The Annals of Probability
1978, Vol. 6, No. 1, 85–93

INSENSITIVITY OF STEADY-STATE DISTRIBUTIONS OF GENERALIZED SEMI-MARKOV PROCESSES. PART II

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In a number of well-known applied probability models certain steady-state probabilities display an insensitivity property: they depend only on the means of certain lifetime distributions entering the definition of the model, not on their exact shapes. This phenomenon has been studied by Matthes and co-workers in a general framework. New and simple proofs are given for their essential results.

1. Introduction. In [15] a certain insensitivity phenomenon was investigated for a class of stochastic processes named generalized semi-Markov processes (GSMP's). The class was introduced by Matthes in [13] for the purpose of studying this phenomenon. The results derived in [15] are due to him with one exception of minor importance (Corollary 4.1 of [15]), the purpose of [15] being a presentation of new and simple proofs of those results. We restricted our attention in [15] to a special case of the insensitivity problem (we varied one lifetime-distribution only).

The first part of the present sequel of [15] deals with the unrestricted insensitivity problem as formulated in [11] and [12]. Using the terminology of [15] (familiarity with [15] being assumed henceforth) this problem reads: find conditions for an irreducible generalized semi-Markov scheme (GSMS) $\Sigma = (G, S, p)$ to be $\Phi_{S'}(\lambda_s;\, s \in S)$-insensitive, where $\emptyset \neq S' \subset S$ and $\Phi_{S'}(\lambda_s;\, s \in S) = \{\phi : \phi \in \Phi(\Sigma),\ \varphi_s$ exponential with mean λ_s^{-1} for $s \notin S'$, φ_s arbitrary with mean λ_s^{-1} for $s \in S'\}$. If $S' = \{s_0\}$ for some $s_0 \in S$, we refer to our phenomenon as $\Phi_{s_0}(\lambda_s;\, s \in S)$-insensitivity.[1] This special case was the topic of [15]. The general case reduces to the special one as seen from Theorem 3.1 below, which implies that Σ is $\Phi_{S'}(\lambda_s;\, s \in S)$-insensitive if and only if it is $\Phi_{s_0}(\lambda_s;\, s \in S)$-insensitive for every $s_0 \in S'$.

Decisive for the proof of this theorem are certain properties exhibited by a $\Phi_{S'}(\lambda_s;\, s \in S)$-insensitive GSMS or the augmented (see [15]) GSMP's based upon such a GSMS. One of these, the property of S'-disconnectedness, is the topic of Section 2. The other one, the so-called product property, has already been important in [15] and will exhibit its scope more fully in Section 3.

The second part of the present paper is devoted to an important extension of

Received January 21, 1977; revised April 18, 1977.

[1] In [15], we spoke of Φ_{s_0}-insensitivity instead. However, the collection of maps ϕ under consideration depends on the means λ_s^{-1}; $s \in S$. We may have Φ_{s_0}-insensitivity for one choice of these means but not for other choices. The notation of the present paper is designed to avoid possible confusion in this respect.

AMS 1970 *subject classification*. Primary 60K99; Secondary 60K15, 60K20.

Key words and phrases. Generalized semi-Markov processes, insensitivity, reliability, queues.

the search for insensitivity. In a GSMP based upon some irreducible $\Sigma = (G, S, p)$ the successive lifetimes of an element $s \in S$ can be viewed as the successive intervals between points of a renewal point process which, when stationary, has intensity λ_s. If Σ is $\Phi_{s_0}(\lambda_s;\ s \in S)$-insensitive, we may thus state: the stationary distribution $\{p_g;\ g \in G\}$ of a GSMP based upon Σ is insensitive towards the choice of the renewal point process generating the lifetimes of s_0, as long as this process has intensity λ_{s_0} and the means λ_s^{-1} of the exponential lifetime distributions φ_s, $s \in S/\{s_0\}$, remain fixed. Analogously $\Phi_{S'}(\lambda_s;\ s \in S)$-insensitivity can be described. It has now been shown in [11] and [12] that, if the notion of a GSMP is generalized such as to allow the successive lifetimes of an element $s \in S$ to be viewed as the successive intervals between points of an arbitrary random point process on the real line, then $\Phi_{s_0}(\lambda_s;\ s \in S)$-insensitivity implies that $\{p_g;\ g \in G\}$ is insensitive even towards the choice of the arbitrary random point process generating the lifetimes of s_0, as long as the latter, when stationary, possesses intensity λ_{s_0}. Thus the insensitivity phenomenon extends considerably beyond the framework of the ordinary GSMP.

