The Annals of Probability
1978, Vol. 6, No. 1, 162–168

ON THE LAW OF THE ITERATED LOGARITHM[1]

BY R. J. TOMKINS

University of Regina

Let $X_1, X_2, \cdots$ be a sequence of independent random variables, each with zero mean and finite variance. Define $S_n = \sum_{i=1}^n X_i$, $s_n^2 = E(S_n^2)$, $t_n^2 = 2 \log\log s_n^2$ and $\Lambda = \limsup_{n\to\infty} S_n/(s_n t_n)$. Suppose that $|X_n| \leqq c_n s_n$ a.s. for all n and some real sequence $\{c_n\}$ and that $s_n \to \infty$, and let $\nu = \limsup_{n\to\infty} t_n c_n$. By Kolmogorov's law of the iterated logarithm, $\Lambda = 1$ if $\nu = 0$. Egorov proved that $0 \leqq \Lambda < \infty$, in every case. In this paper it will be shown that, if $\nu < \infty$, then $0 < \Lambda \leqq 1 + \sum_{j=3}^\infty \nu^{j-2}/j!$. A similar result for certain classes of unbounded random variables will also be presented.

1. Introduction. Let $X_1, X_2, \cdots$ denote a sequence of independent random variables (rv), each of which has mean zero and finite variance. For $n \geqq 1$, define $S_n = \sum_{i=1}^n X_i$ and $s_n^2 = E(S_n^2)$; throughout this article, it will be assumed that

$$s_n \to \infty .$$

Suppose, in addition, the existence of a real sequence $\{c_n\}$ such that

$$|X_n|/s_n \leqq c_n \quad \text{almost surely (a.s.),} \tag{1}$$

and define

$$\nu \equiv \limsup_{n\to\infty} (2 \log\log s_n^2)^{\frac{1}{2}} c_n . \tag{2}$$

This paper is concerned with the value of the constant Λ which satisfies

$$\Lambda \equiv \limsup_{n\to\infty} \frac{S_n}{(2s_n^2 \log\log s_n^2)^{\frac{1}{2}}} \quad \text{a.s.}$$

By Kolmogorov's renowned law of the iterated logarithm (LIL) (see Loève (1963), pages 260–263), $\Lambda = 1$ whenever $\nu = 0$. In fact, Egorov (1969) proved that $0 \leqq \Lambda < \infty$, no matter what the value of ν may be. But it is known that Λ need not equal one when $\nu > 0$ even if ν is close to zero. Indeed, Marcinkiewicz and Zygmund (1937) produced a family of examples in which $\nu > 0$ but $\Lambda < 1$, while Weiss (1959) and Egorov (1972) have concocted situations in which $\nu > 0$ but $\Lambda > 1$. Theorem 5 of Teicher (1974) presents some interesting assertions in the same vein.

This article was motivated by the aforementioned result of Egorov (1969); its purpose will be twofold. First, it will be shown that Λ is, in fact, positive

Received February 16, 1977.

[1] This work was supported by National Research Council of Canada grant A7588.

AMS 1970 *subject classifications.* Primary 60F15, 60G50.

Key words and phrases. Law of the iterated logarithm, exponential bounds, independent random variables.

if $\nu < \infty$. Second, an upper bound on the value of Λ will be established. More precisely, the following theorem will be proved in Section 2.

THEOREM 1. *Suppose that* (1) *holds and that* ν, *as defined in* (2), *is finite. Then*

$$0 < \limsup_{n\to\infty} \frac{S_n}{(2s_n^2 \log\log s_n^2)^{\frac{1}{2}}} \leqq 1 + \textstyle\sum_{j=3}^{\infty} \nu^{j-2}/j! \quad \text{a.s.} \tag{3}$$

To see that $\Lambda \geqq 0$, notice that Chebychev's inequality implies $S_n/(2s_n^2 \log\log s_n^2)^{\frac{1}{2}} \to 0$ in probability as $n \to \infty$ and, hence, this convergence holds a.s. on some subsequence. Furthermore, Egorov's proof that $\Lambda < \infty$ gives no inkling as to the actual value of Λ. For these reasons, techniques different from Egorov's are required to prove Theorem 1. Both of the inequalities in (3) will be established using methods of Kolmogorov, appropriately modified.

Section 3 contains a theorem, similar to Theorem 1, which is valid under conditions less stringent than (1).

