they generate $\mathscr{F}_t$. Or, further, for sets of the form

$$(t_{j-1}, t_j] \times \{X_{t_j} \in A_j, \cdots, X_{t_n} \in A_n\} = (t_{j-1}, t_j] \times \Lambda_j\,, \qquad A_i \in \mathscr{E}_\nabla$$

$j \leqq i \leqq n$. But for such a $\hat{\Lambda}$

$$\{\xi_{s+t} \in B\} \cap \hat{\Lambda} = \{[0, s+t] \times B\} \cap (t_{j-1}, t_j] \times \Lambda_j = (t_{j-1}, t_j] \times (B \cap \Lambda_j)$$

and thus

$$\begin{aligned}\mathscr{P}_x(\xi_{s+t} \in B, \hat{\Lambda}) &= E_x[(\alpha_{t_j} - \alpha_{t_{j-1}})(X_{s+t} \in B, \Lambda_j)] \\ &= E_x(\Lambda_j(\alpha_{t_j} - \alpha_{t_{j-1}})E_{x_t}(X_s \in A)]\,.\end{aligned}$$

As for the right-hand side, let $f(y) = E_y(X_s \in A)$, then it equals

$$\begin{aligned}\mathscr{E}_x(f(\xi_t), \hat{\Lambda}) &= E_x[f(X_t)(\alpha_{t_{j-1}} - \alpha_{t_j}), \Lambda_j] \\ &= E_x[(\alpha_{t_j} - \alpha_{t_{j-1}})\Lambda_j E_{x_t}(X_s \in A)]\,.\end{aligned}$$

The right-hand side of (3.2) was well defined since on $\Omega_t$ $\xi(t) = X(t)$ for $t \geqq \lambda$.

Notice that for the construction of $\mathscr{P}_x$ we have not used the multiplicativity of $\alpha_t$. We used only right continuity and monotonicity. The multiplicativity was necessary only to prove the semigroup property of $Q_t$, $t \geqq 0$.

A shorter existence proof of MPCA, not using right continuity of $\alpha_t$, can easily be produced using Corollary 2.3 of [6] (See Theorem 4.1). The advantage of the proof given here is that it makes clear the notion of creation via the second coordinate of $\hat{\omega} \in \hat{\Omega}$. Also one can get intuitively the "rate of creation" through this proof by calculating the "conditional probability" of $\pi_1 > t$ given no creation until time $t$, to get a creation rate of $d\alpha_t/\alpha_t$.

LEMMA 3.3. *The semigroup $Q_t$, $t \geqq 0$, generated by $\alpha_t$, $t \geqq 0$, is absolutely continuous with respect to $P_t$, $t \geqq 0$.*

PROOF. Indeed $Q_t(x, A) = \int_{\{X(t) \in A\}} \alpha_t \, dP_x$ thus $P_t(x, A) = P_x(X_t \in A) = 0$ implies $Q_t(x, A) = 0$. Denote by $q_t(x, y)$ the Radon-Nikodym derivative of $Q_t(x, \cdot)$ w.r.t. $P_t(x, \cdot)$. $q_t(x, y)$ can be chosen to be $E_\nabla \times \mathscr{E}_\nabla$ measurable.

The following lemma can be found in [3] page 285.

LEMMA 3.4. *Let $S_n = \{0 = s_0 < s_1 < \cdots < s_n = t\}$ be a partition of $[0, t]$. Then $\beta(S_n) = \prod_{i=0}^{n} q_{s_{i+1}-s_i}(X_{s_i}, X_{s_{i+1}})$, $n \geqq 1$, in uniform integrable* w.r.t. $P_x$. *Further $\beta(S_n) \to \alpha_t$* a.e. $P_x$ *as the norm of partition $\|S_n\| \to 0$.*

The following is the converse of Theorem 3.2.