The proof in [11] and [12] for this fact can only be understood by a reader who possesses a certain background in the theory of random point processes. It is, of course, necessary to use this theory for defining the adequate extension of the notion of a GSMP. We found it, however, possible to demonstrate the existence of this extended phenomenon in such a manner that the theory of random point processes enters the argument only in a last step of a weak convergence nature, whereas the essence emerges entirely within the framework of ordinary GSMP's. The latter is shown in Section 4, the weak convergence step being carried out elsewhere and in a more general setting.

We conclude this paper in Section 5 with some comments on related work.

2. The property of S'-disconnectedness. An irreducible GSMS $\Sigma = (G, S, p)$ is said to be S'-disconnected, where S' is a given nonempty subset of S, if $p(g, s, g') = 0$ for all triples (g, s, g') such that either $s \in S'$, $s \in g'$, and $|(g' \cap S')/(g \cap S')| \geqq 1$ or $s \notin S'$ and $|(g' \cap S')/(g \cap S')| \geqq 2$. In words: at most one element of S' can be activated at a time. We now prove

THEOREM 2.1. *If Σ is $\Phi_s(\lambda_s;\ s \in S)$-insensitive for every $s \in S'$, then Σ is S'-disconnected.*

PROOF. The proof can be found in [11] and hence is sketched here only because [11] is hard to obtain. We shall carry out the proof for the case that S' contains just two elements, the generalization following obvious lines. Let then $S' = \{s_1, s_2\}$, and define

$$G_{12} = \{g : g \in G,\ s_1, s_2 \in g\}\,,$$
$$G_1 = \{g : g \in G,\ s_1 \in g,\ s_2 \notin g\}\,,$$
$$G_2 = \{g : g \in G,\ s_1 \notin g,\ s_2 \in g\}\,, \qquad \text{and}$$
$$\bar{G} = G/(G_{12} \cup G_1 \cup G_2)\,.$$

Denote by $\{p_g\}$ the (insensitive) stationary distribution under consideration. Then, for $g \in G_{12}$, the equations (4.1) of [15] may be written as

$$\Lambda_g p_g = c_g + u_g + v_g + \lambda_{s_1}\alpha_{1,g} + \lambda_{s_2}\alpha_{2,g} + \sum_{g' \in G_{12}} p_{g'} \sum_{s \neq s_1, s_2; s \in g' \cap S} p(g', s, g)\lambda_s \,,$$

where

$$c_g = \sum_{g' \in \bar{G}} p_{g'} \sum_{s \in g' \cap S} p(g', s, g)\lambda_s + \sum_{g' \in G_1} p_{g'} p(g', s_1, g)\lambda_{s_1} + \sum_{g' \in G_2} p_{g'} p(g', s_2, g)\lambda_{s_2} \,,$$

$$u_g = \sum_{g' \in G_1} p_{g'} \sum_{s \neq s_1; s \in g' \cap S} p(g', s, g)\lambda_s \,,$$

$$v_g = \sum_{g' \in G_2} p_{g'} \sum_{s \neq s_2; s \in g' \cap S} p(g', s, g)\lambda_s \,,$$

and

$$\alpha_{i,g} = \sum_{g' \in G_{12}} p_{g'} p(g', s_i, g) \,, \qquad i = 1, 2 \,.$$