2. PROOF OF THEOREM 1. Note that (3) is true when $\nu = 0$ by Kolmogorov's LIL, so assume hereinafter that $\nu > 0$. First, let us show that $\Lambda > 0$ when $\nu < \infty$ using the result of Egorov (1969) in conjunction with one of Kolmogorov's exponential bounds, which is stated below in a slightly unusual form.

LEMMA 1 (cf. Loève (1963), pages 254–257). *Assume that* (1) *holds. Let* $t \geqq 0$ *and* $n \geqq 1$. *Then, for any* $\gamma > 0$, *there exist positive numbers* ε_0 *and* η_0 *such that* $P[S_n > ts_n] \geqq \exp\{-(1+\gamma)t^2/2\}$, *provided that* $tc_n \leqq \eta_0$ *and* $t \geqq \varepsilon_0$. *Further, it may be assumed that*

$$\eta_0 \equiv h(g^{-1}(1+\gamma))$$

where g *and* h *are defined on* $(0, 1)$ *as follows*:

$$g(x) = (1 + 2x + x^2/2)/(1-x)^2 \qquad \textit{and} \qquad h(x) = x^2(1-x)/[8(1+x)^2]\,.$$

The preceding statement can be checked by carefully perusing the proof of Lemma 1 given by Loève (1963).

Note that $0 < h(x) < 1$ for all $0 < x < 1$, and that g is strictly increasing, so its inverse g^{-1} is well defined.

Now, for brevity's sake, let $t_n = (2\log\log s_n^2)^{\frac{1}{2}}$. Let $\Lambda^- \equiv \limsup_{n\to\infty}(-S_n/(s_n t_n))$ a.s.; then $0 \leqq \Lambda^- < \infty$ by Egorov (1969), in view of (1). Also, observe that there is no harm in assuming that $\nu > 1$, for, if $\nu \leqq 1$, one can take any number $b > 1$ and set $c_n^* \equiv bc_n/\nu$; then, trivially, (1) remains true with c_n^* in place of c_n.

In view of the preceding remarks, one may choose δ satisfying $0 < \delta < h(g^{-1}(\nu^2))/\nu$. Define $\gamma = \nu^2 - 1$ and $\eta_0 = h(g^{-1}(\nu^2))$. Clearly $\delta\nu < \eta_0$ so one may choose $\varepsilon > 0$ so small that $0 < \varepsilon < \delta$ and $\delta(\nu + \varepsilon) < \eta_0$. Finally, pick a number $c > 1$ so large that

$$\delta(\nu+\varepsilon)c(c^2-1)^{-\frac{1}{2}} < \eta_0 \qquad \text{and} \qquad \varepsilon c > \Lambda^-\,. \tag{4}$$

Define $n_0 = 1$. Since $s_n \to \infty$, there is a sequence $\{n_k\}$ of subscripts such that $s_{n_k} \geqq cs_{n_{k-1}} > s_{n_k-1}$. Since $\nu < \infty$ and $t_n \to \infty$, $c_n \to 0$ so, in view of (1), $E(X_n^2) = o(s_n^2)$ or, equivalently, $s_n \sim s_{n-1}$ (that is, $s_n/s_{n-1} \to 1$) as $n \to \infty$. Therefore $s_{n_k} \sim cs_{n_{k-1}} \sim c^k$ and $t_{n_k} \sim t_{n_{k-1}}$ as $k \to \infty$.

Now let $u_k^2 = s_{n_k}^2 - s_{n_{k-1}}^2$ for $k \geqq 1$. Then $s_{n_k}^2/u_k^2 \to c^2/(c^2-1)$, which fact, together with (2), (4), and the knowledge that $t_n \to \infty$ and that $h(x) < 1$ on $(0, 1)$, ensures the existence of an integer $k_0 > 0$ such that for all $k \geqq k_0$ one has

$$(5)\qquad \delta t_{n_k} c_{n_k} s_{n_k} u_k^{-1} \leqq \eta_0 , \qquad \delta \nu s_{n_k} \leqq u_k \quad \text{and} \quad \delta t_{n_k} s_{n_k} u_k^{-1} \geqq \varepsilon_0 ,$$

where ε_0 is the number corresponding to $\gamma = \nu^2 - 1$ in Lemma 1.

