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A NOTE ON DOMAINS OF PARTIAL ATTRACTION

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We attempt to relate the domains of partial attraction for sums of i.i.d. random variables to some properties of a generalised regular variation. We give a new characterisation of the domain of partial attraction of the normal distribution and new sufficient conditions for a distribution to belong to a domain of partial attraction.

1. Introduction and results. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with distribution F, and let $S_n = X_1 + X_2 + \cdots + X_n$. We say that F is in a *domain of partial attraction*, written $F \in D_p$, if there is a sequence n_i of integers and constants A_i and B_i, $B_i > 0$, $B_i \to +\infty$, for which $(S_{n_i}/B_i) - A_i$ converges in distribution to a nondegenerate (infinitely divisible) random variable. If the limit random variable is normally distributed, we say that F is the *domain of partial attraction of the normal distribution*, written $F \in D_p(2)$.

The domains of partial attraction constitute a generalisation of the domains of attraction, which are the distributions for which the whole sequence $(S_n/B_n) - A_n$ converges to a nondegenerate random variable. The domains of attraction can be described in terms of regularly varying properties of the tail sum, $P(|X| > x)$, and the truncated second moment, $V(x) = \int_{-x}^{x+} u^2 dF(u)$, of F; here X is any random variable with distribution F. In particular, F is in the domain of attraction of the normal distribution if and only if $x^2 P(|X| > x)/V(x) \to 0$ as $x \to +\infty$ (Lévy, 1937, page 113), and this condition is equivalent to the slow variation of $V(x)$; i.e., $V(x\lambda)/V(x) \to 1$ as $x \to +\infty$ for each $\lambda > 0$ (Feller, 1971, page 283). Lévy (1937, page 113) showed that $F \in D_p(2)$ if and only if $\liminf_{x\to+\infty} x^2 P(|X| > x)/V(x) = 0$.

Doeblin (1940, 1947) examined the domains of partial attraction and related problems by classical methods; his results were extended by Mejzler (1974), who looked at the case where the subsequence n_i for which $(S_{n_i}/B_i) - A_i$ converges satisfies $\liminf_{i\to+\infty} n_i/n_{i+1} > 0$. More recently Simons and Stout (1978) gave two new characterisations of $D_p(2)$, and some related results, as a by-product of their studies on invariance theorems. In this note we attempt to show that D_p and $D_p(2)$ can be related to some properties of a generalised type of regular variation of the tail of the distribution. We give an equivalence for $D_p(2)$ (Theorem 1 below) which has no counterpart for the domain of attraction of the normal distribution. A

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property analogous to the slow variation of V which might have been expected to characterise $D_p(2)$ is $\liminf_{x\to+\infty} V(x\lambda)/V(x) = 1$ for $\lambda \geq 1$; we show that, although this is necessary for $F \in D_p(2)$, it is in general not sufficient. In Theorem 2 we give a necessary condition and a sufficient condition (but not one that is necessary and sufficient) for $F \in D_p$. Applying this result, we then derive easily two moment conditions for D_p and $D_p(2)$ originally due to Lévy. As another application of our methods, we give in Theorem 3 a simple proof and extension of a result of Gnedenko.

Our methods are basically the classical ones for convergence of sums of random variables as used by Doeblin (1940) and Gnedenko and Kolmogorov (1968), supplemented with some of the techniques of the theory of regular variation. The sufficiency part of Theorem 2 relies on a result of Drasin and Shea (1972) giving "Pólya peaks" of the second kind for a monotone function.

We remark that the class $D_p(2)$ has been prominent in work by Rogozin (1968), Heyde (1969) and Kesten (1972). The theory of regular variation, along with some generalised aspects, is discussed by Seneta (1976).

THEOREM 1. *$F \in D_p(2)$ if and only if* $\liminf_{x\to+\infty} P(|X| > x\lambda)/P(|X| > x) \leq \lambda^{-2}$ *for* $\lambda \geq 1$.