The assumptions of $\Phi_{s_1}(\lambda_s;\, s \in S)$-insensitivity and $\Phi_{s_2}(\lambda_s;\, s \in S)$-insensitivity imply, respectively, the equations

$$\lambda_{s_1} p_g = c_g + u_g + \alpha_{1,g}\lambda_{s_1}$$

and

$$\lambda_{s_2} p_g = c_g + v_g + \alpha_{2,g}\lambda_{s_2} \,,$$

for $g \in G_{12}$. These are just the equations (4.6) of [15]. Hence, for $g \in G_{12}$,

$$(\Lambda_g - \lambda_{s_1} - \lambda_{s_2})p_g - \sum_{g' \in G_{12}} p_{g'} \sum_{s \neq s_1, s_2; s \in g' \cap S} p(g', s, g)\lambda_s = -c_g \,.$$

Summing up these equations for all $g \in G_{12}$ yields

$$0 = -\sum_{g \in G_{12}} c_g \,.$$

But $c_g \geqq 0$ and hence $c_g = 0$ for all $g \in G_{12}$. This implies immediately the statement of our theorem.

3. The product property. Theorem 2.1 and the results obtained in [15] for $\Phi_s(\lambda_s;\, s \in S)$-insensitivity put us now in a position to prove

THEOREM 3.1. *In an irreducible* GSMS $\Sigma = (G, S, p)$ *is* $\Phi_s(\lambda_s;\, s \in S)$-*insensitive for every* $s \in S'$, *where* S' *is a nonempty subset of* S, *then* Σ *is* $\Phi_{S'}(\lambda_s;\, s \in S)$-*insensitive.*

PROOF (sketch). We start letting

$$F_s(t) = \sum_{i=1}^{K^{(s)}} \pi_i^{(s)} E^i_{\lambda^{(s)}}(t) \,, \qquad s \in S' \tag{3.1}$$

(see [15], relation (5.2), for this notation).

We adopt the point of view, suggested by the form of (3.1), that the life of an element $s \in S'$ always consists, with probability $\pi_i^{(s)}$, of i consecutive independent exponentially distributed time phases of mean length $1/\lambda^{(s)}$, starting with phase i. Accordingly we construct a process which, at any time, reports the state g, $g \in G$, of the system along with the number of the phase in which every element $s \in g \cap S'$ is found. This process is an irreducible Markov chain with finite state space and stationary transition probabilities.

Denoting the state of the chain by $(g;\, (i_s),\, s \in g \cap S')$, where i_s is the number of the phase s is found in, and by (g) if $g \cap S' = \emptyset$, we write down the familiar system of linear equations for the (unique) stationary distribution $(p_{(g)}, p_{(g;(i_s),s\in g\cap S')})$ of our chain. Denoting by (p_g) the stationary distribution of the GSMP based upon Σ by means of the exponential family in $\Phi_{S'}$ we then verify that a probability solution of the above system of linear equations is given by

$$(3.2.1) \qquad p_{(g;(i_s),s\in g\cap S')} = p_g \prod_{s\in g\cap S'} \frac{\lambda_s}{\lambda^{(s)}} (\pi_{i_s}^{(s)} + \pi_{i_s+1}^{(s)} + \cdots + \pi_{K^{(s)}}^{(s)})$$

and

$$(3.2.2) \qquad p_{(g)} = p_g \, .$$

The verification is direct, using the relations (4.6) of [15] for (p_g) and all $s \in S'$, and using the S'-disconnectedness of Σ as derived in Section 2. Both of these facts are consequences of the assumed $\Phi_s(\lambda_s; s \in S)$-insensitivity for all $s \in S'$. Observing then that (3.2.1) implies

$$\sum_{(i_s)} p_{(g;(i_s),s\in g\cap S')} = p_g \, ,$$

we conclude that our theorem is proved under the restriction (3.1). We then drop this restriction by means of the weak convergence argument following the proof of Theorem 5.2 in [15].

The proof just sketched parallels the one given for Theorem 5.2 of [15], whence we skip the details.