Applying Lemma 1 for $k \geqq k_0$ implies

$$P[S_{n_k} - S_{n_{k-1}} > \delta s_{n_k} t_{n_k}] \geqq \exp\{-(1+\gamma)\delta^2 s_{n_k}^2 t_{n_k}^2 u_k^{-2}/2\}$$
$$\geqq (\log s_{n_k}^2)^{-1} \quad \text{by (5)}.$$

But $\log s_{n_k}^2 \sim (2 \log c)k$ and the rv $\{S_{n_k} - S_{n_{k-1}}, k \geqq 1\}$ are independent, so

$$\textstyle\sum_{k=2}^{\infty} P[S_{n_k} - S_{n_{k-1}} > \delta s_{n_k} t_{n_k}] = \infty ,$$

which is tantamount to

$$(6)\qquad P[S_{n_k} - S_{n_{k-1}} > \delta s_{n_k} t_{n_k} \ \text{i.o.}] = 1$$

by the Borel 0-1 Law.

Finally, since $c\varepsilon > \Lambda^-$ by (4), clearly

$$P[-S_{n_{k-1}} > c\varepsilon s_{n_{k-1}} t_{n_{k-1}} \ \text{i.o.}] = 0$$

by the definition of Λ^-. But $\varepsilon s_{n_k} t_{n_k} \geqq \varepsilon c s_{n_{k-1}} t_{n_{k-1}}$ so

$$(7)\qquad P[-S_{n_k} > \varepsilon s_{n_k} t_{n_k} \ \text{i.o.}] = 0 .$$

(6) and (7) together imply

$$P[S_{n_k} > (\delta - \varepsilon) s_{n_k} t_{n_k} \ \text{i.o.}] = 1 ;$$

now it is evident that

$$\Lambda \geqq \limsup_{k\to\infty} S_{n_k}/(s_{n_k} t_{n_k}) \geqq \delta - \varepsilon > 0 \quad \text{a.s.}$$

Now turn to the proof of the right-hand inequality in (3). First, consider the following minor generalization of an exponential bound.

LEMMA 2 (cf. Loève (1963), pages 254–255). *Suppose* (1) *holds and that* $\beta > \nu$ *where* ν *is defined by* (2). *Then there exists an integer* $N = N(\beta)$ *such that*

$$(8)\qquad E(\exp(t_n S_n/s_n)) \leqq \exp\{t_n^2 \textstyle\sum_{j=2}^{\infty} \beta^{j-2}/j!\}$$

for all $n \geqq N$.

PROOF. Choose N so large that $t_n c_n \leqq \beta$ when $n \geqq N$. Then, using (1) (cf. page 255 of Loève (1963)),

$$E(\exp(t_n X_k/s_n)) \leqq 1 + t_n^2 E(X_k^2) s_n^{-2} \textstyle\sum_{j=2}^{\infty} \beta^{j-2}/j!$$

for all $k \leqq n$, provided $n \geqq N$. Now, in view of the inequality $1 + x \leqq e^x$ and the independence of $\{X_1, X_2, \cdots, X_n\}$, (8) is immediate. □

Now, take any $\beta > \nu$, and choose $\delta > 1 + \sum_{j=3}^{\infty} \beta^{j-2}/j!$. Let $\varepsilon > 0$ and choose $c > 1$ such that $c(\delta + \varepsilon) < \delta + 2\varepsilon$. Define the corresponding integral sequence $\{n_k\}$ exactly as in the first part of the proof. Note that there exists an integer $k_1 > 0$ so large that, if $k \geqq k_1$, then

$$t_{n_k} c_{n_k} < \beta \qquad \text{and} \qquad \varepsilon^2 t_{n_k}^2 > 2 . \tag{9}$$

For $k \geqq 1$, define $S_k^* = \max_{n_{k-1} < n \leq n_k} S_n$ and $P_k^* = P[S_k^* \geqq (\delta + \varepsilon) s_{n_k} t_{n_k}]$. If $k \geqq k_1$, the version of Lévy's inequality given on page 248 of Loève (1963) implies

$$\begin{aligned} P_k^* &\leqq 2P[S_{n_k} \geqq (\delta + \varepsilon) s_{n_k} t_{n_k} - 2^{\frac{1}{2}} s_{n_k}] \\ &\leqq 2P[S_{n_k} > \delta s_{n_k} t_{n_k}] \quad \text{by (9)} \\ &= 2P[\exp(t_{n_k} S_{n_k}/s_{n_k}) > \exp(\delta t_{n_k}^2)] \\ &\leqq 2 \exp\{-(t_{n_k}^2/2)(2\delta - 2 \textstyle\sum_{j=2}^{\infty} \beta^{j-2}/j!\} \end{aligned}$$

by Markov's inequality and Lemma 2.

