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CLUSTERING AND DISPERSION RATES FOR SOME INTERACTING PARTICLE SYSTEMS ON $\mathbb{Z}^1$

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It is well known that the voter model on $\mathbb{Z}^d$ under initial product measure will converge weakly to a point process as $t \to \infty$; if $d \geq 3$, convergence will be to a nontrivial point process; if $d = 1, 2$, convergence will be to a linear combination of trivial point processes. Therefore, for $d = 1, 2$, the cluster size of a particular state around any fixed point tends to become arbitrarily large as $t \to \infty$. Here we examine the rate of growth for $d = 1$ of this clustering for the nearest neighbor voter model and the related problem of interparticle distance for nearest neighbor coalescing random walks and annihilating random walks. We show that under spatial renormalization $t^{\frac{1}{2}}$ these cluster sizes/interparticle distances in each case approach a nondegenerate distribution. We examine these distributions and obtain numerical estimates for these and related problems.

1. Introduction. Three of the simplest types of interacting particle systems are the basic voter model, coalescing random walks, and annihilating random walks. The present paper deals with asymptotics for these systems in the one-dimensional case.

Voter models have been studied by Holley and Liggett [13], and independently by Clifford and Sudbury [4]; see also [15] and [9]. The informal description is as follows. Each site of the d-dimensional integer lattice $\mathbb{Z}^d$ is occupied by a person who is either in favor of or opposed to some proposition. The "voter" at $x \in \mathbb{Z}^d$ is influenced by his $2d$ nearest neighbors, and changes his opinion at an exponential rate proportional to the number of neighboring voters with the opposite opinion. If all $2d$ neighbors disagree with the person at x, the flip rate is 1. A typical state for the model is a subset of $\mathbb{Z}^d$: namely, the set A of sites occupied by voters who favor the proposition. Thus, writing $S = \{\text{all subsets of } \mathbb{Z}^d\}$, $\mathfrak{M} = \{\text{all probability measures on } S\}$, the voter model is a family of continuous time Markov processes with state space S, one process for each initial measure $\mu \in \mathfrak{M}$.

We denote the basic voter model by $\{(\zeta_t^\mu)_{t\geq 0}; \mu \in \mathfrak{M}\}$, or simply $\{(\zeta_t^\mu)\}$. The foremost question for this system is: starting from a state of "individual independence", i.e., a Bernoulli product measure $\mu_\lambda \in \mathfrak{M}$ with density $\lambda \in (0, 1)$, does the interaction lead to "eventual consensus" or not? It was shown in [13] and [4] that the answer is "consensus" if $d = 1$ or 2, "no consensus" if $d \geq 3$. More precisely, if $d = 1$ or 2 the distribution of $\zeta_t^{\mu_\lambda}$ converges weakly to a mixture of $\mu_0 =$ "all against" and $\mu_1 =$ "all for" as $t \to \infty$, whereas for $d \geq 3$ the distribution of $\zeta_t^{\mu_\lambda}$

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converges to a nondegenerate steady state ν_λ. In the latter case, ν_λ has interesting properties; see [13] and [3]. When $d = 1$ or 2, the convergence to a mixture of delta measures indicates *clustering*: as time goes on, the connected component of $\zeta_t^{\mu_\lambda}$ or $\mathbb{Z}^d - \zeta_t^{\mu_\lambda}$ which contains any site $x \in \mathbb{Z}^d$ tends to become larger and larger. A natural problem in this setting is to determine the rate of clustering. We will study clustering properties of the basic voter model in one dimension.

Coalescing random walks have appeared in various contexts in [13], [14], [12] and [9]. By a continuous time simple random walk on $\mathbb{Z}^d$, we mean (unless otherwise noted) a Markov process which makes transitions like the familiar discrete time simple random walk after mean-1 exponential holding times. The basic system of coalescing random walks on $\mathbb{Z}^d$, denoted by $\{(\xi_t^\mu)\}$, is described as follows. Particles initially distributed according to $\mu \in \mathfrak{M}$ attempt to execute independent continuous time simple random walks, but an interference mechanism takes place: whenever a particle attempts to jump to a site already occupied by another, the two particles coalesce, i.e., merge into one. $\xi_t^\mu \in S$ is the set of occupied sites at time t.

Annihilating random walks were first considered by Erdös and Ney [7], and later treated in [14], [1], [17], [19] and [9]. They are described in the same manner as the coalescing system, except for the interference mechanism. The processes (η_t^μ) comprising the basic annihilating random walks consist of particles performing independent continuous time simple random walks, but such that *both* particles disappear whenever a collision takes place.

As noted in [9], for *any* $d \geqslant 1$, the coalescing and annihilating random walks, starting from arbitrary $\mu \in \mathfrak{M}$, both converge to μ_0 (the empty state) as $t \to \infty$. Due to collisions, the extant particles *disperse* as $t \to \infty$. For these models, one seeks to determine the rate of dispersion. In one dimension, this amounts to finding the growth rate of the interparticle distance; we are indebted to Peter Ney for posing this problem in connection with the annihilating random walks.

In essence, our results state that when $d = 1$, cluster size in the voter model and interparticle distance in the random walk systems are of order $t^{\frac{1}{2}}$ at time t. After establishing notation and a few preliminary definitions and results in Section 2, we determine the asymptotic (in time) mean cluster size/interparticle distance when averaged over the entire space for the three systems in Section 3. With the aid of Birkhoff's ergodic theorem, these asymptotics can be computed exactly in all three cases, starting from suitable initial μ including the product measures μ_λ. Sections 4 and 5 deal respectively with several variants of (i) the distance between the two particles surrounding the origin in the random walk systems, and (ii) the size of the cluster containing the origin in the voter model. We prove that after spatial renormalization by $t^{\frac{1}{2}}$ at time t, these random variables converge in distribution as $t \to \infty$.

We examine the domain of attraction for each model, that is, the family of initial μ which lead to that limit. In each case, the domain of attraction contains the measures μ_λ. We discuss the coalescing random walk first because it is easiest to handle. In fact, its limiting distribution will be determined explicitly: it is a normal distribution truncated at zero. The limit law for the annihilating model will not be analyzed directly, but will be reduced to a special case of the voter model problem treated in Section 5. The limiting behavior of the voter model is more difficult to analyze than that of the coalescing model. We will proceed indirectly by modifying the model so as to simplify the effect of the interdependence among the different voters. This will be done in such a manner as to leave unchanged the system's ultimate behavior, but so that we will be able to apply a version of the invariance principle to the problem. It will turn out that the limiting distributions encountered are not elementary, but nevertheless have exponential tail.

Our methods of proof rely on various close connections among the three systems at hand, and between them and certain embedded random walks. Some connections are expressed via *duality equations* ([22], [13], [11], [14], [12]), which enable one to reinterpret the processes in such a manner as to considerably simplify conceptualization. We will find it convenient to utilize these relations throughout the paper; after the introduction of some more terminology, we conclude this section with a typical application.

For a given $\mu \in \mathfrak{M}$, we define

$$\varphi^{\mu}(\Lambda) = \mu\{A : A \cap \Lambda = \varnothing\},$$

where $\Lambda \in S_0 = \{\text{finite subsets of } \mathbb{Z}^d\}$. One of the duality equations asserts that

$$P(\xi_t^{\mu} \cap \Lambda = \varnothing) = E\left[\varphi^{\mu}(\zeta_t^{\Lambda})\right] \qquad \text{for} \quad \mu \in \mathfrak{M}, \quad \Lambda \in S_0.$$

(We will use the symbols P and E for the probability law and expectation operator of any of our processes, and write the initial state as a superscript when a process starts deterministically.) Another relevant observation is that when Λ is a *block*, i.e., $\Lambda = [x, y] = \{x, x+1, \cdots, y\}$, then the process $(|\zeta_t^{[x,y]}| - 1)$ ($|\Lambda|$ denoting the cardinality of Λ) is a continuous time simple random walk with *mean*-$\frac{1}{2}$ *holding times* which starts at $y - x \geqslant 0$ and is absorbed at -1. We denote such a random walk by (Y_t^{y-x}). Let (Z_t^x) be a *mean*-$\frac{1}{2}$ simple random walk without absorption, and let τ_y^x be the hitting time of $y \in \mathbb{Z}$ for (Z_t^x). Also, for $A \in S$, let $D_0^+(A)$ be the distance from 0 to the first site to the right of 0 which is occupied by A.

We now apply the preceding observations to compute asymptotics for $E[D_0^+(\xi_t^{\mathbf{Z}})]$, the expected distance at time t from 0 to the first particle to the right of 0, for the coalescing random walks started at "all 1's". Specifically,

$$\begin{aligned} E\left[D_0^+(\xi_t^{\mathbf{Z}})\right] &= \textstyle\sum_{n=0}^{\infty} P\left(D_0^+(\xi_t^{\mathbf{Z}}) > n\right) \\ &= \textstyle\sum_{n=0}^{\infty} P\left(\xi_t^{\mathbf{Z}} \cap [0, n] = \varnothing\right) \\ &= \textstyle\sum_{n=0}^{\infty} P\left(\zeta_t^{[0,n]} = \varnothing\right) \end{aligned}$$

by the preceding duality equation. By the last observation, this equals

$$\begin{aligned} \textstyle\sum_{n=0}^{\infty} P(Y_t^n = -1) &= \textstyle\sum_{n=1}^{\infty} P\left(\tau_n^0 \leqslant t\right) \\ &= E\left[M_t\right], \end{aligned}$$

where $M_t = \max_{0 \le s \le t} Z_s^0$. Since it is well known (cf. problem 4, page 232 of [20]) that $E[M_t] \sim 2(t/\pi)^{\frac{1}{2}}$, we obtain $E[D_0^+(\xi_t^{\mathbb{Z}})] \sim 2(t/\pi)^{\frac{1}{2}}$. Note that the expected distance between the two particles surrounding the origin is therefore $\sim 4(t/\pi)^{\frac{1}{2}}$.

We should emphasize that our methods throughout the paper depend critically on the one-dimensionality of the systems we study, and on our assumption that interaction is restricted to nearest neighbors. Corresponding problems without the nearest neighbor assumption, and *especially* in higher dimensions, are important for applications, but are apparently much more difficult to treat. Related results, in one *and* higher dimensions, have recently been obtained for (i) critical branching particle systems on $\mathbb{R}^d$ ([5], [6], [8]), and (ii) "stepping stone" models on $\mathbb{Z}^d$ with infinitely many types of particles [18].

2. Preliminaries. Let $\{(\zeta_t^\mu)\}$, $\{(\xi_t^\mu)\}$, and $\{(\eta_t^\mu)\}$ be the basic one-dimensional voter model, and coalescing and annihilating random walks, respectively, as defined in the introduction. This section contains the basic equations, terminology, and preliminary results which enter into our analysis of these three systems.

We will use three *duality equations*, the first of which has already been mentioned. These are, for $\mu \in \mathfrak{M}$, $\Lambda \in S_0$, $t \geq 0$,

$$P(\xi_t^\mu \cap \Lambda = \varnothing) = E[\varphi^\mu(\zeta_t^\Lambda)], \tag{1}$$

$$P(\zeta_t^\mu \cap \Lambda = \varnothing) = E[\varphi^\mu(\xi_t^\Lambda)], \tag{2}$$

$$P(|\eta_t^\mu \cap \Lambda| \text{ even}) = E[\psi^\mu(\zeta_t^\Lambda)], \tag{3}$$

where $\varphi^\mu(\Lambda) = \mu\{A : A \cap \Lambda = \varnothing\}$ and $\psi^\mu(\Lambda) = \mu\{A : |A \cap \Lambda| \text{ even}\}$. Proofs of (1) and (2) can be found in [10]–[14], while (3) is proved in [14] and [10]. We also define $\bar{\varphi}^\mu(\Lambda) = \mu\{A : \Lambda \subset A\}$, for which (2) implies that

$$P([0, x] \subset \zeta_t^\mu) = E[\bar{\varphi}^\mu(\xi_t^{[0,x]})]. \tag{2'}$$

Manipulating (3), one also obtains the useful variant

$$P(\eta_t^\mu \cap \Lambda = \varnothing) = 2^{-|\Lambda|}\Sigma_{B \subset \Lambda} E[2\psi^\mu(\zeta_t^B) - 1]. \tag{3'}$$

A *border equation*, first exploited by Schwartz [19], will be useful in conjunction with annihilating random walks. The equation states that the "borders" between voters of opposite opinion execute annihilating random walks. We write $A(x) = 1$ for $x \in A$, and $A(x) = 0$ for $x \notin A$. Then, for any $\mu \in \mathfrak{M}$, $x, y \in \mathbb{Z}$, and $t \geq 0$, the border equation states that

$$P\big(|\eta_t^{\tilde{\mu}} \cap [x+1, y]| \text{ even}\big) = P(\zeta_t^\mu(x) = \zeta_t^\mu(y)), \tag{4}$$

where $\tilde{\mu}$ is the *border measure* corresponding to μ, defined by

$$\varphi^{\tilde{\mu}}(\Lambda) = \mu\{A : A(x-1) = A(x), \quad x \in \Lambda\}.$$

Next, we discuss the connections between our systems and the simple random walks (Y_t^x) (jump rate 2, absorption at -1) and (Z_t^x) (jump rate 2, no absorption).