THEOREM 3.5. *Let $(\hat{\Omega}, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ be an* MPCA *with transition function*

*$P_t$, $t \geqq 0$ and with marginal distribution $Q_t$, $t \geqq 0$, constituting a* QTF. *Suppose $Q_t(x, \cdot)$ is absolutely continuous* w.r.t. *$P_t(x, \cdot)$ for $t \geqq 0$, $x \in E_\nabla$ and denote by $q_t(x, y)$ the Radon–Nikodym derivative. Let $S_n$, $\beta(S_n)$ be defined as in Lemma* 3.4. *Assume that for any sequence $S_n$ of partitions with $\|S_n\| \to 0$, $\beta(S_n)$, $n \geqq 1$, is uniform integrable* w.r.t. *$P_x$. Then there exists an expansion* MF *$\alpha_t$, $t \geqq 0$, that generates $Q_t$, $t \geqq 0$.*

PROOF. The proof is similar to Dynkin's proof to Theorem 9.3. We will give a short outline for the sake of completeness. One first proves that the semigroup property of $Q_t$, $t \geqq 0$, is equivalent to the fact that for $s \leqq t$

$$E_x[q_s(X, X_s)q_{t-s}(X_s, X_t) \mid X_t] = q_t(x, X_t)\,. \tag{3.3}$$

Then let $S_1$, $S_2$ be two partitions of $[0, t]$ such that $S_2$ is a refinement of $S_1$ and denote by $\mathscr{F}(S) = \sigma(X_s\colon s \in S)$ then (3.3) is equivalent to

$$E_x[\beta(S_2) \mid \mathscr{F}(S_1)] = \beta(S_1) \qquad \text{a.e.} \quad P_x \quad \text{on} \quad \Omega_t\,. \tag{3.4}$$

Thus for any sequence $\{S_n\colon n \geqq 1\}$ of partitions of $[0, t]$, each of which is a refinement of its predecessor and for which $\|S_n\| \to 0$ we get by (3.4) that $\{\beta(S_n), \mathscr{F}(S_n)\colon n \geqq 1\}$ is a martingale, and being nonnegative it converges almost everywhere $P_x$. Let $\alpha_t = \lim \beta(S_n)$, then $\alpha_t$ is $F_t$ measurable and is clearly an MF. By (3.4) we get

$$E_x(\beta(S_n), X_t \in A) = E_x(q_t(x, X_t), X_t \in A) = Q_t(x, A)$$

and by uniform integrability of $\beta(S_n)$, $n \geqq 1$, we get, passing to the limit, that $Q_t$, $t \geqq 0$, is generated by $\alpha_t$, $t \geqq 0$.

The condition of uniform integrability is not easily checked in general and it would be useful to find sufficient conditions for that, at least in special cases. For example if $q_t(x, y) \leqq \exp((x - y)f(t))$ for $f$ bounded on compact subintervals of $[0, \infty]$, then clearly $\beta(S_n)$, $n \geqq 1$, is uniform integrable.

**4. Conditionally monotone functionals.** In Section 3 we proved relations between MF and MPCA whose marginal distributions $Q_t$, $t \geqq 0$, satisfy the semigroup property. But the semigroup property of $Q_t$, $t \geqq 0$ is not required in the definition of MPCA. We noticed in [6] that a necessary and sufficient condition for $Q_t$, $t \geqq 0$, to constitute the marginal distributions of a MPCA having transition function $P_t$, $t \geqq 0$, is that $Q_{s+t} \geqq Q_sP_t$ and $Q_t(x, E_\nabla) < \infty$ for $s, t \geqq 0$. The question thus arises what characterizes those functionals that generate the marginal distribution of some MPCA. The result is a new class of functionals called conditionally monotone functionals which are a generalization of MF. In this section we prove two theorems which are analogues of Theorems 3.2 and 3.4 for a general MPCA.