But $\lambda \equiv 2\delta - 2 \sum_{j=2}^{\infty} \beta^{j-2}/j! = 2(\delta - 1 - \sum_{j=3}^{\infty} \beta^{j-2}/j!) + 1 > 1$ so $P_k^* \leqq 2[(2 \log c)k]^{-\lambda}$ and, hence, $\sum_{k=1}^{\infty} P_k^* < \infty$. By the Borel–Cantelli lemma, then, $P[S_k^* \geqq (\delta + \varepsilon) s_{n_k} t_{n_k} \text{ i.o.}] = 0$. But then, by choice of c,

$$0 \leqq P[S_n \geqq (\delta + 2\varepsilon) s_n t_n \ \text{ i.o.}] \leqq P[S_k^* \geqq (\delta + \varepsilon) s_{n_k} t_{n_k} \ \text{ i.o.}] = 0$$

so that $\Lambda \leqq \delta + 2\varepsilon$. A glance at the way in which ε, δ and β were chosen shows that $\Lambda \leqq 1 + \sum_{j=3}^{\infty} \nu^{j-2}/j!$ as claimed. □

REMARKS. 1. While the second inequality in (3) remains true, trivially, when $\nu = \infty$, the first inequality may be false, even if $c_n \to 0$. Consider the following two examples.

First, suppose $X_1 = X_2 = 0$ and that, for $n \geqq 3$, $P[X_n = -1] = P[X_n = 1] = (1 - n^{-3})/2$ and $P[X_n = \pm n^{\frac{1}{2}}(\log \log n)^{-\frac{1}{2}}] = 1/(2n^3)$. Then $E(S_n) = 0$ and $s_n^2 \sim n$, so that $c_n \sim (\log \log n)^{-\frac{1}{2}} \to 0$ and $\nu = \lim_{n\to\infty} (2 \log \log n)^{\frac{1}{2}} c_n = \infty$. Now let $Y_n = X_n I(|X_n| \leqq 1)$, where $I(A)$ denotes the indicator function of an event A. Then $P[X_n \neq Y_n \text{ i.o.}] = 0$ by the Borel–Cantelli lemma. Furthermore, $E(Y_n) = 0$ and $\operatorname{Var}(\sum_{k=1}^{n} Y_k) \sim n$, so $\Lambda = \limsup_{n\to\infty} \sum_{k=1}^{\infty} Y_k/(2n \log \log n)^{\frac{1}{2}}$ a.s. in this example. But Kolmogorov's LIL applies to the sequence $\{Y_n\}$, so $\Lambda = 1$. Therefore, it may happen that $\Lambda > 0$ when $\nu = \infty$.

In the second example, suppose $X_n \equiv 0$ if $n < 7$. For $n \geqq 7$, assume $P[X_n = \pm a_n] = \frac{1}{2}$ where $a_n^2 = \{\log \log n - 1/\log n\} \exp\{n/\log \log n\}/(\log \log n)^2$. Define the function $\psi(x) = \exp\{x/\log \log x\}$ and $\phi(x) = [\psi(x)]^{\frac{1}{2}}$. Since $a_n^2 = \psi'(n)$, it is not hard to check that $s_n^2 \sim \psi(n) = \exp\{n/\log \log n\}$ as $n \to \infty$. Moreover, $5\phi'(n) > \phi(n)/\log \log n$ if $n \geqq 7$, so $|S_n| \leqq \sum_{k=1}^{n} a_k \leqq \sum_{k=1}^{n} \phi(k)/(\log \log k)^{\frac{1}{2}} \leqq (\log \log n)^{\frac{1}{2}} \sum_{k=1}^{n} \phi(k)/\log \log k \leqq 5(\log \log n)^{\frac{1}{2}} \sum_{k=1}^{n} \phi'(k) \sim 5(\log \log n)^{\frac{1}{2}} \phi(n) \sim 5(\log \log n)^{\frac{1}{2}} s_n$. Since $t_n^2 \sim 2 \log n$, it is obvious that $\Lambda = 0$ in this case. But $c_n^2 \sim (\log \log n)^{-1} \to 0$ and $t_n^2 c_n^2 \sim 2 \log n/\log \log n$, so $\nu = \infty$. (The a_n's in this example diverge at a faster rate than in the examples of Marcinkiewicz and

Zygmund (1937) cited earlier. In proving that $\Lambda = 0$, the above procedure is suggested by the proof of Theorem 5(i) of Teicher (1974).)

