The Annals of Probability
1976, Vol. 4, No. 1, 127-132

THE RECONSTRUCTABILITY OF MARKOV CHAINS

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As an extension of the work of Denzel, Kemeny, and Snell on the excessive functions of a continuous time Markov chain, this paper introduces the concept of reconstructability in two forms. First, there is reconstructability from the class of excessive functions, where it is seen that the transition matrix for a transient chain with a finite atomic exit boundary can be written down knowing only the membership of its class of excessive functions. A similar result is true, with the transient condition dropped, for reconstructability from the characteristic operator, based on a natural extension to the boundary of the operator corresponding to the initial derivative matrix.

Historically, the initial work on characterizing the transition matrix of a continuous time Markov chain resulted in a characterization of the range of the corresponding resolvent operator. It was known that when viewed as a mapping of the class of bounded functions defined on the state space of the chain, the Laplace transform of the transition matrix, and hence the transition matrix itself, is uniquely determined. In the work of Feller, Dynkin, Williams, and others (see, e.g., [9], [10], [7], [16], [2]), the range of this operator was specified by conditions formulated in terms of certain entrance or, more commonly, exit boundaries induced on the state space. In [6], Denzel, Kemeny, and Snell focused their attention not on the range of the resolvent, but on the closely related class of excessive functions for the chain. They found that for the transient case, this class was rich enough in information for the transition matrix to be reconstructed from its component parts.

The purpose of the present paper is two-fold. First, there will be a generalization of the results of [6] to transient Markov chains with a more complicated exit boundary. And, secondly, the initial derivative matrix will be extended to the boundary as an operator which, even when the chain is not transient, retains the identity of the corresponding transition matrix. The reader is assumed to be familiar with the notation and fundamental results contained in [3], [4] and [5]. They, with [1], form the general foundation upon which the following results are based. Since this paper should be considered as an addendum to [1], only a sketch of the assumptions in force and the results in use will be given here.

1. Excessive functions. Fix the $\mathbf{I}_\theta \times \mathbf{I}_\theta$ matrix Q so that assumptions A, B', and C_1 of [4] and [5] are fulfilled. That is: Q has finite entries and is conservative;

Received February 5, 1975; revised June 24, 1975.

AMS 1970 *subject classifications.* Primary 60J10, 60J35, 60J50; Secondary 60J45, 60G17.

Key words and phrases. Continuous time Markov chain, excessive functions, reconstructability, characteristic operator.

the minimal transition matrix Φ, which can be analytically constructed from Q (see Section 2.18 of [3]), has all of its recurrent states lumped into the one absorbing state θ; the passable part $\mathbf{A}$ of the Martin exit boundary corresponding to Q is nonvoid, completely atomic, and finite. As explained in Section 18 of [5], Q also determines the natural exit laws $\{l^a(t); a \in \mathbf{A}\}$.

Let $\mathscr{P}(Q)$ be the class of all standard, stochastic transition matrices $P = P(t)$ for chains with state space $\mathbf{I}_\theta$ so that P has initial derivative Q and so that the atoms in $\mathbf{A}$ are distinguishable (assumption D of [5]; a more detailed discussion of the conditions under which we work can be found in [1]). Let $\mathscr{E}(P)$ be the class of excessive functions for P in $\mathscr{P}(Q)$. Then P is said to be reconstructable from $\mathscr{E}(P)$ if P can be written down from a knowledge alone of which Φ-excessive functions belong to $\mathscr{E}(P)$.

From Theorem 3.18 and inequality (3.14) of [1] we have the following characterization of $\mathscr{E}(P)$.

THEOREM. *For P in $\mathscr{P}(Q)$, the class $\mathscr{E}(P)$ of P-excessive functions consists of those Φ-excessive functions which satisfy*

$$(1.1)\qquad \delta^a C(a) - \textstyle\sum_{b \neq a} F^{ab}(\infty)C(b) - \lim_{t\to 0} \langle \eta^a(t), C - \sum_{b \in \mathbf{A}} C(b)L^b(\infty)\rangle \geqq 0$$

for all a in $\mathbf{A}$.