The sufficiency of the condition for $F \in D_p(2)$ in Theorem 1 is implicit in Feller (1969, Theorem 1) where it is not, however, related to the concept of partial attraction. Define a function $Q(\lambda)$ for $\lambda \geq 1$ as follows: $Q(1) = 1$; $Q(\lambda) = \liminf_{x\to+\infty} P(|X| > x\lambda)/P(|X| > x)$ if $P(|X| > x) > 0$ for $x > 0$, $Q(\lambda) = 0$ otherwise, when $\lambda > 1$. In the next theorem, the necessary condition for D_p was proved by Doeblin (1940, Theorem VII).

THEOREM 2. *If $F \in D_p$ then* $\lim_{\lambda\to+\infty} Q(\lambda) = 0$. *If* $\lim_{\lambda\to+\infty} -\log Q(\lambda)/\log\lambda > 0$, *then $F \in D_p$.*

If the nonincreasing function $Q(\lambda)$ is zero for some $\lambda_0 > 1$ (and hence is zero for $\lambda \geq \lambda_0$) then we interpret $-\log Q(\lambda)$ and $\lim_{\lambda\to+\infty} -\log Q(\lambda)/\log\lambda$ as $+\infty$; Theorem 2 is formally correct in this case since $F \in D_p(2)$ by Theorem 1. (If $P(|X| > x) = 0$ for some $x > 0$, F is in the domain of attraction of the normal distribution). If $Q(\lambda) > 0$ for $\lambda > 0$, it is easy to check that for $\lambda, \mu \geq 1$, $Q(\lambda\mu) \geq Q(\lambda)Q(\mu)$, so $-\log Q(e^\lambda)$ is a subadditive function. Hence by Hille and Phillip (1957, page 244) the limit $\lim_{\lambda\to+\infty} -\log Q(\lambda)/\log\lambda$ exists (and is nonnegative). The requirement in Theorem 2 is that this limit be positive, and this is clearly equivalent to the condition that there is a $\rho > 0$ such that for every $\varepsilon \in (0, \rho)$ a constant $\lambda_0(\varepsilon) \geq 1$ exists for which $Q(\lambda) \leq \lambda^{-\rho+\varepsilon}$ whenever $\lambda \geq \lambda_0$; i.e., it is a restriction on the rate of increase of Q. It seems a reasonable conjecture, which we have not been able to establish, that this requirement can be replaced by the simpler condition $\lim_{\lambda\to+\infty} Q(\lambda) = 0$.

Translating a result of Matuszewska (1962, page 320) to our notation shows that $\log Q(\lambda)/\log\lambda \leq \liminf_{x\to+\infty} \log P(|X| > x)/\log x$ for $\lambda > 1$, if $P(|X| > x) > 0$

for $x > 0$, while it is easy to see that for $\alpha > 0$, $E|X|^\alpha < +\infty$ (equivalently $\int_1^\infty x^{\alpha-1}P(|X| > x)\,dx < +\infty$) implies $\liminf_{x\to+\infty}\log P(|X| > x)/\log x \leqslant -\alpha$. Thus we have the following heirarchy of conditions, each of which implies the next, and hence, by Theorem 2, that $F \in D_p$:

(i) $E|X|^\alpha < +\infty, \alpha > 0$;

(ii) $\liminf_{x\to+\infty}\log P(|X| > x)/\log x \leqslant -\alpha$;

(iii) $Q(\lambda) = \liminf_{x\to+\infty}P(|X| > x\lambda)/P(|X| > x) \leqslant \lambda^{-\alpha}$ for $\lambda \geqslant 1$;

(iv) $\lim_{\lambda\to+\infty} -\log Q(\lambda)/\log\lambda \geqslant \alpha$.

We note further that if (iii) holds for every $\alpha < 2$, then it holds for $\alpha = 2$. Hence as an immediate consequence of Theorems 1 and 2 we have the:

COROLLARY TO THEOREM 2. *If $E|X|^\alpha < +\infty$ for $\alpha > 0$ then $F \in D_p$. If $E|X|^\alpha < +\infty$ for every $\alpha \in (0, 2)$ then $F \in D_p(2)$.*

The results of the corollary were deduced by Lévy (1937, pages 213 and 117). Thompson and Owen (1972) proved the special case, that when $\alpha > 1$ and $E|X|^\alpha < +\infty$, then $(S_{n_i}/B_i) - A_i$ converges to a nondegenerate random variable when $B_i = E^{1/\beta}|S_{n_i}|^\beta$ for any $\beta \in (1, \alpha)$. A direct proof of the fact that (i) above implies (iii) can be obtained from Lemma 6 of Maller (1977). A sufficient condition for D_p, given by Simons and Stout (1978), is that $\lim_{\lambda\to+\infty}\limsup_{x\to+\infty}P(|X| > x\lambda)/P(|X| > x) = 0$; this can be shown to imply that $E|X|^\alpha < +\infty$ for some $\alpha > 0$.