If at a given time (3.2.1) and (3.2.2) represent the distribution of the chain constructed above, then, for the corresponding augmented GSMP, it follows that the probability that the state at this time is g, $g \in G$, and the residual lifetimes of the elements $s \in g \cap S'$ do not exceed $x^{(s)}$ is given by

$$(3.3) \qquad p_g \prod_{s\in g\cap S'} \lambda_s \int_0^{x^{(s)}} (1 - F_s(t))\, dt \, ,$$

where $F_s(t)$, $s \in S'$, is given by (3.1), and where $g \cap S' \neq \emptyset$ is assumed.

The corresponding statement for the case of a single-element S' has been made in [15], and the corresponding property of this distribution has been called there the product property. Thus we have shown that $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity implies the product property (3.3) for the stationary distribution of any augmented GSMP based upon Σ by means of a family $\phi \in \Phi_{S'}$.

Now assume the latter and choose, for every fixed $s \in S'$, a ϕ such that all lifetime distributions are exponential except φ_s. By the remark following Theorem 5.3 of [15] Σ is thus seen to be $\Phi_s(\lambda_s; s \in S)$-insensitive for every $s \in S'$. In view of Theorem 3.1 we have therefore

THEOREM 3.2. *Let $\Sigma = (G, S, p)$ be irreducible and S' a nonempty subset of S. Then Σ is $\Phi_{S'}(\lambda_s; s \in S)$-insensitive if and only if the stationary distribution of every augmented* GSMP *based upon Σ by means of a family $\phi \in \Phi_{S'}$ possesses the product property.*

Our proof of this theorem contains a test for $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity: for every fixed $s \in S'$ choose a $\phi \in \Phi_{S'}(\lambda_s; s \in S)$ such that φ_s is nonexponential and all other lifetime distributions are exponential. If the stationary distributions of the resulting augmented GSMP's possess the product property, then Σ is $\Phi_{S'}(\lambda_s; s \in S)$-insensitive. Another test, amounting to checking only one GSMP, is given by

THEOREM 3.3. *Let $\Sigma = (G, S, p)$ be irreducible and S' a nonempty subset of S. Let $\phi \in \Phi_{S'}(\lambda_s; s \in S)$ be a family such that φ_s is nonexponential for every $s \in S'$. If the stationary distribution of the augmented* GSMP *based upon Σ by means of ϕ possesses the product property, then Σ is $\Phi_{S'}(\lambda_s; s \in S)$-insensitive.*

The corresponding statement for the case of a single-element S' is that of Theorem 5.1 in [15]. Our methods seem to allow proofs of these statements only under the restriction that the nonexponential lifetime distributions in ϕ are of type (3.1). The proof of Theorem 3.3 is then sketched as follows: construct a phase process just as the one considered in the proof of Theorem 3.1; write down the linear system for its stationary probabilities; by assumption, the unique probability solution of this system is of the form (3.2), where (p_g) is a probability distribution on G; inserting this solution into the given system of linear equations shows that (p_g) statisfies the equations (4.6) of [15] for every $s \in S'$; hence, by Theorem 5.3 (iii) of [15], Σ is $\Phi_s(\lambda_s; s \in S)$-insensitive for every $s \in S'$; the rest follows by Theorem 3.1.

A complete proof of Theorem 3.3 is found in [12].

4. An extension. We start out with an irreducible GSMS $\Sigma = (G, S, p)$, for which some element $s_0 \in S$ has been singled out, and an irreducible stochastic matrix $R = (r_{ij})_1^K$, $K < \infty$, with stationary law $(q_i)_1^K$. Moreover, we need a set $S_0 = \{s_{01}, \cdots, s_{0K}\}$ satisfying $S \cap S_0 = \emptyset$ and $g \cap S_0 = \emptyset$ for every $g \in G$ and a set $M = \{m_1, \cdots, m_K\}$ satisfying $M \cap (S \cup S_0) = \emptyset$ and $g \cap M = \emptyset$ for every $g \in G_0$, where $G_0 = \{g: g \in G, s_0 \in g\}$. We now define a GSMS $\Sigma^* = (G^*, S^*, p^*)$ by letting (i) G^* be the collection of the sets $g \cup \{m_i\}$ for $g \in G/G_0$, $i = 1, \cdots, K$ and the sets $(g/\{s_0\}) \cup \{s_{0i}\}$ for $g \in G_0$, $i = 1, \cdots, K$; (ii) $S^* = (S/\{s_0\}) \cup S_0$; and (iii) (using the shorter symbolic notation (g, i) for $g \cup \{m_i\}$ or $(g/\{s_0\}) \cup \{s_{0i}\}$) by setting