2. The proof that $\Lambda > 0$ when $\nu < \infty$ demonstrates that, in fact, $\Lambda \geqq h(g^{-1}(\nu^2))/\nu$ if $\nu > 1$, but this inequality is not a sharp one. The function $h(g^{-1}(x^2))/x$ attains its maximal value, which is less than .0027, when $x^2 = g(\alpha)$, where α is the unique root of the equation

$$\alpha^4 + 10\alpha^3 + 19\alpha^2 + 2\alpha - 4 = 0$$

satisfying $0 < \alpha < 1$. Incidentally, $\alpha = .3764\cdots$ while $[g(\alpha)]^{\frac{1}{2}} = 1.456\cdots$.

3. The right-hand inequality in (3) can be quite sharp, at least in the sense that one part of Kolmogorov's LIL (viz., the fact that $\Lambda \leqq 1$ when $\nu = 0$) is an easy consequence of (3). It also sheds some light on the family of counter-examples given by Weiss (1959). Weiss showed that, given any $\eta > 0$, a sequence of rv $\{X_n\}$ exists which obeys (1) and (2) with $\nu < \eta$, but $\Lambda > 1$. From (3), it is evident that as $\nu \downarrow 0$, the corresponding values of Λ converge to unity.

3. A partial generalization of Theorem 1. In proving the right-hand part of (3), good use was made of the assumption (1) to establish Lemma 2. Trying to employ Kolmogorov's exponential bound in its usual form would have restricted the theorem, to, at most, values of $\nu \leqq 1$.

There are, however, sequences of rv for which the exponential bounds are valid but which do not satisfy (1); indeed, such rv may be unbounded. The conclusion of the following theorem is weaker than (3) but applies to a wider variety of sequences $\{X_n\}$ than does Theorem 1.

THEOREM 2. *Suppose that a sequence of independent* rv $\{X_n\}$ *satisfies the following condition: a positive real sequence* $\{c_n\}$ *and an integer* $N > 0$ *exist such that, for each* $n \geqq N$,

$$\exp\{(t^2/2)(1 - tc_n)\} \leqq E\exp(tS_n/s_n) \leqq \exp\{(t^2/2)(1 + tc_n/2)\} \tag{10}$$

whenever $0 \leqq tc_n \leqq 1$. *Let* $v = \limsup_{n\to\infty} (2\log\log s_n^2)^{\frac{1}{2}}c_n$. *If* $2v^2 < 1$ *then*

$$0 < \limsup_{n\to\infty} \frac{S_n}{(2s_n^2\log\log s_n^2)^{\frac{1}{2}}} \leqq r_1(v) < 2^{\frac{1}{2}} \quad \text{a.s.} \tag{11}$$

where $r_1(v)$ *is the smaller positive root of the function*

$$f_v(x) \equiv x^2 - 1 - vx^3/2\,.$$

REMARK. The condition (10) was introduced by Tomkins (1971, 1972) who remarked that Kolmogorov's exponential bounds are valid as stated on page 254 of Loève (1963) when (1) is replaced by the weaker condition (10). With this in mind, it is not hard to modify Egorov's proof that $\Lambda < \infty$ to show that Egorov's result remains true with (1) replaced by (10).

If (1) holds and $4v^2 < 1$, then the middle inequality in (11) is a consequence of a supermartingale theorem of Stout (1973; or see Theorem 5.4.1 of Stout (1974)). Theorem 2 will be proved using methods similar to those of Stout.

PROOF. When $v = 0$, it is known that $\Lambda = 1$ (see Theorem 1 of Tomkins (1972)), so assume $0 < 2v^2 < 1$. In light of the preceding remark, the proof that $\Lambda > 0$ is virtually the same as the proof in Theorem 1. It remains to show that $\Lambda \leqq r_1(v)$.

Notice that $f_v(v^{-1}) > 0$ if and only if $0 < v^2 < \frac{1}{2}$, and that $f_v'(v^{-1}) > 0$. Therefore $r_1(v) < v^{-1}$. Note also that $f_v(1) = -v/2 < 0$ while $f_v'(1) > 0$ so $1 < r_1(v)$. Moreover, evidently $f_v(x) > 0$ whenever $r_1(v) < x < r_2(v)$ where $r_2(v)$ is the larger positive root of f_v; in this connection, observe that f_v has exactly two positive roots if and only if $27v^2 < 16$, so $r_1(v)$ and $r_2(v)$ are well defined under our hypotheses.