If a boundary atom is recurrent, then it is shown in [1] that the boundary condition (1.1) reduces to the requirement that C be constant on that atom's boundary and state space recurrence class. From this it might seem that the existence of P-recurrent boundary atoms could mask the identity of P when investigated by way of $\mathscr{E}(P)$ alone. Indeed, this is the case, and an elementary but tedious construction contained in [2] shows furthermore that infinitely many members of $\mathscr{P}(Q)$ can claim $\mathscr{E}(P)$ as their common class of excessive functions. For this reason, in this section we will consider only transition matrices which have no recurrent boundary atoms, or equivalently, those transition matrices for which all states in $\mathbf{I}$ are transient.

THEOREM 1.2. *If P is transient, then P is reconstructable from $\mathscr{E}(P)$.*

PROOF. By equations (16.15) and (16.20) of [5], in terms of Laplace transforms the vector $\xi(\lambda)$ in the canonical decomposition of P can be written

$$\begin{aligned} \xi(\lambda) &= \{I - F(\infty) + U(\lambda)\}^{-1}\eta(\lambda) \\ &= \{I + [I - F(\infty)]^{-1}U(\lambda)\}^{-1}[I - F(\infty)]^{-1}\eta(\lambda) \end{aligned}$$

since $I - F(\infty)$ is invertible by Theorem (18.4) of [5]. Now column b of the matrix $[I - F(\infty)]^{-1}U(\lambda)$ is the vector $[I - F(\infty)]^{-1}\langle \lambda\eta(\lambda), L^b(\infty)\rangle$. Hence, if $[I - F(\infty)]^{-1}\eta(\lambda)$ can be determined from a knowledge of which Φ-excessive functions belong to $\mathscr{E}(P)$, then obviously so can the resolvent

$$P(\lambda) = \Phi(\lambda) + \textstyle\sum_{a \in \mathbf{A}} l^a(\lambda)\xi^a(\lambda)\,,$$

or P is reconstructable from $\mathscr{E}(P)$ by Lerch's theorem. We show that this is the case in much the same way that Denzel, Kemeny, and Snell did in [6].

Let $K = \{K(a); a \in \mathbf{A}\}$ be a vector on $\mathbf{A}$ with nonnegative constant entries. It is easily established that for fixed j, the function defined on the state space $\mathbf{I}$ as

$$G_{\cdot j} + \textstyle\sum_{a \in \mathbf{A}} K(a) L^a(\infty) \tag{1.3}$$

is Φ-excessive, where G is the Φ-potential $\int_0^\infty \Phi(t)\,dt$. Since G has 0 boundary values (see, e.g., (0.3) in the appendix of [7]), by (1.1) this function is in $\mathscr{E}(P)$ if and only if

$$K(a) - \textstyle\sum_{b \neq a} F^{ab}(\infty) K(b) - \lim_{t \to 0} \langle \eta^a(t), G_{\cdot j} \rangle \geqq 0$$

for a in $\mathbf{A}$, or in a more compact matrix form

$$[I - F(\infty)]K \geqq e_j \tag{1.4}$$

where $e_j = \{e_j{}^a; a \in \mathbf{A}\}$ is the vector formed from the canonical entrance sequences

$$e^a = \int_0^\infty \eta^a(t)\,dt \tag{1.5}$$

from Theorem 14.3 of [5]. But since $[I - F(\infty)]^{-1}$ has nonnegative terms, the last system of inequalities implies

$$K \geqq [I - F(\infty)]^{-1} e_j\,. \tag{1.6}$$

And conversely equality in (1.6) implies equality in (1.4). The vector $[I - F(\infty)]^{-1} e_j$ is then determined as the minimum of the vectors K for which the function (1.3) belongs to $\mathscr{E}(P)$. Upon varying j, $[I - F(\infty)]^{-1}\eta(\lambda)$ can then be determined since $\eta(\lambda) = e - \lambda e \Phi(\lambda)$, by Theorem 13.1 of [5].