(In this paper, (X_t^x) will denote a simple random walk with jump rate 1 and no absorption.) As already noted,

(5) $$(|\zeta_t^{[x,y]}| - 1) \quad \text{is a copy of} \quad (Y_t^{y-x}).$$

Also, if diam $\Lambda = \max_{x,y\in\Lambda}|y - x|$ is the diameter of Λ, then

(6) $$\left(\operatorname{diam} \xi_t^{\{x,y+1\}} - 1\right) \quad \text{is a copy of} \quad (Y_t^{y-x}).$$

Throughout the paper, we will be making use of certain basic properties of random walks, which are simple consequences of the reflection principle, the central limit theorem, and the local limit theorem. Namely, if $x \in \mathbb{Z}^+ \cup \{0\}$ and $\alpha \in \mathbb{R}^+$, then

(7) $$P(Y_t^x \in B) = P(Z_t^x \in B) - P(Z_t^x \in -(2 + B)), \qquad B \subset [0, \infty),$$

(8) $$P\left(Y_t^0 \geqslant 0\right) \sim \frac{1}{(\pi t)^{\frac{1}{2}}} \quad \text{as} \quad t \to \infty,$$

(9) $$P(Y_t^x \geqslant 0) \leqslant c\frac{x+1}{t^{\frac{1}{2}}} \text{ for some } c \text{ independent of } x \text{ and } t,$$

(10) $$\lim_{t\to\infty} P\left(Y_t^{\lfloor \alpha t^{\frac{1}{2}} \rfloor} \geqslant 0\right) = \pi^{-\frac{1}{2}}\int_0^{\alpha} e^{-u^2/4}\, du,$$

where $\lfloor u \rfloor$ denotes the greatest integer less than u, and $\lceil u \rceil$ denotes the least integer greater than u, and if $g(t) = o(t^{\frac{1}{4}})$ as $t \to \infty$, then

(11) $$\lim_{t\to\infty} \sup_{x\geqslant 0} t^{\frac{1}{2}} P\left(Y_t^x \in \left[0, g(t)\right]\right) = 0.$$

In Section 5, we will also find it convenient to make a pathwise comparison of systems of coalescing random walks commencing from different initial states, A and B. (The same basic comparison also has analogues for annihilating random walks and the voter model). We make the comparison by coupling the systems (ξ_t^A) and (ξ_t^B) so that random walks present at the same site of each process undergo the same motion. (ξ_t^A) and (ξ_t^B) may be considered to be simultaneously evolving over the same probability space; clearly, for $A \subset B$, $\xi_t^A \subset \xi_t^B$ for all realizations for all times t. (For a detailed presentation, see [12]).

Various classes of measures in $\mathfrak{M}$ will qualify as "nice" initial states for the limit theorems we have in mind. For the remainder of the paper, we restrict our attention to translation invariant (that is, spatially stationary) μ. The evolution laws of our three particle systems are such that the systems remain translation invariant for all time. $\mu \in \mathfrak{M}$ is called *mixing*, if, in addition,

$$\lim_{|x|\to\infty}|\varphi^{\mu}(B_0 \cup (x + B_1)) - \varphi^{\mu}(B_0)\varphi^{\mu}(x + B_1)| = 0 \qquad \text{for} \quad B_0, B_1 \in S_0,$$

and *n-fold mixing* $(n \geqslant 2)$ if

$$\lim_{|x_{i_1} - x_{i_2}|\to\infty \text{ for all } i_1, i_2}\left|\varphi^{\mu}\left(\bigcup_{i=0}^{n}(x_i + B_i)\right) - \prod_{i=0}^{n} \varphi^{\mu}(x_i + B_i)\right| = 0 \qquad \text{for} \quad B_i \in S_0.$$

Of special interest in our study of annihilating random walks will be the *renewal measures* on $\mathbb{Z}$ ([21]). Given a probability density $f = (f_k;\ k = 1, 2, \cdots)$ such that

$M = \Sigma\, kf_k < \infty$, the renewal measure μ_f determined by f is the (translation invariant) element of $\mathfrak{M}$ with cylinder probabilities

$$\begin{aligned}\mu_f\{A : A(x) = 1, A(x + y_1) = 1, \cdots, A(x + y_1 + \cdots + y_n) = 1,\\ A(z) = 0 \text{ for all other } z \in [x, x + y_1 + \cdots + y_n]\}\\ = M^{-1}\Pi_{l=1}^{n} f_{y_l}.\end{aligned}$$

Note that the product measure μ_λ, $\lambda \in (0, 1]$, is the renewal measure μ_f with $f_k = \lambda(1 - \lambda)^{k-1}$. It is also true that if a renewal measure is aperiodic, then it is mixing (and n-fold mixing for all n).

We conclude this section with several results which will be useful later on. The first result, Lemma 1, is an assertion regarding the parity of a renewal measure.

LEMMA 1. *Let μ_f be a renewal measure with support $f = mA$ for some $m \geq 1$, $A \subset \mathbb{Z}^+$, where $A = \{1\}$ or some two elements of A are relatively prime. If A contains both even and odd integers, then*

(12) $$\psi^{\mu_f}([0, n]) \to \tfrac{1}{2} \quad \text{as} \quad n \to \infty,$$

whereas if A contains only odd integers, then for some p, $0 \leq p \leq 1$,

(13) $$\psi^{\mu_f}([0, i + 2mn]) \to \delta_i \quad \text{as} \quad n \to \infty,$$

where $\delta_i = (|m - i - 1|/m)p + ((m - |m - i - 1|)/m)(1 - p)$ for $0 \leq i \leq 2m - 1$. In particular, $\{\delta_i\}$ satisfies

(14) $$\frac{1}{2m}\Sigma_{i=0}^{2m-1}\, \delta_i = \tfrac{1}{2}.$$

PROOF. Let $\omega \subset \mathbb{Z}$ be μ_f-distributed, and write $N_\Lambda(\omega) = |\omega \cap \Lambda|$ = the number of renewals of ω on $\Lambda \in S_0$. Define $p_n = \mu\{N_{[1,n]} \text{ is even} | \omega(0) = 1\}$, $n \geq 1$, and note that (p_n) satisfies the renewal equation

(15) $$p_n = a_n + \Sigma_{k=1}^{n}\, (f*f)_k p_{n-k},$$

where $a_n = \Sigma_{k=n+1}^{\infty} f_k$. If $m = 1$ and A contains both an even integer and an odd integer, then $f*f$ is aperiodic, and the renewal theorem states that

(16) $$p_n \to \frac{\Sigma_{k=1}^{\infty}\Sigma_{l=k}^{\infty} f_l}{\Sigma_{k=1}^{\infty} k(f*f)_k} = \frac{M}{2M} = \tfrac{1}{2} \quad \text{as} \quad n \to \infty.$$

Since

$$\begin{aligned}&\lim_{n\to\infty} \psi^{\mu_f}([0, n])\\ &= \lim_{n\to\infty} \Sigma_{k=0}^{n-1}\, \mu\{N_{[k+1,n]} \text{ is odd} | \omega(k) = 1, \omega(j) = 0 \quad \text{for} \quad 0 \leq j < k\}\\ &\qquad \cdot \mu\{\omega(k) = 1, \omega(j) = 0 \quad \text{for} \quad 0 \leq j < k\}\\ &= \lim_{n\to\infty} \Sigma_{k=0}^{n-1}\, (1 - p_{n-k}) \cdot \mu\{\omega(k) = 1, \omega(j) = 0 \quad \text{for} \quad 0 \leq j < k\},\end{aligned}$$

this implies (12) in the case where $m = 1$. To demonstrate (12) for general m, note that if a realization of μ_f is concentrated on $m\mathbb{Z}$, then the parity for $[0, mn]$ and

$[0, i + mn]$ are the same for $0 \leqslant i < m$. If the set A contains only odd integers, then an analogue of (16) is still true, although the limiting ratio need no longer be $\frac{1}{2}$. A more complicated version of the above reasoning, together with the remark that μ_f is translation invariant, implies (13). We omit the details.

The final two results deal with the rate at which particular systems of coalescing random walks coalesce. The first (Lemma 2) is used only as an aid for proving the second, whereas the second (Lemma 3) will be applied in Sections 4 and 5.

LEMMA 2. *Let a and b denote positive integers. Then,*

$$P\big(|\xi_T^{\{-a, 0, b\}}| = 3\big) \leqslant Cab/t \tag{17}$$

for some constant C which is independent of a, b and t.

PROOF. Let L and l be piecewise constant right continuous functions with $l < L$. (The functions, in our case, will turn out to be realizations of continuous time simple random walks.) We first show that the simple random walk (X_t^0) satisfies

$$\begin{aligned} (18) \quad & P\big(l(s) < X_s^0 < L(s) \text{ for } 0 \leqslant s \leqslant t\big) \\ & \leqslant P\big(l(s) < X_s^0 \text{ for } 0 \leqslant s \leqslant t\big) \cdot P\big(X_s^0 < L(s) \text{ for } 0 \leqslant s \leqslant t\big). \end{aligned}$$

To see this, we set, for $0 \leqslant r \leqslant t$,

$$u(r, x) = P\big(l(s) < X_s^0 \text{ for } r \leqslant s \leqslant t | X_r^0 = x\big) \tag{19a}$$

and

$$v(r, x) = P\big(X_s^0 < L(s) \text{ for } r \leqslant s \leqslant t | X_r^0 = x\big). \tag{19b}$$

u and v are both space-time harmonic functions: u in the region $x > l(r)$ and v in the region $x < L(r)$. That is, u satisfies

$$u_r(r, x) + \tfrac{1}{2}\big[u(r, x+1) + u(r, x-1)\big] - u(r, x) = 0,$$

and v satisfies the analogous equation. A simple computation shows that the product, $u \cdot v$, is a space-time superharmonic function within the region $l(r) < x < L(r)$; $u \cdot v$ satisfies the boundary conditions

$$u \cdot v(r, l(r)) = u \cdot v(r, L(r)) = 0 \tag{20a}$$

and

$$u \cdot v(t, x) = 1 \qquad \text{for} \quad l(t) < x < L(t). \tag{20b}$$

Next, we observe that

$$w(r, x) = P\big(l(s) < X_x^0 < L(s) \qquad \text{for} \quad r \leqslant s \leqslant t | X_r^0 = x\big) \tag{21}$$

is a space-time harmonic function for $l(r) < x < L(r)$. Since w possesses the same boundary values as given in (20) for $u \cdot v$, it follows that

$$w(r, x) \leqslant u \cdot v(r, x) \qquad \text{for} \quad l(r) < x < L(r).$$

This implies inequality (18).

We now derive (17) from (18). Observe that the independent random walks (X_s^{-a}), (X_s^0) and (X_s^b) satisfy

$$P\big(|\xi_t^{\{-a,0,b\}}| = 3\big) = P\big(X_s^{-a} < X_s^0 < X_s^b \quad \text{for} \quad 0 \leq s \leq t\big).$$

If we let $(\Omega, P) = (\Omega_1, P_1) \times (\Omega_2, P_2)$ denote the product probability space induced by the pair $(X_s^{-a}, X_s^b;\ 0 \leq s \leq t)$, then this equals

$$\int_\Omega P\big(l(s) < X_s^0 < L(s) \quad \text{for} \quad 0 \leq s \leq t\big)\, dP,$$

where (l, L) is viewed as an element of Ω. Equation (18) implies this is at most

$$\int_\Omega P\big(l(s) < X_s^0 \quad \text{for} \quad 0 \leq s \leq t\big) \cdot P\big(X_s^0 < L(s) \quad \text{for} \quad 0 \leq s \leq t\big)\, dP.$$

By the independence of the random walks (X_s^{-a}) and (X_s^b), this equals

$$\begin{aligned} \int_{\Omega_1} P\big(l(s) < X_s^0 \quad \text{for} \quad 0 \leq s \leq t\big)\, dP_1 \cdot \int_{\Omega_2} P\big(X_s^0 < L(s) \quad \text{for} \quad 0 \leq s \leq t\big)\, dP_2 & \\ = P\big(|\xi_t^{\{-a,0\}}| = 2\big) \cdot P\big(|\xi_t^{\{0,b\}}| = 2\big) & \\ \leq Cab/t & \end{aligned}$$

by (9), where $C = c^2$. Thus,

$$P\big(|\xi_t^{\{-a,0,b\}}| = 3\big) \leq Cab/t,$$

which completes the proof.

Lemma 3. *For any positive integer x,*

$$P\big(|\xi_t^{[0,x]}| > 2\big) \leq C'x^2/t \tag{22}$$

for some constant C' which is independent of x and t.

Proof. We consider the case $x = 2^n$. We define the events $E_{i,j}$, for $j = 1, 2, \cdots, 2^i$, and $i = 0, 1, \cdots, n-1$, so that

$$E_{0,1} = \big\{|\xi_t^{\{0,2^{n-1},2^n\}}| = 3\big\},\ E_{1,1} = \big\{|\xi_t^{\{0,2^{n-2},2^{n-1}\}}| = 3\big\},$$
$$E_{1,2} = \big\{|\xi_t^{\{2^{n-1},3\cdot 2^{n-2},2^n\}}| = 3\big\}, \cdots, E_{i,j} = \big\{|\xi_t^{A_{i,j}}| = 3\big\}, \cdots,$$

where $A_{i,j} = \{(j-1)2^{n-i}, (2j-1)2^{n-i-1}, j2^{n-i}\}$. A little contemplation shows that the set $\{|\xi_t^{[0,x]}| > 2\}$ may be rewritten as

$$\bigcup_{i=0}^{n-1} \bigcup_{j=1}^{2^i} E_{i,j}.$$

(In other words, if there still exist more than two distinct random walks by time t, then the random walk commencing at some $y \in (0, x), y = (2j-1)2^{n-i-1}$, has not yet coalesced with either of its two nearest neighbors commencing on the lattice consisting of multiples of 2^{n-i}). Therefore,

$$P\big(|\xi_t^{[0,x]}| > 2\big) \leq \textstyle\sum_{i=0}^{n-1} \sum_{j=1}^{2^i} P(E_{i,j}),$$

which, by Lemma 2, is at most

$$\textstyle\sum_{i=0}^{n-1} \sum_{j=1}^{2^i} C \cdot 2^{2(n-i-1)}/t < 2^{2n-1}C/t = Cx^2/2t.$$

The same estimate holds trivially for arbitrary x, but with $C/2$ replaced by $C' = 2C$.