The above comments allow the choice of $\delta > 0$ such that

$$\delta > r_1(v)\,, \qquad \delta v < 1 \qquad \text{and} \qquad \delta < r_2(v)\,.$$

Since $f_v(\delta) > 0$ for such a δ, one can choose $\varepsilon > 0$ so small that

$$\delta^2(1 + \delta(v + \varepsilon)/2) > 1\,. \tag{12}$$

Now select $c > 1$ so close to 1 that $c(\delta + \varepsilon) < \delta + 2\varepsilon$, and define the sequence $\{n_k\}$ as in the proof of Theorem 1. It is apparent from the proof of Theorem 1 that proving

$$\textstyle\sum_{k=1}^{\infty} P[S_{n_k} > \delta s_{n_k} t_{n_k}] < \infty \tag{13}$$

will suffice to show that $\Lambda < r_1(v)$.

To this end, one need only choose $m > 0$ so large that

$$n_k \geqq N\,, \qquad \delta t_{n_k} c_{n_k} < 1\,, \qquad t_{n_k} c_{n_k} < v + \varepsilon \qquad \text{and} \qquad \varepsilon^2 t_{n_k}^2 > 2 \tag{14}$$

for all $k \geqq m$. Then the exponential bound (Lemma 1(i) of Tomkins (1972)) yields, for $k \geqq m$,

$$\begin{aligned} P[S_{n_k} > \delta s_{n_k} t_{n_k}] &\leqq \exp\{-(\delta^2 t_{n_k}^2/2)(1 - t_{n_k} c_{n_k}/2)\} \\ &\leqq (\log s_{n_k}^2)^{-1-f_{v+\varepsilon}(\delta)} \quad \text{by (14)}. \end{aligned}$$

But $\log s_{n_k}^2 \sim (2 \log c)k$ and $f_{v+\varepsilon}(\delta) > 0$ by (12), so (13) is true.

Finally, note that r_1 increases as v increases. Moreover, $r_1(2^{-\frac{1}{2}}) = 2^{\frac{1}{2}}$, so $r_1(v) < 2^{\frac{1}{2}}$ when $2v^2 < 1$. □

REMARKS. 1. It is easy to check that $r_1(v) \downarrow 1$ as $v \downarrow 0$ so the second inequality can be sharp.

2. If (1) holds then (10) holds, so that $v = \nu$; thus Theorems 1 and 2 overlap when (1) holds and $2\nu^2 < 1$. However, $r_1(v) > 1 + \sum_{j=3}^{\infty} \nu^{j-2}/j!$ so that (4) is the better result than (11) in such cases.

3. It is still true that $\Lambda \leqq r_1(v)$ when only the second inequality of (10) holds.

Acknowledgment. The author thanks the referee for useful comments and suggestions.

REFERENCES

[1] Egorov, V. A. (1969). On the law of the iterated logarithm. *Theor. Probability Appl.* **14** 693–699.

[2] Egorov, V. A. (1972). On Kolmogorov's law of the iterated logarithm. *Vestnik Leningrad. Univ.* **13** 140–142.

[3] Loève, M. (1963). *Probability Theory.* Van Nostrand, Princeton.

[4] Marcinkiewicz, J. and Zygmund, A. (1937). Remarque sur la loi du logarithme itéré. *Fund. Math.* **29** 215–222.

[5] Stout, William F. (1973). Maximal inequalities and the law of the iterated logarithm. *Ann. Probability* **1** 322–328.

[6] Stout, William F. (1974). *Almost Sure Convergence.* Academic Press, New York.

[7] Teicher, H. (1974). On the law of the iterated logarithm. *Ann. Probability* **2** 714–728.

[8] Tomkins, R. James (1971). An iterated logarithm theorem for some weighted averages of independent random variables. *Ann. Math. Statist.* **42** 760–763.

[9] Tomkins, R. J. (1972). A generalization of Kolmogorov's law of the iterated logarithm. *Proc. Amer. Math. Soc.* **32** 268–274.

[10] Weiss, M. (1959). On the law of the iterated logarithm. *J. Math. Mech.* **8** 121–132.

Department of Mathematics
University of Regina
Regina, Saskatchewan
Canada S4S 0A2