The Annals of Probability
1978, Vol. 6, No. 1, 159-161

CONDITIONS FOR A CLASS OF STATIONARY GAUSSIAN PROCESSES TO BE KOLMOGOROV MIXING

BY HARRY DYM

The Weizmann Institute of Science, Rehovot, Israel

Necessary and sufficient conditions are given for a class of stationary Gaussian processes to be mixing in the sense of Kolmogorov.

1. Introduction. Let $\mathfrak{x}(t)$ be a real, zero mean, finite variance, stationary Gaussian stochastic process with spectral density $|h(\gamma)|^2$:

$$E[\mathfrak{x}(s)\mathfrak{x}(t)] = \int_{-\infty}^{\infty} e^{i\gamma(s-t)}|h(\gamma)|^2\, d\gamma ,$$

in which h is an outer function belonging to the Hardy class

$$H^2 = \left\{f \in L^2(R^1, d\gamma) : f^\vee(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\gamma)e^{-i\gamma x}\, d\gamma = 0 \ \text{ for } \ x < 0\right\} .$$

Let $\mathfrak{p}$ denote the orthogonal projection of $L^2(R^1, d\gamma)$ onto H^2, let $\mathfrak{q} = I - \mathfrak{p}$ and let $\chi(\gamma) = e^{i\gamma}$. It is known (see, e.g., Sections 3.3—3.6 and 4.12 of Dym–McKean [1] and the references cited there) that $\mathfrak{x}$ is mixing in the sense of Rosenblatt if and only if

$$\rho(T) = ||\mathfrak{q}\chi^T(h^*/h)\mathfrak{p}|| = o(1) ,$$

as $T \uparrow \infty$, and that $\mathfrak{x}$ is mixing in the sense of Kolmogorov if and only if

$$\kappa(T) = |\mathfrak{q}\chi^T(h^*/h)\mathfrak{p}|_2 = o(1) ,$$

as $T \uparrow \infty$. Here $||\ \ ||$ denotes the operator norm, $|\ \ |_2$ denotes the Hilbert–Schmidt norm and $h^* : \gamma \to [h(\gamma)]^*$ in which $[h(\gamma)]^*$ stands for the complex conjugate of the complex number $h(\gamma)$; $||\ \ ||_2$ will denote the standard L^2 norm.

The present objective is to establish

THEOREM 1. *If h/h^* agrees* a.e. *on the line with an inner function j, then $\kappa(T) \to 0$ as $T \uparrow \infty$ if and only if*

(i) *j has no singular component,*

(ii) *the roots $\omega_n = a_n + ib_n$, $b_n > 0$, of j meet the constraint*

$$\sum \frac{b_n}{|\gamma - \omega_n|^2} \leqq M < \infty$$

for all $\gamma \in R^1$, and

(iii) *$\sum e^{-b_n t} < \infty$ for some $t > 0$.*

2. Auxiliary facts. The proof of Theorem 1 depends upon the following results:

Received November 1, 1976.

AMS 1970 *subject classifications*. Primary 60G10, 60G15; Secondary 47B10.

Key words and phrases. Mixing, stationary processes, Gaussian processes, Hilbert-Schmidt, Blaschke product.

THEOREM 2. *If h/h^* agrees* a.e. *on the line with an inner function j, then $\rho(T) \to 0$ as $T \uparrow \infty$ if and only if conditions* (i) *and* (ii) *of Theorem* 1 *are in force.*

LEMMA. *If*

$$B(\gamma) = \prod \left(\frac{1 - \gamma/\omega_n}{1 - \gamma/\omega_n^*} \right)$$

is a convergent (i.e., *nonzero*) *Blaschke product with roots $\omega_n = a_n + ib_n$, $b_n > 0$, and if P is the orthogonal projection of $L^2(R^1, d\gamma)$ onto $H^2 \ominus BH^2$, then*

$$|\mathfrak{q}(\chi^T/B)\mathfrak{p}|_2 = |P\chi^{-T}P|_2$$

for every $T \geqq 0$.

THEOREM 3. *If P and B are as in the statement of the preceding lemma, then in order for $P\chi^{-T}P$ to be of Hilbert–Schmidt class it is necessary that*

$$T \geqq \max\,[\beta/2, \alpha]$$

and it is sufficient that

$$T > [(\beta/2)^{\frac{1}{2}} + (\alpha)^{\frac{1}{2}}]^2$$

where

$$\alpha = \limsup_{|\gamma| \to \infty} \sum \frac{b_n}{|\gamma - \omega_n|^2} \qquad \gamma \in R^1$$

and

$$\beta = \text{the abscissa of convergence of } \sum e^{-b_n t}$$
$$= \inf\{t > 0 : \sum e^{-b_n t} < \infty\}\,.$$

For a proof of Theorems 2 and 3 see pages 132–134 of Dym–McKean [1] and Theorem 2 of Dym–Shapiro [2], respectively. The statement of the former should be compared with the statement of Theorem 1.