THEOREM 4.1. *Let $\alpha_t$, $t \geqq 0$, be a conditionally monotone expansion functional. Let $(\Omega, F, F_t, X_t, \theta_t, P_x)$ be a Markov process with transition function $P_t$, $t \geqq 0$, and state space $(E_\nabla, \mathscr{E}_\nabla)$. Let $Q_t(x, A) = E_x(\alpha_t, X_t \in A)$, $x \in E_\nabla$, $A \in \mathscr{E}_\nabla$. Then* *there exists an* MPCA *having transition function* $P_t$, $t \geqq 0$, *and marginal distributions* $Q_t$, $t \geqq 0$.

PROOF. $Q_t$, $t \geqq 0$, is a family of kernels on $E_\nabla \times \mathscr{E}_\nabla$ satisfying $Q_t(x, E_\nabla) = E_x(\alpha_t, X_t \in E_\nabla) = E_x(\alpha_t) < \infty$. Further $Q_{s+t} \geqq Q_s P_t$ for $s, t \geqq 0$. Indeed

$$\begin{aligned} Q_{s+t}(x, A) &= E_x(\alpha_{s+t}, X_{s+t} \in A) \geqq E_x[E_x(\alpha_s \mid X_{s+t}), X_{s+t} \in A] \\ &= E_x E_x(\alpha_s, X_{s+t} \in A \mid X_{s+t}) = E_x(\alpha_s, X_{s+t} \in A) . \end{aligned}$$

On the other hand

$$\begin{aligned} Q_s P_t(x, A) &= E_x(\alpha_s P_t(X_s, A)) \\ &= E_x(\alpha_s E_x(X_{s+t} \in A)) \\ &= E_x E_x(\alpha_s, X_{s+t} \in A \mid X_s) = E_x(\alpha_s, X_{s+t} \in A) . \end{aligned}$$

Thus the conditions of Theorem 2.1 of [6] hold and there exists an MPCA satisfying the required properties.

THEOREM 4.2. *Let* $Q_t$, $t \geqq 0$, *be a family of finite kernels on* $E_\nabla \times \mathscr{E}_\nabla$. *Let* $(\Omega, F, F_t, X_t, \theta_t, P_x)$ *be a Markov process with transition function* $P_t$, $t \geqq 0$, *and with a state space* $(E_\nabla, \mathscr{E}_\nabla)$ *such that* $Q_{s+t} \geqq Q_s P_t$, $s, t \geqq 0$, *and with* $Q_t(x, \cdot)$ *absolute continuous* w.r.t. $P_t(x, \cdot)$, $t \geqq 0$. *Then there exists a conditionally monotone functional* $\alpha_t$, $t \geqq 0$, *generating* $Q_t$, $t \geqq 0$.

PROOF. Let $q_t(x, y)$ satisfy $q_t(x, A) = \int_A q_t(x, y) P_t(x, dy)$. Let $\alpha_t = q_t(x, X_t)$. Then $\alpha_t$, $t \geqq 0$, is certainly a functional of the process. For conditional monotonicity we must prove

$$E_x(\alpha_{s+t}, X_{s+t} \in A) \geqq E_x(\alpha_s, X_{s+t} \in A) .$$

But

$$\begin{aligned} E_x(\alpha_s, X_{s+t} \in A) &= E_x(q_s(x, X_s), X_{s+t} \in A) \\ &= E_x[q_s(x, X_s) E_x(X_{s+t} \in A \mid X_s)] \\ &= Q_s P_t(x, A) . \end{aligned}$$

Notice that by Theorem 2.1 of [6] there exists MPCA with $P_t$ and $Q_t$ as defined above, and by the last theorem its $Q_t$, $t \geqq 0$, is generated by $\alpha_t$, $t \geqq 0$.

**5. Application and examples.** Let us see some examples of expansion functionals of Markov processes and examine their application to perturbation theory of Markov processes.