It is interesting to notice that an upper bound on, e.g., $\liminf_{x\to+\infty}P(|X| > x\lambda)/P(|X| > x)$ for $\lambda > 1$ is equivalent to a lower bound on the 'limsup' of the same ratio for $\lambda < 1$; if we recall that the limit of a distribution in D_p is an infinitely divisible distribution with a normal component $\sigma^2 \geqslant 0$, we can apply this observation to prove

THEOREM 3. *The following are equivalent*:

(i) *$F \in D_p(2)$*;

(ii) *F is in the domain of partial attraction of a distribution with finite variance*;

(iii) *F is in the domain of partial attraction of a distribution with nonzero normal component.*

The equivalence of (i) and (ii) above was also proved by Gnedenko (cf. Gnedenko and Kolmogorov (1968, page 189)).

We conclude with some counter examples constructed like one due to Feller. Choose a symmetric distribution with $P(|X| > x) = c_n$ for $2^{2^n} \leqslant x < 2^{2^{n+1}}$, where $c_n \downarrow 0$ as $n \to +\infty$. Let $x_n = 2^{2^n}$ and take any $\lambda > 1$; then if n_0 is so large that $2^{2^{n_0}} > \lambda$, the interval $[x_n, \lambda x_n]$ is contained in $[2^{2^n}, 2^{2^{n+1}}) = [2^{2^n}, 2^{2^n}.2^{2^n})$, for $n \geqslant n_0$. Since $P(|X| > x)$ is constant on this interval we have

$$V(\lambda x_n) - V(x_n) = -\int_{x_n}^{\lambda x_n} u^2 dP(|X| > u) = 0,$$

so $\liminf_{x\to+\infty}V(x\lambda)/V(x) = 1$ for $\lambda \geqslant 1$. Following the proof of Theorem 1 we show that this is a necessary condition for $D_p(2)$—but it is not sufficient. To see this, let $c_n = n^{-1}$. The maximum difference that $P(|X| > \lambda y_n)$ and $P(|X| > y_n)$

can attain occurs when $y_n = 2^{2^n} - 1$, so that $P(|X| > \lambda y_n) = n^{-1}$, $P(|X| > y_n) = (n-1)^{-1}$, ultimately. Thus $\liminf_{x \to +\infty} P(|X| > x\lambda)/P(|X| > x) = 1$ for $\lambda \geq 1$, so $F \notin D_p(2)$; in fact by Theorem 2, $F \notin D_p$. We remark that the integral $2\int_0^x uP(|X| > u)\,du$ is larger than $V(x)$, and it is true that $F \in D_p(2)$ if and only if $\liminf_{x \to +\infty} \int_0^{x\lambda} uP(|X| > u)\,du / \int_0^x uP(|X| > u)\,du = 1$ for $\lambda \geq 1$. (In fact this condition is easily seen to be the opposite of Simons and Stout's condition A1$''$ for $F \notin D_p(2)$.)

It is easy to see that the above example has $\limsup_{x \to +\infty} V(x\lambda)/V(x) = +\infty$ for $\lambda > 1$; the condition $\limsup_{x \to +\infty} V(x\lambda)/V(x) < +\infty$ for $\lambda > 1$ appears in Feller (1968, page 345).