REMARK. In the application of Lemma 3 to Theorems 3 and 5, we will see that the presence of the factor x^2, rather than x, is crucial.

3. Mean dispersion and clustering rates. Given a configuration $A \subset \mathbb{Z}$, we let

$$D(A) = \text{the mean interparticle distance for } A$$

in the case of the coalescing and annihilating random walks, and

$$C(A) = \text{the mean cluster size for } A$$

in the case of the voter model. D and C are defined as limits of their natural restrictions to $[-n, n]$ as $n \to \infty$ (provided the limits exist). Note that the clusters of configuration A are the connected components of *either* A *or* A^c. In this section, we compute asymptotics for $D(\xi_t^\mu)$, $D(\eta_t^\mu)$, and $C(\zeta_t^\mu)$ as $t \to \infty$, starting from appropriate $\mu \in \mathfrak{M}$.

THEOREM 1. *As* $t \to \infty$,

(a) $$\frac{D(\xi_t^\mu)}{t^{\frac{1}{2}}} \to \pi^{\frac{1}{2}} \quad \textit{in probability}$$

for all mixing $\mu \in \mathfrak{M}$, $\mu \neq \mu_0$ (μ_0 *assigns measure one to the empty set*),

(b) $$\frac{D(\eta_t^\mu)}{t^{\frac{1}{2}}} \to 2\pi^{\frac{1}{2}} \quad \textit{in probability}$$

for all renewal measures $\mu_f \in \mathfrak{M}$, *and*

(c) $$\frac{C(\zeta_t^\mu)}{t^{\frac{1}{2}}} \to \frac{\pi^{\frac{1}{2}}}{2\lambda(1-\lambda)} \quad \textit{in probability}$$

for all mixing $\mu \in \mathfrak{M}$ *with density* $\lambda \in (0, 1)$.

PROOF. For convenience, we divide the proof into two steps; in combination, they yield the results.

Step I. If $\mu \in \mathfrak{M}$ is mixing, then with probability one

(Ia) $$D(\xi_t^\mu) = \left[P(0 \in \xi_t^\mu)\right]^{-1} \quad \text{for} \quad \mu \neq \mu_0,$$

(Ib) $$D(\eta_t^\mu) = \left[P(0 \in \eta_t^\mu)\right]^{-1} \quad \text{for} \quad \mu \neq \mu_0,$$

(Ic) $$C(\zeta_t^\mu) = \left[P(\zeta_t^\mu(0) \neq \zeta_t^\mu(1))\right]^{-1} \quad \text{for} \quad \mu \neq \mu_0, \mu_1.$$

The argumentation for (Ia)–(Ic) is in all cases similar, so we will derive only (Ic). As in the beginning of Section 2, we say that A has a border at x, $A \subset \mathbb{Z}$, $x \in \mathbb{Z}$, if $A(x-1) \neq A(x)$; as before, ζ_t^μ induces a border measure. Now, since μ is mixing, so is ζ_t^μ (see [10] for a proof). It immediately follows that the border measure associated with ζ_t^μ is also mixing. Consequently, Birkhoff's ergodic theorem implies that with probability one,

$$\lim_{n\to\infty}\frac{|\{\text{borders of } \zeta_t^\mu \text{ in } [-n,n]\}|}{2n} = P(\zeta_t^\mu \text{ has a border at } 1),$$

which is positive. Since the number of clusters of the set A in $[-n, n]$ differs from the number of borders by at most one, it follows that, with probability one,

$$C(\zeta_t^\mu) = [P(\zeta_t^\mu \text{ has a border at } 1)]^{-1} = [P(\zeta_t^\mu(0) \neq \zeta_t^\mu(1))]^{-1},$$

demonstrating (Ic).

Step IIa. If $\mu(\{\varnothing\}) = 0$, then

(22) $$P(0 \in \xi_t^\mu) \sim \frac{1}{(\pi t)^{\frac{1}{2}}} \quad \text{as} \quad t \to \infty.$$

(Of course, if $\mu \neq \mu_0$ is mixing, then $\mu(\{\varnothing\}) = 0$.) To obtain (22), we first note that (5) and (8) imply that

(23) $$P(\zeta_t^{\{0\}} \neq \varnothing) \sim \frac{1}{(\pi t)^{\frac{1}{2}}} \quad \text{as} \quad t \to \infty.$$

We now show that the difference of the probabilities in (22) and (23) goes to zero faster than $1/t^{\frac{1}{2}}$. Applying (1), we write

$$P(\zeta_t^{\{0\}} \neq \varnothing) - P(0 \in \xi_t^\mu) = E[\varphi^\mu(\zeta_t^{\{0\}}), \zeta_t^{\{0\}} \neq \varnothing],$$

which is nonnegative. Let $g(t)$ be a function satisfying $g(t) = o(t^{\frac{1}{4}})$ and $g(t) \to \infty$ as $t \to \infty$. By decomposing the above expectation based on whether $|\zeta_t^{\{0\}}|$ is less than or greater than $g(t)$, we obtain the inequality

(24)
$$t^{\frac{1}{2}} E[\varphi^\mu(\zeta_t^{\{0\}}), \zeta_t^{\{0\}} \neq \varnothing] \leqslant t^{\frac{1}{2}} P(0 < |\zeta_t^{\{0\}}| \leqslant g(t))$$
$$+ t^{\frac{1}{2}} P(\zeta_t^{\{0\}} \neq \varnothing) \cdot [\sup_{[x,y]\,:\, y-x > g(t)} \varphi^\mu([x,y])].$$

(Here we use the fact that $\zeta_t^{\{0\}} = [x, y]$ for some $x, y \in \mathbb{Z}$ whenever $\zeta_t^{\{0\}} \neq \varnothing$). By (5) and (11), the first term on the right hand side of (24) tends to 0 as $t \to \infty$, whereas by (23) and $\mu(\{\varnothing\}) = 0$, the second term also tends to 0. Therefore, the left hand side of (24) tends to 0 as $t \to \infty$; this establishes (22).

Step IIb. If $\mu = \mu_f$ is a renewal measure, then

(25) $$P(0 \in \eta_t^\mu) \sim \frac{1}{2(\pi t)^{\frac{1}{2}}} \quad \text{as} \quad t \to \infty.$$

We use the same basic strategy as that employed in the derivation of (22): we estimate $P(0 \in \eta_t^\mu)$ by $P(\zeta_t^{\{0\}} \neq \varnothing)$, whose asymptotic behavior is known to us through (23). A simple computation with the aid of (3) shows that

(26) $$\tfrac{1}{2} P(\zeta_t^{\{0\}} \neq \varnothing) - P(0 \in \eta_t^\mu) = E[\psi^\mu(\zeta_t^{\{0\}}) - \tfrac{1}{2}, \zeta_t^{\{0\}} \neq \varnothing].$$

As before, we choose $g(t)$ so that $g(t) = o(t^{\frac{1}{4}})$ and $g(t) \to \infty$ as $t \to \infty$. We first cover the (simpler) case where $f*f$ is aperiodic (or, more generally, where (12) holds). In this case, we employ the upper bound

$$(27) \quad t^{\frac{1}{2}}E\left[|\psi^{\mu}(\zeta_t^{\{0\}}) - \tfrac{1}{2}|, \zeta_t^{\{0\}} \neq \varnothing\right] \leqslant t^{\frac{1}{2}}P(0 < |\zeta_t^{\{0\}}| \leqslant g(t)) + t^{\frac{1}{2}}P(\zeta_t^{\{0\}} \neq \varnothing) \cdot \left[\sup_{[x,y]:y-x>g(t)}|\psi^{\mu}([x,y]) - \tfrac{1}{2}|\right].$$

As in Step IIa, it follows from (5) and (11) that the first term on the right hand side of (27) tends to 0 as $t \to \infty$, whereas by (23) and Lemma 1, the second term also tends to 0. Therefore, the left hand side of (27) tends to 0 as $t \to \infty$, which, together with (23), implies that $P(0 \in \eta_t^{\mu}) \sim \frac{1}{2}(\pi t)^{\frac{1}{2}}$. For general $f*f$ with period q, we replace (27) with the more complicated expression

$$(28) \quad t^{\frac{1}{2}}\left|E\left[\psi^{\mu}(\zeta_t^{\{0\}}) - \tfrac{1}{2}, \zeta_t^{\{0\}} \neq \varnothing\right]\right| \leqslant t^{\frac{1}{2}}P(0 < |\zeta_t^{\{0\}}| \leqslant g(t)) + t^{\frac{1}{2}} \max_{k=0,1} \left[\Sigma_{i=0}^{q-1} P(|\zeta_t^{\{0\}}| = (i+1) \bmod q, |\zeta_t^{\{0\}}| > g(t)) \cdot \left[(-1)^k(\delta_i - \tfrac{1}{2}) + \sup_{[x,y]=y-x>g(t);\, y-x=i \bmod q}|\psi^{\mu}([x,y]) - \delta_i|\right]\right],$$

with δ_i defined as in (13). Since $\Sigma_{i=0}^{q-1}(\delta_i - \frac{1}{2}) = 0$ and $\sup_{[x,y]:y-x\geqslant g(t);\, y-x=i \bmod q}|\psi^{\mu}([x,y]) - \delta_i| \to 0$ as $t \to \infty$ for all i, the last term on the right hand side of (28) will approach 0 if

$$t^{\frac{1}{2}}\max_{i,j}|P(|\zeta_t^{\{0\}}| = i \bmod q, |\zeta_t^{\{0\}}| > g(t)) - P(|\zeta_t^{\{0\}}| = j \bmod q, |\zeta_t^{\{0\}}| > g(t))|$$

approaches 0 as $t \to \infty$. Equations (5), (8) and (11) imply that this is at most equal to

$$(29) \quad t^{\frac{1}{2}}\max_{i,j}\Sigma_{k=0}^{\infty} P\left(Y_{t-\tilde{g}(t)}^{0} = k\right) \cdot \left|P\left(Z_{\tilde{g}(t)}^{k} = i \bmod q\right) - P\left(Z_{\tilde{g}(t)}^{k} = j \bmod q\right)\right| + o(1),$$

where $\tilde{g}(t)$ is a copy of $g(t)$. Since the random walk (Z_t^k) has undergone a Poisson-distributed number of jumps with mean $2\tilde{g}(t)$ by time $\tilde{g}(t)$, with $\tilde{g}(t) \to \infty$ as $t \to \infty$, it is not hard to show that

$$(30) \quad \max_{i,j}\left|P\left(Z_{\tilde{g}(t)}^{k} = i \bmod q\right) - P\left(Z_{\tilde{g}(t)}^{k} = j \bmod q\right)\right| \to 0 \qquad \text{as} \quad t \to \infty$$

uniformly in k. Since $P(Y_{t-\tilde{g}(t)}^{0} \geqslant 0) \sim 1/(\pi t)^{\frac{1}{2}}$, this implies that (29) $\to 0$ as $t \to \infty$; hence (28) $\to 0$, which implies that $P(0 \in \eta_t^{\mu}) \sim \frac{1}{2}(\pi t)^{\frac{1}{2}}$ in the periodic case as well.

Step IIc. If μ is mixing with density $\lambda \in (0, 1)$, then

$$(31) \qquad P(\zeta_t^{\mu}(0) \neq \zeta_t^{\mu}(1)) \sim \frac{2\lambda(1-\lambda)}{(\pi t)^{\frac{1}{2}}} \qquad \text{as} \quad t \to \infty.$$

To obtain (31), we first rewrite $P(\zeta_t^\mu(0) \neq \zeta_t^\mu(1))$ as

$$P(0 \notin \zeta_t^\mu) + P(1 \notin \zeta_t^\mu) - 2P(\zeta_t^\mu \cap \{0, 1\} = \varnothing). \tag{32}$$

Each of the first two terms equals $(1 - \lambda)$; it therefore follows from (2) that (32) equals

$$2((1 - \lambda) - E[\varphi^\mu(\xi_t^{\{0,1\}})]),$$

or after more algebra,

$$\begin{aligned} 2\big((1 - \lambda) &- (1 - \lambda)P(|\xi_t^{\{0,1\}}| = 1) - E[\varphi^\mu(\xi_t^{\{0,1\}}), |\xi_t^{\{0,1\}}| = 2]\big) \\ &= 2\lambda(1 - \lambda)P(|\xi_t^{\{0,1\}}| = 2) + 2E\big[(1 - \lambda)^2 - \varphi^\mu(\xi_t^{\{0,1\}}), |\xi_t^{\{0,1\}}| = 2\big]. \end{aligned}$$

Statements (6) and (8) together imply that

$$P(|\xi_t^{\{0,1\}}| = 2) \sim \frac{1}{(\pi t)^{\frac{1}{2}}} \qquad \text{as} \quad t \to \infty. \tag{33}$$

Thus, we will have demonstrated (31) once we show that

$$t^{\frac{1}{2}} E\big[|(1 - \lambda)^2 - \varphi^\mu(\xi_t^{\{0,1\}})|, |\xi_t^{\{0,1\}}| = 2\big] \to 0 \qquad \text{as} \quad t \to \infty. \tag{34}$$

Proceeding as in Steps IIa and IIb, and choosing $g(t)$ as before, the left hand side of (34) is at most

$$\begin{aligned} (35)\quad & t^{\frac{1}{2}} P(0 < \operatorname{diam} \xi_t^{\{0,1\}} \leqslant g(t)) \\ &\qquad + t^{\frac{1}{2}} P(|\xi_t^{\{0,1\}}| = 2) \cdot \big[\sup_{x, y = |y - x| > g(t)} |(1 - \lambda)^2 - \varphi^\mu(\{x, y\})|\big]. \end{aligned}$$

It follows from (6) and (11) that the first term of (35) tends to 0 as $t \to \infty$, and from (33) and the hypothesis that μ is mixing that the second term also tends to 0. Therefore, (34) tends to 0 as $t \to \infty$, which yields (31).