Let $u$ be an integrable nonnegative function on $(-\infty, \infty)$ satisfying $E_x(\exp \int_0^t u(X_s)\, ds) < \infty$ for some Markov process $(\Omega, F, F_t, X_t, \theta_t, P_x)$. Then $\alpha_t = \exp \int_0^t u(X_s)\, ds$ is clearly a right continuous expansion MF. Thus there exists an MPCA $(\hat{\Omega}, \mathscr{F}, \mathscr{F}_t, \xi_t, \theta_t, \mathscr{P}_x)$ satisfying $\mathscr{P}_x(\xi_t \in A) = E_x[\exp \int_0^t u(X_s)\, ds, x_t \in A]$. If $u$ is continuous and bounded then the infinitesimal generator of the new process is $A + u \times I$ where $A$ is the infinitesimal generator of the original process. This result was first proved, by a completely different method by Helms [4] for Brownian motion and for $u$ satisfying $\lim_{x \to \infty} u(x) = 0$.

An example of an expansion conditionally monotone functional which is not multiplicative is $\alpha_t = 1 + \int_0^t u(X_s)\,ds$ where $u$ is nonnegative and satisfies $E_x[\int_0^t u(X_s)\,ds] < \infty$. This functional being monotone in $t$ is also conditionally monotone. Thus there exists a MPCA corresponding to it. For this functional

$$\begin{aligned} Q_t(x, A) &= P_t(x, A) + E_x[\textstyle\int_0^t u(X_s)\,ds,\ X_t \in A] \\ &= P_t(x, A) + \textstyle\int_0^t \int_\Omega \chi_A(X_t)u(X_s)\,dP_x\,ds \\ &= P_t(x, A) + \textstyle\int_0^t \int_{E_\nabla} P_{t-s}(y, A)\phi_x(ds, dy) \end{aligned}$$

where $\phi_x(ds, dy) = u(y)P_s(x, dy)\,ds$ is a measure on $[0, \infty) \times E_\nabla$ satisfying $\phi_x([0, t] \times E_\nabla) = \int_0^t \int_{E_\nabla} u(y)P_s(x, dy) = E_x \int_0^t u(X_s)\,ds < \infty$. Thus we obtain an explicit representation for the creation measure $\phi_x$ as defined in [4]. Of course the marginal distribution of the resulting MPCA does not constitute a semigroup.

The theory of probabilistic solutions to perturbation of partial differential equations was developed in [5]. Let us only notice relations to MF. For a given Markov process with infinitesimal generator $A$, the introduction of a MF results (under some continuity condition) in a new MPCA whose infinitesimal generator can be denoted by $A + B$, $B \geqq 0$. Thus the MF provides a probabilistic solution to a new differential equation. The question then arises, if $A$ is the infinitesimal generator of some Markov process, to what kind of perturbation of $A$ (by $B$) can we find a probabilistic solution by means of MF on the process. We know by [2] Chapter VIII.1 and [6] Theorem 3.1, that if $B$ satisfies, for example, $\int_0^t \|BP_s\|\,ds < \infty$, for some $t > 0$, then there exists an MPCA corresponding to $A + B$ satisfying

$$(5.1)\qquad Q_t(x, A) = \textstyle\sum_{n=0}^{\infty} S_n{}^t(x, A) \quad \text{where} \quad S_0{}^t = P_t\,, \quad S_n{}^t = \int_0^t S_{n-1}^s BP_{t-s}\,ds\,.$$

Now Theorem 3.5 gives sufficient conditions for the existence of a corresponding MPCA. Thus, for example, if the original process a is Brownian motion, then all measures $P_s(x, \cdot)$ $s > 0$ are absolutely continuous with respect to each other and thus by (5.1) $Q_t(x, \cdot)$ is absolutely continuous with w.r.t. $P_t(x, \cdot)$. So for each $B$ the only condition remains to be checked is uniform integrability to get a probabilistic solution via MF to the perturbed equation.

As in [3] it might be interesting to find relations between the resolvent families of the original process and the one generated by $\alpha_t$, $t \geqq 0$.

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