$$
\begin{aligned}
p^*((g, i), s^*, (g', j) &= p(g, s^*, g') && g, g' \in G_0, \quad s^* \in S^*/S_0, \quad j = i, \quad \text{or} \\
& && g, g' \in G/G_0, \quad j = i \\
&= p(g, s_0, g')r_{ij} && g, g' \in G_0, \quad s^* = s_{0i} \\
&= p(g, s_0, g') && g \in G_0, \quad g' \in G/G_0, \quad s^* = s_{0i}, \quad j = i \\
&= p(g, s^*, g')r_{ij} && g \in G/G_0, \quad g' \in G_0 \\
&= 0 && \text{all other choices.}
\end{aligned}
$$

It is easily seen that Σ^* is again irreducible. This GSMS allows us to modify a GSMP based upon Σ as follows: the successive lifetimes of s_0 are taken to be the successive holding times of the states of a semi-Markov process with state space

$\{1, \cdots, K\}$, jump matrix R and holding time distributions which do not depend on the state to be jumped to next; the states $g \in G$ are marked; the mark m_i of a state $g \in G/G_0$ represents the fact that the most recent lifetime of s_0 was of type i, i.e., corresponded to a holding time of the state i of the semi-Markov process; the mark s_{0i} of a state $g \in G_0$ signalizes that the current lifetime of s_0 is of type i; the mark m_i is added as a passive component to the states $g \in G/G_0$; the mark s_{0i}, however, is an active component of the marked state, replacing the former component s_0: it might be interpreted as being the old s_0 in type-i manifestation. We now have

THEOREM 4.1. *If Σ is $\Phi_{s_0}(\lambda_s; s \in S)$-insensitive with stationary distribution $\{p_g; g \in G\}$, then Σ^* is $\Phi_{S_0}(\mu_s; s \in S^*)$-insensitive for every choice of $\{\mu_s; s \in S^*\}$ satisfying $\mu_s = \lambda_s$, $s \in S^*/S_0$, and $\mu_{s_{0i}} = \lambda_{s_{0i}}$, $i = 1, \cdots, K$, with*

$$\textstyle\sum_{i=1}^{K} q_i \lambda_{s_{0i}}^{-1} = \lambda_{s_0}^{-1} .$$

The stationary distribution $\{p^*_{(g,i)}; g \in G, i = 1, \cdots, K\}$ of the states of G^* is given by

$$(*) \qquad \begin{aligned} p^*_{(g,i)} &= p_g \frac{\lambda_{s_0}}{\lambda_{s_{0i}}} q_i \quad && g \in G_0 \\ &= p_g q_i && g \in G/G_0 . \end{aligned}$$

REMARK 4.1. We note that $\sum_{i=1}^{K} p^*_{(g,i)} = p_g$, $g \in G$. Thus the theorem implies the following statement: suppose Σ is $\Phi_{s_0}(\lambda_s; s \in S)$-insensitive with stationary distribution $\{p_g; g \in G\}$. Then this distribution remains unchanged, if the renewal process generating the lifetimes of s_0 is replaced by a semi-Markov generated point process of the same intensity λ_{s_0}. Once this fact has been established, one would expect that an arbitrary point process of intensity λ_{s_0} in place of the original renewal point process would not affect $\{p_g; g \in G\}$ either. For, as has been shown by Herrmann in [4], every stationary point process on the real line of finite intensity can be obtained as the weak limit of semi-Markov generated stationary point processes of the same intensities. Hence, a weak convergence argument will supply the desired extension of Theorem 4.1. Our point is that, in view of Herrmann's theorem, the extended insensitivity phenomenon is essentially established by Theorem 4.1.