It is an easy consequence of this theorem that if with positive probability the boundary is not reached in finite time by the minimal chain from any state (or, analytically, $L(\infty) < 1$), then each member P of $\mathscr{P}(Q)$ is transient and hence reconstructable from $\mathscr{E}(P)$.

2. The characteristic operator. We now drop the transience assumption and, in the spirit of Dynkin and Yushkevich [8], consider what will be called the characteristic operator of a chain, X_t, defined in terms of a derivative of the form

$$\lim_{t \to 0} \frac{\mathbf{E}^x(C(X_t{}^x)) - C(x)}{\mathbf{E}^x(A_t{}^x)}\,. \tag{2.1}$$

Roughly speaking, x is any member of the extended state space $\mathscr{I} = \mathbf{I}_\theta \cup \mathbf{A}$ and $X_t{}^x$ is X_t absorbed at all states other than x, if x is in $\mathbf{I}$, or at all boundary atoms other than x, if x is in $\mathbf{A}$. The random variable $A_t{}^x$ measures in some sense the time $X_t{}^x$ spends at x before absorption. Let us look more closely at these definitions when x is either a state or a boundary atom.

For i in $\mathbf{I}$, let τ^i be the first entrance time of i. Define $X^i(t)$ to be the post-τ^i process until the first exit from i, at which time $X^i(\cdot)$ is absorbed at the new state. Let $A^i(t)$ be the amount of time $X^i(\cdot)$ spends at i until time t. And $P^i(\cdot)$

will be the conditional probability measure induced by the post-τ^i chain (see Sections 2.9 and 2.15 of [3]).

LEMMA 2.2. *Assume C is a function defined on $\mathbf{I}_\theta$. Then for i in $\mathbf{I}$, $\mathbf{E}^i(C(X_t^i))$ is defined for $t > 0$ if and only if*

$$\langle Q, |C|\rangle(i) < +\infty , \tag{2.3}$$

in which case the limit (2.1) *equals $QC(i)$ when x is i.*

PROOF. If C is nonnegative, then by Theorems 2.15.2 and 2.15.6 of [3],

$$\begin{aligned} \mathbf{E}^i(C(X_t^i)) &= \textstyle\sum_{j\neq i} C(j)\mathbf{P}^i(X_t^i = j) + C(i)\mathbf{P}^i(X_t^i = i) \\ &= \textstyle\sum_{j\neq i} C(j)(1 - e^{-q_i t})q_{ij}/q_i + C(i)e^{-q_i t} , \end{aligned} \tag{2.4}$$

and so for any C, the expectation is defined if and only if (2.3) holds. The distribution of $A^i(t)$ can be derived from 2.15.5 of [3], enabling us to compute $\mathbf{E}^i(A^i(t))$ as $(1 - e^{-q_i t})/q_i$. Upon subtracting $C(i)$ from both sides of (2.4) and dividing by $\mathbf{E}^i(A^i(t))$ we have $QC(i)$ exactly.

So far, the derivative operator (2.1) evaluated on $\mathbf{I}$ is the same for each member of $\mathscr{P}(Q)$. By modifying the notion of the "a-process" in [5], we will extend this operator to the boundary for each P in $\mathscr{P}(Q)$, paralleling the extension of the minimal chain through the boundary to the chain associated with P. Analogous to the definition of $X^i(t)$ above, for any $a \in \mathbf{A}$ the chain $X^a(t)$ is the post τ^a-process until the time β^a of a "switch of banners," at which time $X^a(t)$ is absorbed at the new boundary atom. Then $X^a(t)$ has as its state space $\mathscr{I} - \{a\}$ with transition probabilities given by (14.8) and (14.30) of [5]. Moreover, from Definition 14.2 and the subsequent discussion in [5],

$$\mathbf{P}^a(X_t^a = b) = F^{ab}(t) \tag{2.5}$$

and

$$\mathbf{P}^a(X_t^a = j) = \rho_j^a(t) \tag{2.6}$$

for $t > 0$, $b \neq a$, and $j \in \mathbf{I}_\theta$.