REMARKS. Theorem 1 indicates that for the voter model, and the coalescing and annihilating random walks, the asymptotic dispersion and clustering rates are precisely the same for any initial measure μ belonging, in each case, to a certain class of measures. The classes for these three models are qualitatively different. The asymptotics for the voter model depend on the initial density λ, in contrast to the asymptotics for the coalescing and annihilating random walks. On the other hand, the asymptotics for the annihilating random walks are more sensitive to the structure of the initial measure μ than for the other models, in the sense that whereas it is sufficient for the voter model and coalescing random walk to commence from mixing μ in order for Theorem 1 to be valid, there are mixing μ which violate the conclusions of Theorem 1 in the case of the annihilating random walks. For example consider the border measure $\tilde{\mu}_\lambda$ corresponding to the product measure μ_λ, $\lambda \in (0, 1)$. Using the border equation (4), it is easy to show that

$$\frac{D(\eta_t^{\tilde{\mu}_\lambda})}{t^{\frac{1}{2}}} \to \frac{\pi^{\frac{1}{2}}}{2\lambda(1 - \lambda)} \qquad \text{in probability as} \quad t \to \infty,$$

which does not equal $2\pi^{\frac{1}{2}}$ for $\lambda \neq \frac{1}{2}$. Thus, something like the renewal assumption is apparently necessary to guarantee a strict enough degree of uniformity in the spacing of particles undergoing annihilating random walks so as to assure the conclusions of the theorem. (Of course, part (b) holds for mixtures of renewal measures, in particular for any exchangeable μ).

4. Limit theorems for interacting random walks. For $A \subset \mathbb{Z}$, we let

$$D_0^+(A) = \min\{x \geq 0 : x \in A\},\ D_0^-(A) = \min\{x \geq 0 : -x \in A\},$$
$$D_0(A) = D_0^+(A) + D_0^-(A).$$

In this section, we show that, after normalization by $t^{\frac{1}{2}}$, the distance D_0^+ in the random configurations ξ_t^μ and η_t^μ converges in distribution as $t \to \infty$ for suitable initial μ. By symmetry, the identical results, of course, remain valid for D_0^-. It is also possible, by means of a bit of manipulation involving the results for D_0^+, to demonstrate convergence in distribution for D_0 as well; we present the results for D_0 as corollaries to those for D_0^+. In this section, we also obtain estimates for the expectations of the limiting distributions of ξ_t^μ and η_t^μ (the expectations corresponding to D_0 are clearly double those corresponding to D_0^+ and D_0^-); these results may then be compared with those of Theorem 1 in Section 3. Recall that in this paper all measures μ are assumed translation invariant.

THEOREM 2. *Let $\mu(\{\varnothing\}) = 0$. Then, for $\alpha \in [0, \infty)$,*

$$\lim_{t\to\infty} P\left(\frac{D_0^+(\xi_t^\mu)}{t^{\frac{1}{2}}} \leq \alpha\right) = \pi^{-\frac{1}{2}}\int_0^\alpha e^{-u^2/4}\,du. \tag{36}$$

Moreover, if

$$\int D_0^+(A)\mu(dA) < \infty,$$

then

$$\lim_{t\to\infty} E\left[\frac{D_0^+(\xi_t^\mu)}{t^{\frac{1}{2}}}\right] = \frac{2}{\pi^{\frac{1}{2}}}. \tag{37}$$

PROOF. Fix α, and put $z = z(t) = \lfloor \alpha t^{\frac{1}{2}} \rfloor$. We first check (36) for $\mu = \mu_1$:

$$P\left(\frac{D_0^+(\xi_t^{\mathbb{Z}})}{t^{\frac{1}{2}}} \leq \alpha\right) = P(\xi_t^{\mathbb{Z}} \cap [0, z] \neq \varnothing)$$
$$= P(\zeta_t^{[0,z]} \neq \varnothing)$$

by (1). By (5) and (9), this equals

$$P(Y_t^z \geq 0) \sim \pi^{-\frac{1}{2}}\int_0^\alpha e^{-u^2/4}\,du.$$

By comparison with the case μ_1, we now show that (36) holds for general μ.

Equation (1) yields

$$(38)\qquad 0 \leqslant P\left(\frac{D_0^+(\xi_t^{Z})}{t^{\frac{1}{2}}} \leqslant \alpha\right) - P\left(\frac{D_0^+(\xi_t^{\mu})}{t^{\frac{1}{2}}} \leqslant \alpha\right)$$
$$= E\left[\varphi^{\mu}(\zeta_t^{[0,z]}), \zeta_t^{[0,z]} \neq \varnothing\right].$$

Let the function $g(t)$ satisfy $g(t) = o(t^{\frac{1}{4}})$ and $g(t) \to \infty$ as $t \to \infty$. Decomposing the above expectation based on whether $|\zeta_t^{[0,z]}|$ is less than/greater than $g(t)$, it follows from (5) that (38) is at most

$$P(Y_t^z \in [0, g(t)]) + \sup_{x,y\,:\,y-x>g(t)} \varphi^{\mu}([x,y]).$$

By (11), the first term tends to 0 as $t \to \infty$, and since $\mu(\{\varnothing\}) = 0$, the second term also tends to 0. Therefore, (38) tends to 0, and (36) is proved.

To derive (37), it suffices to show that $G(\alpha) = \inf_{t\geqslant 1} P(D_0^+(\xi_t^{\mu}) \leqslant \alpha t^{\frac{1}{2}})$ has finite mean; our assertion then follows from (36) and dominated convergence. We first note that

$$P\left(D_0^+(\xi_t^{\mu}) \leqslant \alpha t^{\frac{1}{2}}\right) = P(\xi_t^{\mu} \cap [0, \alpha t^{\frac{1}{2}}] \neq \varnothing)$$
$$\geqslant P\left(\xi_t^{\mu} \cap \left[-\frac{\alpha}{2}t^{\frac{1}{2}}, \frac{\alpha}{2}t^{\frac{1}{2}}\right] \neq \varnothing\right)$$

by the translation invariance of μ. This is at least

$$(39)\quad P\left(\xi_t^{\{D_0^+(\mu)\}} \cap \left[-\frac{\alpha}{2}t^{\frac{1}{2}}, \frac{\alpha}{2}t^{\frac{1}{2}}\right] \neq \varnothing\right)$$
$$\geqslant P\left(\xi_t^{\{0\}} \cap \left[-\frac{\alpha}{4}t^{\frac{1}{2}}, \frac{\alpha}{4}t^{\frac{1}{2}}\right] \neq \varnothing\right) - P\left(D_0^+(\mu) > \frac{\alpha}{4}t^{\frac{1}{2}}\right).$$

By applying a Chebyshev estimate to

$$E\left[\exp(\lambda X_t^0)\right] = \exp\left\{t\left(\frac{e^{\lambda} + e^{-\lambda}}{2} - 1\right)\right\},$$

one sees that the first term on the right side of (39) has at most an exponential tail, i.e., is $\geqslant 1 - a\exp(-b\alpha)$ for some positive a, b independent of α and t. Since $D_0^+(\mu)$ is assumed to have finite mean, the second term on the right side has mean which is bounded for $t \geqslant 1$; together, these statements imply that $G(\alpha)$ has finite mean. This concludes the proof.

COROLLARY. *Let $\mu(\{\varnothing\}) = 0$. Then, for $\alpha \in [0, \infty)$,*

$$\lim_{t\to\infty} P\left(\frac{D_0(\xi_t^{\mu})}{t^{\frac{1}{2}}} \leqslant \alpha\right) = \frac{1}{\pi^{\frac{1}{2}}}\int_0^{\alpha} e^{-u^2/4}\,du - \frac{\alpha}{\pi^{\frac{1}{2}}}e^{-\alpha^2/4}$$
$$= \frac{1}{2\pi^{\frac{1}{2}}}\int_0^{\alpha} u^2 e^{-u^2/4}\,du.$$

PROOF. Since μ is assumed to be translation invariant, so is ξ_t^μ for all t. Therefore, we may decompose D_0 as

$$\begin{aligned} &P(D_0(\xi_t^\mu) > z) \\ &\quad = P(\xi_t^\mu \cap [0, z] = \varnothing) + \Sigma_{k=0}^{z-1} P(\xi_t^\mu \cap [-k, z-k-1] = \varnothing, z-k \in \xi_t^\mu) \\ &\quad = P(\xi_t^\mu \cap [0, z] = \varnothing) + z \cdot P(\xi_t^\mu \cap [0, z-1] = \varnothing, z \in \xi_t^\mu), \end{aligned}$$

where z is a positive integer. Choosing z to satisfy $z \sim \alpha t^{\frac{1}{2}}$ as $t \to \infty$, we see that (36) implies that

$$P(\xi_t^\mu \cap [0, z] = \varnothing) \to \frac{1}{\pi^{\frac{1}{2}}} \int_\alpha^\infty e^{-u^2/4}\, du \qquad \text{as} \quad t \to \infty.$$

Next, observe that by translation invariance,

$$\begin{aligned} P(\xi_t^\mu \cap [0, z_1 - 1] = \varnothing, z_1 \in \xi_t^\mu) &= P(\xi_t^\mu \cap [z_2 - z_1, z_2 - 1] = \varnothing, z_2 \in \xi_t^\mu) \\ &\leqslant P(\xi_t^\mu \cap [0, z_2 - 1] = \varnothing, z_2 \in \xi_t^\mu) \end{aligned}$$

for $z_1 \geqslant z_2$. Therefore $P(\xi_t^\mu \cap [0, z-1] = \varnothing, z \in \xi_t^\mu)$ is decreasing in z. By (36), it follows after a little manipulation that

$$z \cdot P(\xi_t^\mu \cap [0, z-1] = \varnothing, z \in \xi_t^\mu) \to \frac{\alpha}{\pi^{\frac{1}{2}}} e^{-\alpha^2/4} \qquad \text{as} \quad t \to \infty.$$

Therefore,

$$\lim_{t\to\infty} P\left(\frac{D_0(\xi_t^\mu)}{t^{\frac{1}{2}}} > \alpha \right) = \frac{1}{\pi^{\frac{1}{2}}} \int_\alpha^\infty e^{-u^2/4}\, du + \frac{\alpha}{\pi^{\frac{1}{2}}} e^{-\alpha^2/4};$$

integrating by parts, we obtain

$$\frac{1}{2\pi^{\frac{1}{2}}} \int_\alpha^\infty u^2 e^{-u^2/4}\, du,$$

which establishes the corollary.

REMARK. A reasonable question is to what extent the preceding result about interparticle distances for coalescing random walks, and the analogous results for annihilating random walks and the voter model (which will both be presented later on) can be generalized. In particular, do the distributions of these particle systems tend, under the same normalization as before, to limiting point processes? The answer is yes, although the exact nature of these processes is still uncertain. Since these limiting processes can have no "double points" (for the voter model, no blocks of zero length), it suffices, as before, to analyze the probability that, after a finite time, a finite set A, $A \subset \mathbb{Z}$, contains no particles (only one block); one proceeds by applying the duality equations, letting time run to infinity, and so on. Now, however, A may contain gaps, rather than comprising (the integral values of) an interval. The procedures of the proofs for the annihilating random walks and the voter model are almost verbatim replicas of the arguments employed in this paper to analyze the interparticle distance/block size at the origin; the formerly simpler proof for the coalescing random walks must be enhanced, and must now employ the same basic arguments used in the proofs of the other two particle systems. One may also inquire as to whether a space-time normalization is possible, so that these particle systems approach limiting particle systems. The answer should again be in the affirmative, although the technical complications are now considerably greater.

The corresponding theorem for annihilating random walks is not so immediate: more work is required to show convergence, and the limiting distribution seems not to be of a standard form; we are only able to compute upper and lower bounds for its expectation. In contrast to the limit law for coalescing random walks, which is a truncated normal distribution, the limit law in this case has *exponential* tail. To treat the annihilating random walks, we use the fact that $\mu_{\frac{1}{2}}$ is its own border measure to conclude from (4) that $D_0^+(\eta_t^{\mu_{\frac{1}{2}}})$ has the same distribution as $C_0^+(\zeta_t^{\mu_{\frac{1}{2}}})$ = the number of members in the cluster containing the origin which lie to the right of the origin for the voter model. We therefore defer the crux of the proof to the next section, which contains the limit laws for the clustering of $\{(\zeta_t^{\mu})\}$, and restrict ourselves here to computing $D_0^+(\eta_t^{\mu_f})$ based on our knowledge of $D_0^+(\eta_t^{\mu_{\frac{1}{2}}})$.