REMARK 4.2. Theorem 4.1 exhibits insensitivity with respect to the various manifestations s_{0i} of s_0. This fact has been proved in [11] and [12], too.

REMARK 4.3. We restrict ourselves in Theorem 4.1 to the case of $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity for a set S' containing just one element, s_0. The general case leads to a statement analogous to that of Theorem 4.1. In particular, if the renewal processes generating the lifetimes of the elements $s \in S'$ are replaced by semi-Markov generated point processes of intensities λ_s, $s \in S'$, then $\{p_g; g \in G\}$ remains again unaffected. The construction of the corresponding extension Σ^* of Σ is obvious and the rest of the argument follows that of the proof of Theorem 4.1.

For instance, if $S' = \{s_1, s_2\}$, we have to use two marks for each state $g \in G$ and obtain the stationary distribution

$$\begin{aligned} p^*_{(g,i_1,i_2)} &= p_g \frac{\lambda_{s_1}}{\lambda_{s_1,i_1}} q_{1,i_1} \frac{\lambda_{s_2}}{\lambda_{s_2,i_2}} q_{2,i_2}\,, && s_1, s_2 \in g \\ &= p_g \frac{\lambda_{s_1}}{\lambda_{s_1,i_1}} q_{1,i_1} q_{2,i_2}\,, && s_1 \in g\,, \quad s_2 \notin g \\ &= p_g\, q_{1,i_1} \frac{\lambda_{s_2}}{\lambda_{s_2,i_2}} q_{2,i_2}\,, && s_1 \notin g\,, \quad s_2 \in g \\ &= p_g q_{1,i_1} q_{2,i_2}\,, && s_1, s_2 \notin g\,, \end{aligned}$$

where the notation will be understood. The details are omitted.

PROOF OF THEOREM 4.1. In view of Theorem 3.1 we just have to show that Σ^* is $\Phi_{s_{0i}}(\mu_s;\, s \in S^*)$-insensitive for every i, $i = 1, \cdots, K$, with $p^*_{(g,i)}$ as stated. By Theorem 5.3 (iii) of [15] this, in turn, is true if the distribution $(p^*_{(g,i)};\, g \in G$, $i = 1, \cdots, K\}$ statisfies the two equilibrium conditions (4.1) and (4.6) of [15]. For $g \in G_0$ and $i = 1, \cdots, K$, (4.1) of [15] is satisfied if

$$(\lambda_{s_{0i}} + \Lambda_g - \lambda_{s_0}) p^*_{(g,i)} = \textstyle\sum_{g' \in G} \sum_{j=1}^K p^*_{(g',j)} \sum_{s^*} p^*((g', j), s^*, (g, i)) \lambda_{s^*}$$

where

$$\Lambda_g = \textstyle\sum_{s \in g \cap S} \lambda_s\,.$$

Inserting (*), we have to show that

$$\begin{aligned} (\lambda_{s_{0i}} + \Lambda_g - \lambda_{s_0}) p_g \frac{\lambda_{s_0}}{\lambda_{s_{0i}}} q_i &= \textstyle\sum_{g' \in G_0} p_{g'} \frac{\lambda_{s_0}}{\lambda_{s_{0i}}} q_i \sum_{s^* \in S^*/S_0} p(g', s^*, g) \lambda_{s^*} \\ &\quad + \textstyle\sum_{g' \in G_0} p_{g'} \sum_{j=1}^K \frac{\lambda_{s_0}}{\lambda_{s_{0j}}} q_j p(g', s_0, g) r_{ji} \lambda_{s_{0j}} \\ &\quad + \textstyle\sum_{g' \in G/G_0} p_{g'} \sum_{j=1}^K q_j \sum_{s^*} p(g', s^*, g) \lambda_{s^*} r_{ji}\,. \end{aligned}$$