3. Proof of lemma.

PROOF. It should be clear that

$$\mathfrak{q}(\chi^T/B)\mathfrak{p} = \mathfrak{q}(\chi^T/B)P$$

for $T \geqq 0$, since for such T

$$\mathfrak{q}(\chi^T/B)Bf = 0$$

for $f \in H^2$. Therefore if $\varphi_n : n = 0, 1, 2, \cdots$, is an orthonormal basis for H^2 alias the range of $\mathfrak{p}$ and $\varphi_n : n = -1, -2, \cdots$, is an orthonormal basis for $L^2 \ominus H^2$ alias the range of $\mathfrak{q}$, then

$$\begin{aligned} |\mathfrak{q}(\chi^T/B)\mathfrak{p}|_2^2 &= |\mathfrak{q}(\chi^T/B)P|_2^2 \\ &= |P\chi^{-T}B\mathfrak{q}|_2^2 \\ &= \textstyle\sum_{n=-\infty}^{-1} \|P\chi^{-T}B\varphi_n\|_2^2 \\ &= \textstyle\sum_{n=-\infty}^{-1} \|P\chi^{-T}(\mathfrak{p} + \mathfrak{q})B\varphi_n\|_2^2 \\ &= \textstyle\sum_{n=-\infty}^{-1} \|P\chi^{-T}\mathfrak{p}B\varphi_n\|_2^2 \\ &= \textstyle\sum_{n=-\infty}^{-1} \|P\chi^{-T}PB\varphi_n\|_2^2\,. \end{aligned}$$

The last line depends upon the identity

$$\mathfrak{p}B\varphi_n = P\mathfrak{p}B\varphi_n = PB\varphi_n$$

which is valid for $n \leqq -1$ since in that case φ_n is orthogonal to H^2 and so $\mathfrak{p}B\varphi_n \in H^2 \ominus BH^2$. But now as

$$PB\varphi_n = 0$$

for $n \geqq 0$, it follows that

$$\begin{aligned}\textstyle\sum_{n=-\infty}^{-1} \|P\chi^{-T}PB\varphi_n\|_2^2 &= \textstyle\sum_{n=-\infty}^{\infty} \|P\chi^{-T}PB\varphi_n\|_2^2 \\ &= |P\chi^{-T}P|_2^2\end{aligned}$$

and the proof is complete.

4. Proof of Theorem 1.

NECESSITY. Conditions (i) and (ii) are plainly necessary in view of Theorem 2 and the fact that $\rho(T) \leqq \kappa(T)$. Moreover, if $\kappa(T) \to 0$, then the lemma implies that $P\chi^{-T}P$ is of Hilbert–Schmidt class for T large enough and hence, by Theorem 3, β must be finite. Therefore condition (iii) is necessary.

SUFFICIENCY. Let ψ_k: $k = 1, 2, \cdots$, be an orthonormal basis for $H^2 \ominus BH^2$ and fix $T > 0$ so that $P\chi^{-T}P$ is of Hilbert–Schmidt class, as is permissible by Theorem 3. Then

$$\mathfrak{p}\chi^{-T}P\psi_k = P\chi^{-T}P\psi_k$$

since $\mathfrak{p}\chi^{-T}P\psi_k$ belongs to $H^2 \ominus BH^2$, and

$$\begin{aligned}\textstyle\sum_{k=1}^{\infty} \|\mathfrak{p}\chi^{-T} P\psi_k\|_2^2 &= \textstyle\sum_{k=1}^{\infty} \|P\chi^{-T}P\psi_k\|_2^2 \\ &= |P\chi^{-T}P|_2^2 < \infty\,.\end{aligned}$$

Now as

$$\begin{aligned}\|\mathfrak{p}\chi^{-(S+T)}P\psi_k\|_2 &= \|\mathfrak{p}\chi^{-S}(\mathfrak{p} + \mathfrak{q})\chi^{-T}P\psi_k\|_2 \\ &= \|\mathfrak{p}\chi^{-S}\mathfrak{p}\chi^{-T}P\psi_k\|_2 \\ &\leqq \|\mathfrak{p}\chi^{-T}P\psi_k\|_2\end{aligned}$$

for $S \geqq 0$, the principle of dominated convergence permits you to conclude that

$$\begin{aligned}\lim_{S\uparrow\infty} |P\chi^{-(S+T)}P|_2^2 &= \lim_{S\uparrow\infty} \textstyle\sum_{k=1}^{\infty} \|\mathfrak{p}\chi^{-(S+T)}P\psi_k\|_2^2 \\ &= \textstyle\sum_{k=1}^{\infty} \lim_{S\uparrow\infty} \|\mathfrak{p}\chi^{-(S+T)}P\psi_k\|_2^2 \\ &= \textstyle\sum_{k=1}^{\infty} \lim_{S\uparrow\infty} 2\pi \int_{S+T}^{\infty} |\psi_k^{\vee}(x)|^2\, dx \\ &= 0\,.\end{aligned}$$

An application of the lemma serves to complete the proof.

REFERENCES

[1] DYM, H. and MCKEAN, H. P. (1976). *Gaussian Processes, Function Theory, and the Inverse Spectral Problem*. Academic Press, New York and London.

[2] DYM, H. and SHAPIRO, I. (1977). Conditions for restricted translation operators to belong to S_p. *Proc. Amer. Math. Soc.* **63** 251–258.

DEPARTMENT OF MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT, ISRAEL