Theorem 3. *Let $\mu = \mu_f$ be a renewal measure. Then there is a distribution function $F^{\frac{1}{2}}$ such that*

$$\lim_{t\to\infty} P\left(\frac{D_0^+(\eta_t^{\mu})}{t^{\frac{1}{2}}} \leqslant \alpha\right) = F^{\frac{1}{2}}(\alpha) \qquad \textit{for} \quad \alpha \in [0, \infty). \tag{40}$$

$F^{\frac{1}{2}}$ has monotone decreasing continuous density on the positive half-line, with $F^{\frac{1}{2}}(0) = 0$, and has exponential tail in the sense that

$$1 - F^{\frac{1}{2}}(\alpha) = \exp\{-c_{\frac{1}{2}}(\alpha)\cdot\alpha\},$$

where $c_{\frac{1}{2}}(\alpha) \to c_{\frac{1}{2}} > 0$ as $\alpha \to \infty$.

Proof. As stated above, when $\mu = \mu_{\frac{1}{2}}$, (40) is equivalent to a special case of Theorem 5 of the next section. The properties of $F^{\frac{1}{2}}$ will also be established in Theorems 5 and 6. What remains to be shown is that

$$\lim_{t\to\infty}\left|P\left(\frac{D_0^+(\eta_t^{\mu_f})}{t^{\frac{1}{2}}} \leqslant \alpha\right) - P\left(\frac{D_0^+\left(\eta_t^{\mu_{\frac{1}{2}}}\right)}{t^{\frac{1}{2}}} \leqslant \alpha\right)\right| = 0$$

for μ_f satisfying the hypothesis.

We set $\Lambda = [0, \alpha t^{\frac{1}{2}}]$. By duality equation (3′), the above absolute difference equals

$$2^{-|\Lambda|}\left|\Sigma_{B\subset\Lambda}\left(E\left[2\psi^{\mu_f}(\zeta_t^B) - 1\right] - E\left[2\psi^{\mu_{\frac{1}{2}}}(\zeta_t^B) - 1\right]\right)\right|.$$

Since

$$E\left[2\psi^{\mu\frac{1}{2}}(\zeta_t^B) - 1\right] = P(\zeta_t^B = \varnothing), \tag{41}$$

this reduces to

$$2^{-|\Lambda|}\left|\Sigma_{B\subset\Lambda} E\left[2\psi^{\mu}(\zeta_t^B) - 1, \zeta_t^B \neq \varnothing\right]\right|, \tag{42}$$

which we will show tends to 0 as $t \to \infty$.

To assist us with the remainder of the proof, we next make two assertions.

ASSERTION I. Let $\beta(B)$ denote the minimal cluster length for B, where $B \in S_0$. Then,

$$P\left(\beta(\zeta_t^B) < t^{\frac{1}{8}}\right) \to 0 \qquad \text{uniformly for} \quad B \subset \Lambda \qquad \text{as} \quad t \to \infty. \tag{43}$$

(The exponent $\frac{1}{8}$ is not crucial, but is chosen for convenience). We proceed to demonstrate the assertion by first defining

$$\gamma_t(\xi_\cdot^a) = \min\{z - x : x, y, z \in A, x < y < z, \text{ and the random walks commencing at } x, y, \text{ and } z \text{ of the process } \xi_\cdot^A \text{ are still distinct by time } t\},$$

and setting $\underline{A} = (-\lfloor t^{\frac{5}{8}} \rfloor, \lfloor t^{\frac{5}{8}} \rfloor)$, $\overline{A} = [-\lceil t^{\frac{5}{8}} \rceil, \lceil t^{\frac{5}{8}} \rceil]$. γ measures the tendency of random walks starting from some initial configuration to avoid coalescence; it follows from (1) that β and γ satisfy the relation

$$P\left(\beta(\zeta_t^B) < t^{\frac{1}{8}}, \zeta_t^B \subset \underline{A}\right) \leqslant P\left(\gamma_t\left(\xi_\cdot^{\overline{A}}\right) < t^{\frac{1}{8}} + 1\right). \tag{44}$$

Decomposing the set $\overline{A}$, we see that

$$\begin{aligned} P\left(\gamma_t\left(\xi_\cdot^{\overline{A}}\right) < t^{\frac{1}{8}} + 1\right) &\leqslant \Sigma_{k = -\lceil t^{\frac{1}{2}} \rceil}^{\lceil t^{\frac{1}{2}} \rceil} P\left(\left|\xi_t^{[kt^{\frac{1}{8}}, (k+2)t^{\frac{1}{8}} + 1]}\right| > 2\right) \\ &\leqslant C'' t^{-\frac{1}{4}} \end{aligned} \tag{45}$$

by Lemma 3, for some constant C''. Since $P(\zeta_t^B \not\subset \underline{A}) \leqslant P(\xi_t^\Lambda \not\subset \underline{A}) \to 0$ as $t \to \infty$, (43) follows from (44) and (45).

ASSERTION II. Let x, y be integers satisfying $y - x \geqslant t^{\frac{1}{8}}$. Then,

$$E\left[2\psi^{\mu}(\zeta_{t^{\frac{1}{8}}}^{[x,y]}) - 1\right] \to 0 \qquad \text{uniformly in} \quad x, y \qquad \text{as} \quad t \to \infty.$$

We proceed with the demonstration by noting that since the random walk (Z_t^0) undergoes a Poisson-distributed number of jumps with mean $2t^{\frac{1}{8}}$ by time $t^{\frac{1}{8}}$, for any $q \in \mathbb{Z}^+$,

$$\max_{i,j}\left|P\left(Z_{t^{\frac{1}{8}}}^0 = i \bmod q\right) - P\left(Z_{t^{\frac{1}{8}}}^0 = j \bmod q\right)\right| \to 0 \qquad \text{as} \quad t \to \infty.$$

Also, we note that $|\zeta_{t^{\frac{1}{8}}}^{[x,y]} - 1|$ has the same law as $Y_{t^{\frac{1}{8}}}^{y-x}$, and that

$$P(Y_{t^{\frac{1}{8}}}^{y-x} \neq Z_{t^{\frac{1}{8}}}^{y-x}) \to 0 \qquad \text{uniformly for} \quad y - x \geqslant t^{\frac{1}{8}} \qquad \text{as} \quad t \to \infty;$$

hence

$$\max_{i,j}\left|P\left(\left|{}_{t^{\frac{1}{8}}}^{[2,y]}\right| = i \bmod q\right) - P\left(\left|\zeta_{t^{\frac{1}{8}}}^{[x,y]}\right| = j \bmod q\right)\right| \to 0 \tag{46}$$

uniformly for $y - x \geqslant t^{\frac{1}{8}}$ as $t \to \infty$. Equation (14) of Lemma 1, together with (46), implies that

$$E\left[2\psi^{\mu}\left(\zeta_{t^{\frac{1}{8}}}^{[x,y]}\right) - 1\right] \to 0 \qquad \text{uniformly for} \quad y - x \geqslant t^{\frac{1}{8}} \qquad \text{as} \quad t \to \infty.$$

We are now ready to conclude our demonstration of Theorem 3. We let $\{A_i(t)\}_{i \in I_t}$, $A_i(t) \subset \zeta_t^B$, denote the distinct clusters of voters "for" in the voter model at time t (I_t is the index set), and denote by A_1 the cluster with the smallest coordinates; $A_1(t)$ induces a probability measure $P(t)$. It follows from Assertions I and II that the voter model commencing at random initial state $A_1(t)$ and evolving for time $t^{\frac{1}{8}}$ satisfies

$$\int_{I_t \neq \varnothing} E\left[2\psi^{\mu}\left(\zeta_{t^{\frac{1}{8}}}^{A_1(t)}\right) - 1\right] dP(t) \to 0 \tag{47}$$

uniformly for $B \subset \Lambda$ as $t \to \infty$. Moreover, it also follows from Assertion I that $P(\min_{i \neq 1} \operatorname{dist}(A_1(t), A_i(t)) < t^{\frac{1}{8}}) \to 0$ uniformly in B as $t \to \infty$, and therefore that

P(the cluster commencing from $A_1(t)$ at time 0 has coalesced
with any other clusters A_i by time $t^{\frac{1}{8}}$)
$\to 0$ uniformly in $B \subset \Lambda$ as $t \to \infty$.

Thus the evolution of the cluster commencing at $A_1(t)$ is (up until time $t^{\frac{1}{8}}$) "asymptotically independent" of the other clusters. Consequently, from (47) it follows that

$$E\left[2\psi^{\mu}\left(\zeta_{t+t^{\frac{1}{8}}}^{B}\right) - 1, I_t \neq \varnothing\right] \to 0 \qquad \text{uniformly in} \quad B \subset \Lambda \qquad \text{as} \quad t \to \infty.$$

In view of Assertion I, this is equivalent to showing that (42) $\to 0$ as $t \to \infty$, completing the proof.

In the following corollary, we drop the superscript from $F^{\frac{1}{2}}$.

COROLLARY. *Let $\mu = \mu_f$ be a renewal measure. Then,*

$$\lim_{t \to \infty} P\left(\frac{D_0(\eta_t^{\mu})}{t^{\frac{1}{2}}} \leqslant \alpha\right) = F(\alpha) - \alpha F'(\alpha).$$

PROOF. The proof is similar to that of the corollary to Theorem 2. Since η_t^{μ} is translation invariant,

$$P(D_0(\eta_t^{\mu}) > z) = P(\eta_t^{\mu} \cap [0, z] = \varnothing) + z \cdot P(\eta_t^{\mu} \cap [0, z - 1] = \varnothing, z \in \eta_t^{\mu}),$$

where z is a positive integer. For z chosen so that $z \sim \alpha t^{\frac{1}{2}}$ as $t \to \infty$, (40) implies that

$$P(\eta_t^{\mu} \cap [0, z] = \varnothing) \to 1 - F(\alpha) \qquad \text{as} \quad t \to \infty.$$

On the other hand, $P(\eta_t^{\mu} \cap [0, z] = \varnothing, z \in \eta_t^{\mu})$ is decreasing in z; therefore, since F' is continuous, it follows that

$$z \cdot P(\eta_t^{\mu} \cap [0, z - 1] = \varnothing, z \in \eta_t^{\mu}) \to \alpha F'(\alpha).$$

Consequently,

$$\lim_{t\to\infty} P\left(\frac{D_0(\eta_t^\mu)}{t^{\frac{1}{2}}} > \alpha\right) = 1 - F(\alpha) + \alpha F'(\alpha),$$

which demonstrates the corollary.

NOTE. Figuratively,

$$\lim_{t\to\infty} P\left(\frac{D_0(\eta_t^\mu)}{t^{\frac{1}{2}}} \leqslant \alpha\right) = \text{“}\textstyle\int_0^\alpha - uF''(u)\,du\text{”}.$$

The last result of this section gives upper and lower bounds for the expectation of the normalized limiting distribution of the annihilating random walks.

THEOREM 4. *Let $M^{\frac{1}{2}}$ denote the expectation of $F^{\frac{1}{2}}$. Then,*

$$\frac{4}{\pi^{\frac{1}{2}}} \leqslant M^{\frac{1}{2}} \leqslant 2\pi^{\frac{1}{2}}.$$

PROOF. The upper bound is a special case of the upper bound in Theorem 7. For the lower bound, we use the inequality

$$P(\zeta_t^B = \varnothing) \geqslant P(\zeta_t^{[0,n-1]} = \varnothing) \tag{48}$$

for all B such that $|B| = n$. This inequality is intuitively obvious, since, until absorption, $(|\zeta_t^B|)$ jumps at least as fast as $(|\zeta_t^{[0,n-1]}|)$ because of its greater number of borders, and will therefore tend to reach (and be absorbed by) 0 first; a rigorous proof can be fashioned after the comparison lemma in [9]. Now, set $x = x(t) = \lfloor \alpha t^{\frac{1}{2}} \rfloor$ and $y = y(t) = \lfloor \alpha t^{\frac{1}{2}}/2 \rfloor$. It follows from (3′), (41) and (48), that

$$P\left(\eta_t^{\mu_{\frac{1}{2}}} \cap [0, x] = \varnothing\right) \geqslant 2^{-(x+1)} \textstyle\sum_{n=0}^{x+1} \binom{x+1}{n} P\left(\zeta_t^{[0,n-1]} = \varnothing\right).$$

If we apply the weak law of large numbers to the binomial distribution on the right side of the above inequality, then (5) and (10) imply that

$$\begin{aligned} 1 - F^{\frac{1}{2}}(\alpha) &= \lim_{t\to\infty} P\left(\eta_t^{\mu_{\frac{1}{2}}} \cap [0, x(t)] = \varnothing\right) \\ &\geqslant \lim_{t\to\infty} P\left(Y_t^{y(t)} = -1\right) \\ &= \pi^{-\frac{1}{2}} \textstyle\int_{\alpha/2}^{\infty} e^{-u^2/4}\,du. \end{aligned}$$

Integration of α from 0 to ∞ yields the lower bound.

5. Limit theorems for the voter model.

Introduction. For $A \subset \mathbb{Z}$, we let

$$C_0^+(A) = \min\{x > 0 : A(0) \neq A(x)\},\quad C_0^-(A) = \min\{x > 0 : A(0) \neq A(-x)\},$$
$$C_0(A) = C_0^+(A) + C_0^-(A).$$

C_0^+ denotes the number of consecutive individuals lying directly to the right of 0 which vote the same way as 0, C_0^- denotes the number of individuals lying to the left, and C_0 denotes the complete block size of individuals voting the same way as 0 (0 is counted twice). In this section, we show that, after normalization by $t^{\frac{1}{2}}$, C_0^+ in the random configuration ζ_t^μ converges in distribution as $t \to \infty$, for suitable initial μ. By symmetry, the same result obviously holds for C_0^-. As in Section 4, a bit of manipulation allows one to derive analogous results for C_0, which we state in the form of a corollary. Unlike the limiting distributions for the coalescing random walks, the limiting distributions for the voter model do not appear to be elementary. We nevertheless show that F^λ, the distribution associated with C_0^+ for initial measures μ of density λ, has a monotone decreasing density which is continuous, and has an exponential right-hand tail (as opposed to a normal tail). In contrast to the behavior for the coalescing random walks, the limiting distribution for the voter model is sensitive to the distribution of the initial configurations; note that the limiting distribution for the annihilating random walks is a special case here, being equal to $F^{\frac{1}{2}}$.