Here, the last two terms on the right-hand side can be written as

$$q_i \big(\textstyle\sum_{g' \in G/G_0} p_{g'} \sum_{s \in g' \cap S} p(g', s, g) \lambda_s + \sum_{g' \in G_0} p_{g'} p(g', s_0, g) \lambda_{s_0} \big)\,,$$

which equals $q_i \lambda_{s_0} p_g$ on account of (4.6) of [15], since Σ is $\Phi_{s_0}(\lambda_s;\, s \in S)$-insensitive. Thus it remains to show that

$$(\Lambda_g - \lambda_{s_0}) p_g = \textstyle\sum_{g' \in G_0} p_{g'} \sum_{s \in g' \cap S;\, s \neq s_0} p(g', s, g) \lambda_s$$

which, again on account of the $\Phi_{s_0}(\lambda_s;\, s \in S)$-insensitivity of Σ, follows directly from (4.1) and (4.6) of [15]. It is similarly easy to check that the $p^*_{(g,i)}$ as given by (*) satisfy (4.1) of [15] for $g \in G/G_0$ and $i = 1, \cdots, K$. Finally, the $p^*_{(g,i)}$ satisfy (4.6) of [15], if

$$\begin{aligned} \lambda_{s_{0i}} p^*_{(g,i)} &= \textstyle\sum_{g' \in G/G_0} \sum_{j=1}^K p^*_{(g',j)} \sum_s p(g', s, g) \lambda_s r_{ji} \\ &\quad + \textstyle\sum_{g' \in G_0} \sum_{j \neq i} p^*_{(g',j)} p(g', s_0, g) \lambda_{s_{0j}} r_{ji} \\ &\quad + \textstyle\sum_{g' \in G_0} p^*_{(g',i)} p(g', s_0, g) \lambda_{s_{0i}} r_{ii} \end{aligned}$$

for every $g \in G_0$ and every i, $i = 1, \cdots, K$ (note that the first two terms (last term) on the right-hand side above correspond to the first term (last term) on the right-hand side of (4.6) in [15]). Again, using (4.1) and (4.6) of [15] for $\{p_g; g \in G\}$, it is easily checked that the above relation holds. This finishes the proof.

5. Conclusion. The present report together with [15] contain new proofs of essential results on $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity. We believe that these proofs are considerably simpler than those of [11] and [12] and hence will enable the reader to penetrate quickly into a most interesting area of research. We also believe that our methods might prove useful for discovering further insensitivity phenomena. Furthermore, in Section 4 of Part II, we have indicated how the widely known and well understood methods of analysis based upon renewal or, more generally, semi-Markov renewal processes, may be used to derive results which might then readily be extended, using Herrman's theorem, to hold for models involving arbitrary point processes. This idea, although not new (see, e.g., [4] and [3]), has, to the best of our knowledge, not been fully applied to any particular model and should also prove useful for discovering new phenomena and relations for many models of applied probability.

We are now going to conclude this paper by pointing out some further results on $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity.

It is shown in [12] (by simple means) that, if some Σ is $\phi_{S'}(\lambda_s; s \in S)$-insensitive, then its stationary distribution is given by the formula

$$p_g = aq_g \prod_{s \in g \cap S'} \lambda_s^{-1}, \qquad g \in G,$$

where a is a norming constant and q_g does not depend in any way on the distributions φ_s, $s \in S'$. Furthermore, it is shown in [12], also by simple means, that $\Phi_{S'}(\lambda_s; s \in S$-)-insensitivity implies $\Phi_{S'}(\mu_s; s \in S)$-insensitivity for any collection $\{\mu_s; s \in S\}$ with $\mu_s = \lambda_s$, $s \in S/S'$. For a wealth of important examples exhibiting $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity, such as the Erlang and Engset schemes from teletraffic theory and certain reliability schemes, we also refer to [11] and [12]. More recently, Koenig, Jansen and co-workers have investigated further model classes with respect to insensitivity and have also generalized the theory to allow for the lifetimes of active components to be realized with nonnegative speeds depending on the states $g \in G$ (see [6], [8], [9], and [10]). Brumelle [2], Kelly [7], Barbour [1], Oakes [14], and the author [16], [17], have also made contributions in this direction.