Now define $\mathscr{A}C(x)$ on $\mathscr{I}$ to be the limit (2.1) if x is a state in $\mathbf{I}$, to be 0 if x is θ, and to be

$$\lim_{t\to 0} \frac{\mathbf{E}^a(C(X_t^a)) - C(a)}{E^a(t)} \tag{2.7}$$

if x is the boundary atom a. Moreover, define the domain $\mathscr{D}(\mathscr{A})$ of $\mathscr{A}$ to be the family of functions defined on $\mathscr{I}$ for which these defining limits exist and are finite. From (19.1) of [5] (which can be extended to the recurrent trap case with some effort), we see that for nonsticky a, if A_t^a is the number of times that X_s^i visits a before time t, then $E^a(t)$ is $\mathbf{E}^a(A_t^a)$, up to a multiplicative constant. And even when a is sticky, Smythe [14] has shown that in certain situations, again except possibly for a multiplicative constant, $E^a(t)$ is the expected value starting from a of a local time at a for X_t killed at β^a.

Remembering from [1] that the Φ-excessive functions are those nonnegative functions C on $\mathbf{I}_\theta$ for which QC is nonpositive, the next theorem provides a natural extension to the boundary for P-excessive functions.

THEOREM 2.8. *$\mathscr{E}(P) = \{C \in \mathscr{D}(\mathscr{A}) : C \geqq 0$ and $\mathscr{A}C \leqq 0\}$.*

PROOF. Assume C is Φ-excessive. Then by (2.5) and (2.6) we have for any a in $\mathbf{A}$,

$$C(a) - \mathbf{E}^a(C(X_t^a)) = C(a) - \sum_{b \neq a} F^{ab}(t)C(b) - \langle \rho^a(t), C \rangle ,$$

whereas by line (3.12) in the proof of the crucial Theorem (3.6) of [1], this last quantity equals

$$E^a(t)\{\delta^a C(a) - \sum_{b \neq a} F^{ab}(\infty)C(b)\} - \langle \eta^a(\cdot), C - \sum_{b \in \mathbf{A}} C(b)L^b(\infty) \rangle * E^a(t) .$$

Dividing by $E^a(t)$ and letting t tend to zero, we have that $\mathscr{A}C(a)$ equals the negative of the boundary expression in (1.1). The proof then follows quickly since C is in $\mathscr{E}(P)$ if and only if C is Φ-excessive and the inequality (1.1) is true for all a in $\mathbf{A}$ if and only if C is nonnegative and in $\mathscr{D}(\mathscr{A})$ with $\mathscr{A}C$ nonpositive.

Together Theorem (1.2), under the transience assumption, and Theorem (2.8) tell us that transient members of $\mathscr{P}(Q)$ can be identified from their characteristic operators. Actually, if the entire range of a characteristic operator is known, then there is no need to assume transience or even a knowledge of the initial derivative matrix Q. We say that P is reconstructable from its characteristic operator $\mathscr{A}$ if P can be written down from a knowledge alone of the values of $\mathscr{A}C$ for C in $\mathscr{D}(\mathscr{A})$.

THEOREM 2.9. *Any transition matrix is reconstructable from its characteristic operator.*

PROOF. By first taking C to be 0 everywhere on $\mathscr{I}$ except at j in $\mathbf{I}$, where C is 1, we can determine $\mathscr{A}C$ at i as q_{ij}; hence Q and therefore Φ, G, and $\{L^b(\infty); b \in \mathbf{A}\}$ can be determined. Next, at a in $\mathbf{A}$, $\mathscr{A}L^b(\infty)$ equals $F^{ab}(\infty)$ and $\mathscr{A}G_{\cdot j}$ is the e_j^a of (1.5). From these canonical quantities, $\eta^a(\cdot)$ and δ^a can be derived, and P can then be found in its canonical form.

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