Although the duality equations apply here just as well as for the coalescing random walks, the technique required is fundamentally more difficult. Whereas for the coalescing random walks, application of the basic equations (5)–(11) was sufficient to conclude the proof of Theorem 2, the behavior of the dual process for the voter model is sufficiently complicated to necessitate an indirect approach.

Our basic procedure is as follows. Rather than investigate the dual process itself, which is of the form $(\xi_s^{[0,x]}, 0 \leqslant s \leqslant t)$, with $x = \lfloor \alpha t^{\frac{1}{2}} \rfloor$, we examine a sequence of processes $(\xi_s^{B^n}, 0 \leqslant s \leqslant t)$, with $B^n \sim t^{\frac{1}{2}} A^n$ for fixed sets A^n. We choose B^n so that $B^n \subset B^{n+1}$, $|B^n| < \infty$, and $\lim_n B^n = [0, x]$. The point of the modification is that the behavior of $(\xi_t^{B^n})$ as t approaches infinity (convergence to a point process) may be analyzed by means of an invariance principle (Lemma 4). As n approaches infinity, the asymptotic behavior of $(\xi_t^{B^n})$ should approximate that of $(\xi_t^{[0,x]})$, with both being nontrivial under the same spatial normalization; that this is indeed true, and that the implicit interchange of limits involved here is valid, will be demonstrated with the aid of Lemma 3. The statement of the convergence result and the final computations comprise Theorem 5.

In Theorem 6, we derive the basic properties of the limiting distributions F^λ (monotone decreasing continuous density, exponential tail); in Theorem 7, we compute upper and lower bounds for the means of F^λ. As the technique involved in deriving Theorem 5 is not that informative regarding quantitative aspects of the asymptotics, we apply various ad hoc estimates to derive Theorems 6 and 7.

Existence of F^λ. In the proof of Lemma 4, we will freely employ the standard results and notation of Billingsley [2]. $(D[0, 1], \mathfrak{B})$ denotes the space of piecewise continuous functions on [0, 1] (not necessarily assuming the value 0 at time 0). Here, we will use $(D^n[0, 1], \mathfrak{B}^n)$ (or D^n, for short) to denote the induced product space of functions on $[0, 1]^n$. A slight variation of the standard functional convergence result for random walks shows that

$$\frac{X_{st}^{x}}{t^{\frac{1}{2}}} \Rightarrow W_s^{\alpha} \qquad \text{for} \quad s \in [0, 1] \qquad \text{as} \quad t \to \infty,$$

where (W_s^{α}) denotes standard Brownian motion commencing at position α, "$\Rightarrow$" designates weak convergence (in D or the uniform topology; since W_s^{α} is concentrated on the continuous paths, the two are equivalent), and $x = \lfloor \alpha t^{\frac{1}{2}} \rfloor$. (See [2], pages 137 and 145.) Therefore, if $(\vec{X}_s^{\vec{x}})$ and $(\vec{W}_s^{\vec{\alpha}})$ denote n-tuples of independent random walks and Brownian motions commencing at $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n)$, with $x_i = \lfloor \alpha_i t^{\frac{1}{2}} \rfloor$, then

$$\frac{\vec{X}_{st}^{\vec{x}}}{t^{\frac{1}{2}}} \Rightarrow \vec{W}_s^{\vec{\alpha}} \qquad \text{for} \quad s \in [0, 1] \qquad \text{as} \quad t \to \infty, \tag{49}$$

where convergence is in the product topology of D^n. If we denote the probability measures induced on D^n by $\vec{X}_{st}^{\vec{x}}/t^{\frac{1}{2}}$ and $\vec{W}_s^{\vec{\alpha}}$ as P^t and P^{∞}, then (49) states that $P^t \Rightarrow P^{\infty}$. We also employ the result that weak convergence is transmitted from the domain to the image of a function which is almost surely continuous on the measure induced by the limit distribution. In our particular case, if f is a.s. continuous on P^{∞}, then

$$f(P^t) \Rightarrow f(P^{\infty}) \qquad \text{as} \quad t \to \infty. \tag{50}$$

We will also make use of several properties of the sample paths of Brownian motion in Lemma 4. Assume that Brownian motions corresponding to different indices are independent. Then,

$$\begin{aligned} &\text{if } j < k, \text{ and } T = \min\{r : W_r^{\alpha(j)} = W_r^{\alpha(k)}\}, \\ &\text{for all } \varepsilon > 0 \text{ there exist } 0 < \varepsilon', \varepsilon'' < \varepsilon \\ &\text{with } W_{T+\varepsilon'}^{\alpha(j)} > W_{T+\varepsilon'}^{\alpha(k)} \text{ and } W_{T+\varepsilon''}^{\alpha(j)} < W_{T+\varepsilon''}^{\alpha(k)}, \end{aligned} \tag{51}$$

$$\text{if } j < k, \text{ then } P\big(W_1^{\alpha(j)} = W_1^{\alpha(k)}\big) = 0, \tag{52}$$

and

$$\begin{aligned} &\text{if } j < k, j' < k', \text{ and } (j, k) \neq (j', k'), \text{ then} \\ &P\big(W_r^{\alpha(j)} = W_r^{\alpha(k)}, W_r^{\alpha(j')} = W_r^{\alpha(k')} \text{ for some } 0 < r \leqslant 1\big) = 0. \end{aligned} \tag{53}$$

None of the above assertions is difficult to prove. Equation (51) is well known, and follows from the strong Markov property for Brownian motion. Assertion (52) is trivial, while to demonstrate (53), one notes that the probability of a 3 or 4-dimensional Brownian motion ever hitting a 2 codimensional plane is zero.

Just as (ξ_t^A) denotes a system of coalescing random walks with initial state A, we let $(\tilde{\xi}_t^A)$ denote a system of coalescing (standard) Brownian motions with initial state $A \subset \mathbb{R}$, where A is locally finite. (It is assumed throughout the paper that random walks begin at integer-valued states, whereas Brownian motions need not).

We also find it convenient to define a metric on the class of finite subsets (or configurations) $A \subset \mathbb{R}$ for Lemma 4. For $A_1 = \{\alpha_1^1, \alpha_2^1, \cdots, \alpha_{n(1)}^1\}$, $A_2 = \{\alpha_1^2, \alpha_2^2, \cdots, \alpha_{n(2)}^2\}$, $\alpha_1^i < \alpha_2^i < \cdots < \alpha_{n(i)}^i$, $i = 1, 2$, we set

$$\begin{aligned} \mathcal{C}(A_1, A_2) &= \min\{1, \max\{|\alpha_k^1 - \alpha_k^2| : k = 1, \cdots, n(i)\}\} && \text{if } n(1) = n(2), \\ &= 1 && \text{if } n(1) \neq n(2). \end{aligned}$$

Note that for $n(1) = n(2)$, the topology induced by $\mathcal{C}$ is simply the symmetrized product topology. The metric $\mathcal{C}$, as defined, is clearly complete.

Now, finally, we state Lemma 4.

LEMMA 4. *Let B_t, $t \in \mathbb{R}^+$, be finite subsets of $\mathbb{Z}$, A a finite subset of $\mathbb{R}$, where all B_t and A have the same cardinality (say n). Assume that*

$$B_t / t^{\frac{1}{2}} \to_{\mathcal{C}} A \quad \text{as} \quad t \to \infty.$$

Then,

$$\xi_t^{B_t} / t^{\frac{1}{2}} \Rightarrow_{\mathcal{C}} \tilde{\xi}_1^A \quad \text{as} \quad t \to \infty,$$

where $\to_{\mathcal{C}}$ ($\Rightarrow_{\mathcal{C}}$) means convergence (weak convergence) in the metric $\mathcal{C}$.

PROOF. We assume that the coordinates of the product measures defined on $D^n[0, 1]$ by P^t and P^∞ are ordered so that for $\omega = (\omega_1, \omega_2, \cdots, \omega_n) \in D^n$, $\omega_j(0) < \omega_k(0)$ w.p. 1 for $j < k$. We define stopping times

$$S_{j,k}(\omega) = \inf\{r : \omega_j(r) \geq \omega_k(r), 0 < r \leq 1\}$$

and

$$T_{j,k}(\omega) = \inf\{r : \omega_j(r) = \omega_k(r), 0 < r \leq 1\}$$

for $1 \leq j < k \leq n$. (Set $S_{j,k}, T_{j,k} = \infty$ if the sets are void.) From (51) and (52) it is not difficult to see that $S_{j,k}$ is an a.s. continuous map on P^∞. Therefore, by (49),

$$(P^t, S_{j,k}, 1 \leq j < k \leq n) \Rightarrow (P^\infty, S_{j,k}, 1 \leq j < k \leq n) \quad \text{as} \quad t \to \infty$$

on the induced product topology on $D^n[0, 1] \times ([0, 1] \cup \{\infty\})^{\binom{n}{2}}$. Since nearest neighbor random walks defined on the same lattice cannot cross without coinciding at some point (the same conclusion is of course valid for Brownian motions),

$$T_{j,k} = S_{j,k} \quad \text{w.p. 1 for each} \quad P^t \text{ (and } P^\infty).$$

Therefore,

$$(54) \quad (P^t, T_{j,k}, 1 \leq j < k \leq n) \Rightarrow (P^\infty, T_{j,k}, 1 \leq j < k \leq n) \quad \text{as} \quad t \to \infty.$$

The values $T_{j,k}$, $1 \leq j < k \leq n$, induce (by ordering) a permutation $\mathcal{P}$ on $\{1, 2, \cdots, \binom{n}{2} + 1\}$, where the extra element is included to distinguish finite and infinite stopping times. By (52) and (53), the finite $T_{j,k}$ are almost surely distinct, so the permutation $\mathcal{P}$ is uniquely determined on these pairs (j, k). We reduce (54) to the form

$$(55) \quad (P^t, \mathcal{P}) \Rightarrow (P^\infty, \mathcal{P}) \quad \text{as} \quad t \to \infty.$$

Now, P^t and P^∞ induce coalescing systems $(\xi_{st}^{B_t}/t^{\frac{1}{2}})$ and $(\tilde{\xi}_s^A)$, if upon intersection of two "particles" (or paths), we remove the one with the larger initial coordinate. Since $\mathcal{P}$ determines which particles have disappeared by time 1, a little thought shows that (55) implies convergence of the corresponding coalescing systems at time 1. That is,

$$\xi_t^{B_t}/t^{\frac{1}{2}} \Rightarrow_{\mathcal{C}} \tilde{\xi}_1^A \qquad \text{as} \quad t \to \infty.$$

This concludes the proof.

We are now equipped to demonstrate Theorem 5.

THEOREM 5. *Let μ be n-fold mixing for all n, with density $\lambda \in (0, 1)$. Then there is a nontrivial distribution function F^λ such that*

$$\lim_{t\to\infty} P\left(\frac{C_0^+(\zeta_t^\mu)}{t^{\frac{1}{2}}} \leqslant \alpha\right) = F^\lambda(\alpha) \qquad \textit{for} \quad \alpha \in [0, \infty). \tag{56}$$

PROOF. By (2),

$$\begin{aligned} P\left(\frac{C_0^+(\zeta_t^\mu)}{t^{\frac{1}{2}}} > \alpha\right) &= P(\zeta_t^\mu \cap [0, x] = \varnothing) + P([0, x] \subset \zeta_t^\mu) \\ &= E\left[\varphi^\mu(\xi_t^{[0,x]}) + \bar{\varphi}^\mu(\xi_t^{[0,x]})\right], \end{aligned} \tag{57}$$

where $x = \lfloor \alpha t^{\frac{1}{2}} \rfloor$. Rather than taking the bull by the horns and tackling $(\xi_t^{[0,x]})$ directly as $t \to \infty$, we instead first analyze $(\xi_t^{B_t^n})$ as $t \to \infty$, where

$$A^n = \{\beta \in [0, \alpha] : 2^n\beta \in \mathbb{Z} \text{ or } \beta = \alpha\},$$
$$B_t^n = \lfloor t^{\frac{1}{2}} A^n \rfloor = \{y : y = \lfloor t^{\frac{1}{2}}\beta \rfloor \text{ for } \beta \in A^n\},$$

and

$$A = \lim_{n\to\infty} A^n, \; B_t = [0, x].$$

The sets B_t^n are defined so that Lemma 4 is applicable; since

$$B_t^n/t^{\frac{1}{2}} \to_{\mathcal{C}} A^n \qquad \text{as} \quad t \to \infty,$$

it follows that

$$\xi_t^{B_t^n}/t^{\frac{1}{2}} \Rightarrow_{\mathcal{C}} \tilde{\xi}_1^{A^n} \qquad \text{as} \quad t \to \infty. \tag{58}$$

If the processes $(\xi_s^{B_t^n}, 0 \leqslant s \leqslant t)$ $((\tilde{\xi}_s^{A^n}, 0 \leqslant s \leqslant 1))$, $n = 1, 2, \cdots$, are all defined on the same probability space in the canonical manner (that is, are all coupled together as mentioned in Section 2), then $\xi_t^{B_t^n}$ $(\tilde{\xi}_1^{A^n})$ is clearly increasing as $n \to \infty$. Since $B_t^n \uparrow^{\mathcal{C}} B_t$ as $n \to \infty$, it is tempting to conclude from (58) that $\xi_t^{B_t}/t^{\frac{1}{2}}$ has a (hopefully finite) distributional limit as $t \to \infty$, which we may think of as "$\tilde{\xi}_1^A$". With the aid of Lemma 3, we now make this reasoning precise.