There are, of course, many further types of insensitivity which deserve study. We have already mentioned in Part I the paper of Jacobi ([5]) who gave an example of a GSMS for which $\Phi_{S'}(\lambda_s; s \in S)$-insensitivity holds only under the additional condition that the distributions φ_s, $s \in S'$ are all the same. This example does not possess the product property. Wolff and Wrightson ([18]) have recently found another such example.

REFERENCES

[1] BARBOUR, A. O. (1976). Networks of queues and the method of stages. *Advances in Appl. Probability* **8** 584–591.

[2] BRUMELLE, S. (1976). A generalization of Erlang's loss system to state dependent arrival and service rates. Report ORC 76-13, Operations Research Center, Berkeley.

[3] FRANKEN, P. (1968). Erlangsche Formeln fuer semimarkovschen Eingang. *Elektron. Informationsverarbeit. Kybernetik.* **4** 197–204.

[4] HERRMANN, U. (1965). Ein Approximationssatz fuer Verteilungen stationaerer zufaelliger Punktfolgen. *Math. Nachr.* **30** 377–381.

[5] JACOBI, H. (1965). Eine Unempfindlichkeitseigenschaft fuer geordnete Buendel ungeordneter Teilbuendel. *Wiss. Z. Friedrich-Schiller-Univ. Jena Math.-Natur. Reihe* **14** 251–260.

[6] JANSEN, U. and KOENIG, D. (1976). Invariant stationaere Zustandswahrscheinlichkeiten fuer eine Klasse stochastischer Modelle mit funktionellen Abhaengigkeiten. *Math. Operationsforsch. Statist.* **7** 497–522.

[7] KELLY, F. P. (1976). Networks of queues. *Advances in Appl. Probability* **8** 416–432.

[8] KOENIG, D. and JANSEN, U. (1974). Stochastic processes and properties of invariance for queuing systems with speeds and temporary interruptions. *Trans. 7th Prague Conf. on Inf. Theory, Statist. Decision Functions, Random Processes.*

[9] KOENIG, D. and JANSEN, U. (1976). Eine Invarianzeigenschaft zufaelliger Bedienungsprozesse mit positiven Geschwindigkeiten. *Math. Nachr.* **70** 321–364.

[10] KOENIG, D., JANSEN, U., KOTZUREK, M. and RABE, H. (1976). Ergebnisse von Invarianzuntersuchungen fuer ausgewaehlte Bedienungs-und Zuverlaessigkeitssysteme. *Math. Operationsforsch. Statist.* **7** 523–557.

[11] KOENIG, D., MATTHES, K. and NAWROTZKI, K. (1967). Verallgemeinerungen der Erlangschen und Engsetschen Formeln. *Schr. Inst. Math. Berlin, B.*

[12] KOENIG, D., MATTHES, K. and NAWROTZKI, K. (1974). Unempfindlichkeitseigenschaften von Bedienungsprozessen. Appendix in *Introduction to Queuing Theory* (B. W. Gnedenko and I. N. Kovalenko, eds.). Akademie Verlag, Berlin.

[13] MATTHES, K. (1962). Zur Theorie der Bedienungsprozesse. *Trans. 3rd Prague Conf. on Inf. Theory, Statist. Decision Functions, Random Processes.*

[14] OAKES, D. (1976). Random overlapping intervals—a generalization of Erlang's loss formula. *Ann. Probability* **4** 940–946.

[15] SCHASSBERGER, R. (1977). Insensitivity of steady state-distributions of generalized semi-Markov processes. Part I. *Ann. Probability* **5** 87–99.

[16] SCHASSBERGER, R. (1977). Insensitivity of stationary probabilities in networks of queues. Res. Paper No. 332, Dept. of Mathematics and Statistics, Univ. of Calgary.

[17] SCHASSBERGER, R. (1977). Insensitivity of steady-state distributions of generalized semi-Markov processes with speeds. Res. Paper No. 329, Dept. of Mathematics and Statistics, Univ. of Calgary.

[18] WOLFF, R. W. and WRIGHTSON, C. W. (1975). An extension of Erlang's loss formula. Report ORC 75-19, Operations Research Centre, Univ. of Calif., Berkeley.

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