We decompose B_t into subintervals D_k,

$$B_t = \bigcup_{k=1}^{z_n+1} D_k,$$

where for $z_n = \lfloor 2^n\alpha \rfloor$,

$$D_k = \left[\lfloor 2^{-n}t^{\frac{1}{2}}k \rfloor, \lfloor 2^{-n}t^{\frac{1}{2}}(k+1) \rfloor\right] \quad \text{for} \quad k = 1, \cdots, z_n,$$

and

$$D_{z_n+1} = \left[\lfloor 2^{-n}t^{\frac{1}{2}}z_n \rfloor, x\right].$$

Note that the endpoints of each interval D_k are contained in B_t^n. Lemma 3 states that

$$P\left(|\xi_t^{D_k}| > 2\right) \leq C'/2^{2n}$$

for all $1 \leq k \leq z_n + 1$, where C' is independent of t. Coupling $(\xi_s^{B_t}, 0 \leq s \leq t)$ together with $(\xi_s^{B_t^n}, 0 \leq s \leq t)$, this implies that

$$P\left(\xi_t^{B_t^n} \neq \xi_t^{B_t}\right) \leq \Sigma_{k=1}^{z_n+1} P\left(|\xi_t^{D_k}| > 2\right) \leq C'(2^n\alpha + 1)/2^{2n} \sim C'\alpha/2^n, \tag{59}$$

and hence

$$\lim_{n\to\infty} P\left(\xi_t^{B_t^n} \neq \xi_t^{B_t}\right) = 0 \quad \text{uniformly in} \quad t. \tag{60}$$

Since $\mathcal{C}$ is a complete metric, by the fundamentals of weak convergence (60) and (58) imply that

$$\xi_t^{B_t}/t^{\frac{1}{2}} \Rightarrow_{\mathcal{C}} \tilde{\xi}_1^A \quad \text{as} \quad t \to \infty \tag{61}$$

for some probability measure $\tilde{\xi}_1^A$.

We are now almost finished. We take advantage of the scaling factor $t^{\frac{1}{2}}$ and the assumption that μ is n-fold mixing to assert that

$$\lim_{t\to\infty} E\left[\varphi^\mu(\xi_t^{[0,x]}) + \bar{\varphi}^\mu(\xi_t^{[0,x]})\right] = \lim_{t\to\infty} \Sigma_{m=1}^\infty \left[\lambda^m + (1-\lambda)^m\right] P\left(|\xi_t^{B_t}| = m\right).$$

By (61) and bounded convergence this equals

$$\Sigma_{m=1}^\infty \left[\lambda^m + (1-\lambda)^m\right] P\left(|\tilde{\xi}_1^A| = m\right).$$

Together with (57), the equality demonstrates convergence in (56).

It remains only to show that F^λ is a nontrivial distribution function. It is simple to show that for all m,

$$P\left(|\tilde{\xi}_1^A| \leq m\right) \to 0 \quad \text{as} \quad \alpha \to \infty,$$

and, therefore, $F^\lambda(\alpha) \to 1$ as $\alpha \to \infty$. The existence of the limit $\tilde{\xi}_1^A$ implies that F^λ is nontrivial. This concludes the proof of the theorem.

Properties of F^λ. We next turn our attention to investigating the properties of F^λ. For this purpose, it suffices to restrict ourselves to those initial measures which are product measures, $\mu = \mu_\lambda$. We will make repeated use of the equality

$$P\left(\zeta_t^{\mu_\lambda} \cap \left[0, \lceil \alpha t^{\frac{1}{2}} \rceil\right) = \varnothing\right) = \Sigma_{m=1}^\infty \lambda^m P\left(|\xi_t^{[0,x)}| = m\right), \tag{62}$$

where $x = \lceil \alpha t^{\frac{1}{2}} \rceil$, which is based on the second duality equation and our restriction to the product measure μ_λ; version of (62) has already been applied in Theorem 5.

We will also find it convenient to couple different systems of coalescing random walks to derive certain estimates. Recall that, as first mentioned in Section 2, systems of coalescing random walks commencing from different initial states can be considered to be simultaneously evolving on the same probability space if the random walks present at the same sites of each undergo the same motion. Therefore, $\xi_t^A \subset \xi_t^B$ for $A \subset B$. Also note that for $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \varnothing$,

$$|\xi_t^A| \leqslant |\xi_t^{A_1}| + |\xi_t^{A_2}|. \tag{63}$$

Inequality (63) still holds if the processes $(\xi_t^{A_1})$ and $(\xi_t^{A_2})$ are independent; we may induce a copy of (ξ_t^A) from the pair by deleting a particle of $(\xi_t^{A_2})$ upon coincidence with a particle of $(\xi_t^{A_1})$. In the following, when confusion is not likely, we omit explicit mention of the coupling.

We now demonstrate a pair of lemmas that will be useful for analyzing F^λ. The first provides estimates for the tail of F^λ, whereas the second provides smoothness estimates.

LEMMA 5. *If x_1, x_2, and y denote nonnegative integers, $x_2 \geqslant y$, then*

(a) $P(\zeta_t^\mu \cap [0, x_1 + x_2) = \varnothing) \geqslant P(\zeta_t^\mu \cap [0, x_1) = \varnothing) \cdot P(\zeta_t^\mu \cap [0, x_2) = \varnothing)$;

(b) $P(\zeta_t^\mu \cap [0, x_1 + x_2) = \varnothing) \leqslant P(\zeta_t^\mu \cap [0, x_1) = \varnothing) \cdot P(\zeta_t^\mu \cap [0, x_2 - y) = \varnothing) + N_t(y/t^{\frac{1}{2}})$,

where, if $y/t^{\frac{1}{2}} \to \beta$ as $t \to \infty$,

$$N_t(y/t^{\frac{1}{2}}) \to \pi^{-\frac{1}{2}} \int_\beta^\infty e^{-u^2/4}\, du.$$

PROOF. (a) Setting $A = [0, x_1 + x_2)$, $A_1 = [0, x_1)$, and $A_2 = [x_1, x_1 + x_2)$, and coupling (ξ_t^A), $(\xi_t^{A_1})$, and $(\xi_t^{A_2})$ as mentioned above, with $(\xi_t^{A_1})$ and $(\xi_t^{A_2})$ independent, we see that (63) holds:

$$|\xi_t^A| \leqslant |\xi_t^{A_1}| + |\xi_t^{A_2}|.$$

Therefore, since by (62), $P(\zeta_t^{\mu_\lambda} \cap [0, \cdot) = \varnothing)$ is the generating function of $|\xi_t^{[0, \cdot)}|$ (in λ), (a) follows.

(b) We set $A = [0, x_1 + x_2)$, $A_1 = [0, x_1)$, and $A_2 = [x_1 + y, x_1 + x_2)$. Then

$$\begin{aligned} P(\zeta_t^\mu \cap [0, x_1 + x_2) = \varnothing) &= \textstyle\sum_{m=1}^\infty \lambda^m P(|\xi_t^A| = m) \\ &\leqslant \textstyle\sum_{m=1}^\infty \lambda^m P(|\xi_t^{A_1 \cup A_2}| = m) \\ &= \textstyle\sum_{m=1}^\infty \lambda^m P(|\xi_t^{A_1 \cup A_2}| = m, |\xi_t^{\{x_1 - 1, x_1 + y\}}| = 2) \\ &\quad + \textstyle\sum_{m=1}^\infty \lambda^m P(|\xi_t^{A_1 \cup A_2}| = m, |\xi_t^{\{x_1 - 1, x_1 + y\}}| = 1). \end{aligned}$$

If $(\xi_t^{A_1})$ and $(\xi_t^{A_2})$ denote independent processes, then the first sum of the last equality is at most

$$\begin{aligned} &\textstyle\sum_{m=1}^\infty \lambda^m P(|\xi_t^{A_1}| + |\xi_t^{A_2}| = m) \\ &\qquad = P(\zeta_t^\mu \cap [0, x_1) = \varnothing) \cdot P(\zeta_t^\mu \cap [0, x_2 - y) = \varnothing). \end{aligned} \tag{64}$$

The second sum is at most

$$(65)\qquad P\left(|\xi_t^{\{x_1-1,\,x_1+y\}}| = 1\right) \to \pi^{-\frac{1}{2}} \int_0^\beta e^{-u^2/4}\,du \qquad \text{as} \quad t \to \infty$$

for $y/t^{\frac{1}{2}} \to \beta$, by (6) and (10). Together, (64) and (65) imply (b).

LEMMA 6. *For x_1, x_2, and y nonnegative integers, $x_1 \leqslant x_2$,*

(a) $P(\zeta_t^\mu \cap [0, x_1) = \varnothing,\ \zeta_t^\mu \cap [0, x_1 + y) \neq \varnothing)$

$\qquad\qquad \geqslant P(\zeta_t^\mu \cap [0, x_2) = \varnothing,\ \zeta_t^\mu \cap [0, x_2 + y) \neq \varnothing)$;

(b) $P(\xi_t^{[0,x_1)} \subsetneq \xi_t^{[x_1-x_2,\,x_1)},\ \xi_t^{[0,x_1)} \subsetneq \xi_t^{[0,\,x_1+y)})$

$\qquad\qquad \leqslant P(\xi_t^{[0,x_1)} \subsetneq \xi_t^{[x_1-x_2,\,x_1)}) \cdot P(\xi_t^{[0,x_1)} \subsetneq \xi_t^{[0,\,x_1+y)})$,

where inclusion in (b) *means under the usual coupling.*

PROOF. (a) Since μ is translation invariant, so is ζ_t^μ. Therefore,

$$\begin{aligned} P(\zeta_t^\mu \cap [0, x_1) = \varnothing, \zeta_t^\mu \cap [0, x_1 + y) \neq \varnothing)& \\ \geqslant P(\zeta_t^\mu \cap [x_1 - x_2, x_1) &= \varnothing, \zeta_t^\mu \cap [x_1 - x_2, x_1 + y) \neq \varnothing) \\ = P(\zeta_t^\mu \cap [0, x_2) &= \varnothing, \zeta_t^\mu \cap [0, x_2 + y) \neq \varnothing). \end{aligned}$$

(b) The assertion is equivalent to showing that

$$\begin{aligned} (66)\quad P\left(|\xi_t^{\{x_1-x_2,\,0\}}| = |\xi_t^{\{x_1-1,\,x_1+y-1\}}| = 2\right)& \\ \leqslant P\left(|\xi_t^{\{x_1-x_2,\,0\}}| = 2\right) &\cdot P\left(|\xi_t^{\{x_1-1,\,x_1+y-1\}}| = 2\right). \end{aligned}$$

Inequality (66) is actually a generalization of Lemma 2, and can be proved in the same basic manner. One fixes the paths of the two outer particles, those commencing at $x_1 - x_2$ and $x_1 + y - 1$, which are denoted by l and L; under this realization, one computes the probability that (the particles of) the system of coalescing random walks $(\xi_t^{\{0,\,x_1-1\}})$ hits neither l nor L. (In Lemma 2, we started with a single initial particle instead of two). As in Lemma 2, one constructs space-time harmonic functions possessing specified boundary data; only now, the domain of the associated process is contained in $\mathbb{Z}^2$ rather than $\mathbb{Z}$. (The diagonal $x = y$ serves as a trap). The proof proceeds in an analogous manner; we omit the details.

We now characterize F^λ.

THEOREM 6. *The limit distribution F^λ defined in Theorem 5 has the following properties*:

(a) *$F^\lambda(\alpha)$ has monotone decreasing continuous density on the positive half-line, with $F^\lambda(0) = 0$, and*

(b) *$F^\lambda(\alpha)$ has an exponential tail in the sense that*

$$1 - F^\lambda(\alpha) = \exp\{-c_\lambda(\alpha) \cdot \alpha\},$$

where $c_\lambda(\alpha) \to c_\lambda > 0$ as $\alpha \to \infty$.

PROOF. (a) It follows from Lemma 6 (a) and (8), that for α_1, α_2, and β nonnegative, $\alpha_1 \leqslant \alpha_2$,

(67)

$$\begin{aligned} F^\lambda(\alpha_1+\beta) - F^\lambda(\alpha_1) &= \lim_{t\to\infty}\Big[P\Big(\zeta_t^\mu \cap \Big[0, \Big\lceil \alpha_1 t^{\frac12}\Big\rceil\Big) = \varnothing, \zeta_t^\mu \cap \Big[0, \Big\lceil(\alpha_1+\beta)t^{\frac12}\Big\rceil\Big) \neq \varnothing\Big) \\ &\quad + P\Big(\Big[0, \Big\lceil \alpha_1 t^{\frac12}\Big\rceil\Big) \subset \zeta_t^\mu, \Big[0, \Big\lceil(\alpha_1+\beta)t^{\frac12}\Big\rceil\Big) \not\subset \zeta_t^\mu\Big)\Big] \\ &\geqslant \lim_{t\to\infty}\Big[P\Big(\zeta_t^\mu \cap \Big[0, \Big\lceil \alpha_2 t^{\frac12}\Big\rceil\Big) = \varnothing, \zeta_t^\mu \cap \Big[0, \Big\lceil(\alpha_2+\beta)t^{\frac12}\Big\rceil\Big) \neq \varnothing\Big) \\ &\quad + P\Big(\Big[0, \Big\lceil \alpha_2 t^{\frac12}\Big\rceil\Big) \subset \zeta_t^\mu, \Big[0, \Big\lceil(\alpha_2+\beta)t^{\frac12}\Big\rceil\Big) \not\subset \zeta_t^\mu\Big)\Big] \\ &= F^\lambda(\alpha_2+\beta) - F^\lambda(\alpha_2). \end{aligned}$$

From (67) it is not difficult to show that F^λ has monotone one-sided derivatives at every point; the density, therefore, exists and is continuous except at at most countably many points. To complete the demonstration of (a), it suffices to show that

(68)

$$\frac{1}{\beta}\big[\big(F^\lambda(\alpha_1+\beta) - F^\lambda(\alpha_1)\big) - \big(F^\lambda(\alpha_2+\beta) - F^\lambda(\alpha_2)\big)\big] \to 0 \qquad \text{as} \quad \alpha_2 \to \alpha_1.$$

at a uniform rate independent of β. We employ the processes $(\xi_t^{[0,x_1)})$, $(\xi_t^{[0,x_1+y)})$, $(\xi_t^{[x_1-x_2,x_1)})$ and $(\xi_t^{[x_1-x_2,x_1+y)})$, and observe the inclusions induced by the usual coupling. Applying (62) and the translation invariance of ζ_t^μ for all t, one can reexpress the left-hand side of (68) after a little manipulation as

(69)
$$\frac{1}{\beta}\lim_{t\to\infty}\Sigma_{m_1,m_2,k}\, b_{m_1,m_2,k} P\big(|\xi_t^{[0,x_1)}| = m_1, |\xi_t^{[x_1-x_2,x_1)}| = m_2, |\xi_t^{[0,x_1+y)}| = m_1+k\big),$$

where $x_i = \Big\lceil \alpha_i t^{\frac12}\Big\rceil$, $y = \Big\lceil \beta t^{\frac12}\Big\rceil$, and

$$b_{m_1,m_2,k} = (\lambda^{m_1} - \lambda^{m_2})(1-\lambda^k) + (\bar\lambda^{m_1} - \bar\lambda^{m_2})(1-\bar\lambda^k),$$

for $\bar\lambda = 1-\lambda$. Since $b_. \leqslant 1$ and since $b_. = 0$ for $k = 0$ or $m_1 = m_2$, (69) is at most

$$\frac{1}{\beta}\lim_{t\to\infty} P\Big(\xi_t^{[0,x_1)} \subsetneq \xi_t^{[x_1-x_2,x_1)}, \xi_t^{[0,x_1)} \subsetneq \xi_t^{[0,x_1+y)}\Big),$$

which, by Lemma 6 (b), is at most

$$\frac{1}{\beta}\lim_{t\to\infty} P\Big(\xi_t^{[0,x_1)} \subsetneq \xi_t^{[x_1-x_2,x_1)}\Big)\cdot P\Big(\xi_t^{[0,x_1)} \subsetneq \xi_t^{[0,x_1+y)}\Big).$$

Applying (6) and (9), we rewrite this as

$$\frac{1}{\beta}\lim_{t\to\infty} P\big(Y_t^{x_2-x_1-1} \geqslant 0\big)\cdot P\big(Y_t^{y-1} \geqslant 0\big) \leqslant c^2(\alpha_2-\alpha_1),$$

which, since c is constant, goes to 0 as $\alpha_2 \to \alpha_1$. This demonstrates (68), and hence (a).

(b) We set $G^\lambda(\alpha) = \lim_{t\to\infty} P(\zeta_t^\mu \cap [0, \lceil \alpha t^{\frac{1}{2}} \rceil))$. Since $1 - F^\lambda(\alpha) = G^\lambda(\alpha) + G^{1-\lambda}(\alpha)$, it suffices to analyze the tail of G^λ. From Lemma 5, we obtain

$$G^\lambda(\alpha_1 + \alpha_2) \geqslant G^\lambda(\alpha_1) \cdot G^\lambda(\alpha_2) \tag{70}$$

and

$$G^\lambda(\alpha_1 + \alpha_2) \leqslant G^\lambda(\alpha_1) \cdot G^\lambda(\alpha_2 - \beta) + \pi^{-\frac{1}{2}} \textstyle\int_\beta^\infty e^{-u^2/4}\, du, \tag{71}$$

for α_1, α_2, β nonnegative, $\alpha_2 \geqslant \beta$. Somewhat more convenient is

$$G^\lambda(\alpha_1 + \alpha_2) \leqslant G^\lambda(\alpha_1) \cdot G^\lambda(\alpha_2)/G^\lambda(\beta) + \pi^{-\frac{1}{2}} \textstyle\int_\beta^\infty e^{-u^2/4}\, du. \tag{71$'$}$$

Now, it follows from (70) that $-\log G^\lambda$ is subadditive; therefore, since G^λ is monotone,

$$c'_\lambda = \lim_{\alpha\to\infty} \frac{-\log G^\lambda(\alpha)}{\alpha}$$

exists. Since $G^\lambda(\alpha) \to 0$ as $\alpha \to \infty$, a rather messy iteration of (71′) for appropriate β shows that $c'_\lambda > 0$. (Iterate along multiplicative lattices of the form $2^n\alpha$, while setting $\beta_n = (2^n\alpha)^\delta$, where $\frac{1}{2} < \delta < 1$ is fixed). If we set $c_\lambda = \min\{c'_\lambda, c'_{1-\lambda}\}$, (b) follows.

REMARK. The tail of F^λ is not precisely exponential because of the interference of the coalescing random walks commencing from neighboring blocks, e.g., the rightmost particles of $[-x, 0)$ and the leftmost particles of $[0, y)$ will have most likely coalesced after a moderate period of time (say t), and therefore $|\xi_t^{[-x,y)}|$ is typically strictly less than $|\xi_t^{[-x,0)}| + |\xi_t^{[0,y)}|$. However, the loss of particles due to this interference tends to a limit (for fixed t) as $x, y \to \infty$; therefore, it should be the case that $F^\lambda(\alpha) = b_\lambda(\alpha) \cdot \exp\{-c_\lambda \alpha\}$, where $b_\lambda(\alpha) \to b_\lambda$ and $b_\lambda > 0$ and $c_\lambda > 0$ are constants. Most likely, b_λ is increasing.

In the following corollary, we state the limiting behavior for C_0 in terms of F^λ; we drop the superscript λ.

COROLLARY. *Let μ be as defined in Theorems 5 and 6. Then,*

$$\lim_{t\to\infty} P\left(\frac{C_0(\zeta_t^\mu)}{t^{\frac{1}{2}}} \leqslant \alpha \right) = F(\alpha) - \alpha F'(\alpha).$$

PROOF. The proof is the same as that of the corollary to Theorem 3. We make use of the fact that the limit distribution $F(\alpha)$ has a continuous derivative, and that the process is translation invariant for all t; these properties are equally valid for (ζ_t^μ) as well as for (η_t^μ).

Estimates for the expectation of F^λ. We conclude this section with upper and lower bounds for the expectation of F^λ. We first state Lemma 7, which will be used in the computation of the upper bound. As before, when meaningful in the context, we assume that systems of coalescing random walks commencing from different initial states are coupled together in the obvious manner.

LEMMA 7. *For x a positive integer*, $0 < \lambda < 1$,

$$\Sigma_{k=1}^{x+1} \lambda^k P(|\xi_t^{[0,x]}| = k)$$
$$\leq \left[P(|\xi_t^{\{x-1,x\}}| = 1) + \lambda P(|\xi_t^{\{x-1,x\}}| = 2) \right] \cdot \Sigma_{k=1}^{x} \lambda^k P(|\xi_t^{[0,x-1]}| = k).$$

PROOF. The proof involves a slight extension of the technique used in the proof of Lemma 2, and will only be sketched here. We first fix the paths of all the coalescing random walks, except the one commencing at $x - 1$. (Denote this realization by $\mathfrak{L}$). As in Lemma 2, one can show that the conditional probability of this random walk hitting none of these fixed paths is less than the product of the probabilities of it not hitting any of the lower paths and not hitting the upper path. Therefore, since λ^k is convex in k,

$$\lambda^{C(\mathfrak{L})} \Sigma_{k=1}^{3} \lambda^{k-1} P(|\xi_t^{\{x-2,x-1,x\}}| = k|\mathfrak{L})$$
$$\leq \left[P(|\xi_t^{\{x-1,x\}}| = 1|\mathfrak{L}) + \lambda P(|\xi_t^{\{x-1,x\}}| = 2|\mathfrak{L}) \right]$$
$$\cdot \lambda^{C(\mathfrak{L})} \Sigma_{k=1}^{2} \lambda^{k-1} P(|\xi_t^{\{x-2,x-1\}}| = k|\mathfrak{L}),$$

where $C(\mathfrak{L}) = |\xi_t^{[0,x-2]}|$ given $\mathfrak{L}$. Upon integrating over the measure space induced by $\mathfrak{L}$, we obtain the assertion.

We now state Theorem 7; the results are analogous to (37) of Theorem 2 and to Theorem 4.

THEOREM 7. *Let M^λ denote the expectation of F^λ. Then,*

$$\frac{\pi^{\frac{1}{2}}}{4\lambda(1-\lambda)} \leq M^\lambda \leq \left(\frac{\lambda}{1-\lambda} + \frac{1-\lambda}{\lambda} \right) \pi^{\frac{1}{2}}.$$

Hence,

$$\frac{\pi^{\frac{1}{2}}}{2\lambda(1-\lambda)} \leq \lim_{t\to\infty} E\left[\frac{C_0(\zeta_t^{\mu_\lambda})}{t^{\frac{1}{2}}} \right] \leq 2\left(\frac{\lambda}{1-\lambda} + \frac{1-\lambda}{\lambda} \right) \pi^{\frac{1}{2}}.$$

PROOF. To obtain the lower bound for M^λ, we make use of the fact that the expected size of the cluster containing the origin in $\zeta_t^{\mu_\lambda}$ is greater than the expected mean cluster size of $\zeta_t^{\mu_\lambda}$ in the sense of Section 3. A similar estimate was exploited in [15]. To show this, we note that by Birkhoff's ergodic theorem, $p_n = E$[mean frequency of clusters of size n in $\zeta_t^{\mu_\lambda}$] is well defined, and that

$$np_n = P(0 \in \text{cluster of size } n \text{ in } \zeta_t^{\mu_\lambda}) \cdot E\left[C(\zeta_t^{\mu_\lambda}) \right]. \tag{72}$$

The Schwarz inequality then states that

$$\Sigma_n n^2 p_n \geq (\Sigma_n np_n)^2,$$

which, in view of (72), reduces to

$$E\left[C_0(\zeta_t^{\mu_\lambda}) \right] \geq E\left[C(\zeta_t^{\mu_\lambda}) \right].$$

Using Theorem 1 (c), we now conclude that

$$\begin{aligned} M^\lambda &= \lim_{t\to\infty} E\left[\frac{C_0^+(\zeta_t^{\mu_\lambda})}{t^{\frac{1}{2}}}\right] = \tfrac{1}{2}\lim_{t\to\infty} E\left[\frac{C_0(\zeta_t^{\mu_\lambda})}{t^{\frac{1}{2}}}\right] \\ &\geqslant \tfrac{1}{2}\lim_{t\to\infty} E\left[\frac{C(\zeta_t^{\mu_\lambda})}{t^{\frac{1}{2}}}\right] \\ &= \frac{\pi^{\frac{1}{2}}}{4\lambda(1-\lambda)}. \end{aligned}$$

To obtain the upper bound for M^λ, we note that Lemma 7, together with (6) and (62), implies that

$$P(\zeta_t^{\mu_\lambda} \cap [0, x] = \varnothing) \leqslant [P(Y_t^0 = -1) + \lambda P(Y_t^0 \geqslant 0)] \cdot P(\zeta_t^{\mu_\lambda} \cap [0, x-1] = \varnothing)$$

for any positive integer x. By induction, this is at most

$$\begin{aligned} (73) \quad [P(Y_t^0 = -1) + \lambda P(Y_t^0 \geqslant 0)]^x \cdot P(\zeta_t^{\mu_\lambda} \cap \{0\} = \varnothing) \\ = \lambda[P(Y_t^0 = -1) + \lambda P(Y_t^0 \geqslant 0)]^x. \end{aligned}$$

If we choose x so that $x \sim t^{\frac{1}{2}}\alpha$ as $t \to \infty$ for some fixed α, it follows from (8) that (73) approaches

$$\lambda e^{(\lambda-1)\alpha/\pi^{\frac{1}{2}}} \quad \text{as} \quad t \to \infty.$$

Therefore,

$$\begin{aligned} 1 - F^\lambda(\alpha) &= \lim_{t\to\infty} [P(\zeta_t^{\mu_\lambda} \cap [0, x] = \varnothing) + P(\zeta_t^{\mu_{1-\lambda}} \cap [0, x] = \varnothing)] \\ &\leqslant \lambda e^{(\lambda-1)\alpha/\pi^{\frac{1}{2}}} + (1-\lambda)e^{-\lambda\alpha/\pi^{\frac{1}{2}}}. \end{aligned}$$

Integrating, we obtain

$$M^\lambda \leqslant \left(\frac{\lambda}{1-\lambda} + \frac{1-\lambda}{\lambda}\right)\pi^{\frac{1}{2}},$$

which concludes the proof.

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