By varying c_n in the above example, other interesting behaviour can be obtained; e.g. $c_n = 2^{-n}$ has $\liminf_{x \to +\infty} P(|X| > x\lambda)/P(|X| > x) = \frac{1}{2}$ for $\lambda > 1$, $c_n = 2^{-2^{n+2}}$ has $\liminf_{x \to +\infty} P(|X| > x\lambda)/P(|X| > x) = 0$ for $\lambda > 1$. The latter also has a finite variance, so F is in the domain of attraction of the normal distribution—but clearly $\limsup_{x \to +\infty} P(|X| > x\lambda)/P(|X| > x) = 1$ for $\lambda \geq 1$ in all these examples (see Feller (1971, page 288) for a similar result).

It is not hard to show that $F \in D_p$ implies $\liminf_{x \to +\infty} x^2 P(|X| > x)/V(x) < +\infty$; but since $x^2 P(|X| > x)/V(x) \to +\infty$ is equivalent to the slow variation of $P(|X| > x)$ (Feller (1971, page 283), the case $c_n = 2^{-n}$ above shows that it is possible to have $\liminf_{x \to +\infty} x^2 P(|X| > x)/V(x) < +\infty$, with $F \notin D_p$.

2. Proofs.

PROOF OF THEOREM 1. We use the abbreviation $H(x) = P(|X| > x)$ for $x > 0$. Note first that if $EX^2 < +\infty$, Theorem 1 follows from the corollary to Theorem 2. Hence we need only consider the case $EX^2 = +\infty$, equivalently, $\int_0^\infty xH(x)\,dx = +\infty$. We take Lévy's condition as known, and show that $\liminf_{x \to +\infty} H(x\lambda)/H(x) \leq \lambda^{-2}$ for $\lambda \geq 1$ is equivalent to it. In fact, suppose this fails; then, letting $Q(\lambda) = \liminf_{x \to +\infty} H(x\lambda)/H(x)$ for $\lambda > 1$, there is a $\lambda_0 > 1$ for which $Q(\lambda_0) > \lambda_0^{-2}$. Given $\varepsilon > 0$ and $\lambda > 1$ choose $x_0 = x_0(\varepsilon, \lambda)$ so large that $x \geq x_0$ implies $H(x\lambda) \geq [Q(\lambda) - \varepsilon]H(x)$. Then (the following argument sharpens the result of the lemma on page 82 of Doeblin (1940)):

$$\begin{aligned}\int_0^x uH(u)\,du &= \int_0^{x_0} uH(u)\,du + \int_{x_0}^x uH(u)\,du \\ &\leq o\left[\int_0^x uH(u)\,du\right] + \int_{x_0}^x uH(u\lambda)\,du/\left[Q(\lambda) - \varepsilon\right] \\ &\leq o\left[\int_0^x uH(u)\,du\right] + \lambda^{-2}\int_0^x uH(u)\,du/\left[Q(\lambda) - \varepsilon\right] \\ &\quad + \lambda^{-2}\int_x^{x\lambda} uH(u)\,du/\left[Q(\lambda) - \varepsilon\right]\end{aligned}$$

so that, since H is nonincreasing,

$$Q(\lambda) - \varepsilon \leq o(1) + \lambda^{-2} + \lambda^{-2}(\lambda^2 - 1)x^2H(x)/2\int_0^x uH(u)\,du$$

and

$$\begin{aligned}\liminf_{x \to +\infty} x^2H(x)/2\int_0^x uH(u)\,du &\geq \left[\lambda^2 Q(\lambda) - 1\right]/(\lambda^2 - 1), \text{ for } \lambda > 1 \\ &> 0, \text{ for } \lambda = \lambda_0.\end{aligned}$$

But this means $\liminf_{x\to+\infty} x^2H(x)/V(x) > 0$, contradicting Lévy's condition. Conversely, suppose $\liminf_{x\to+\infty} x^2H(x)/V(x) \geq a > 0$, so that $\liminf_{x\to+\infty} x^2H(x)/2\int_0^x uH(u)\,du \geq b = a/(a+1) > 0$. Then for $\lambda > 1$, $\varepsilon > 0$, $\varepsilon < b$, and $x \geq x_0(\varepsilon, \lambda)$,

$$\int_0^{x\lambda} uH(u)\,du \Big/ \int_0^x uH(u)\,du = \exp 2\int_x^{x\lambda} \frac{uH(u)\,du}{2\int_0^u yH(y)\,dy}$$

$$= \exp 2\int_1^{\lambda} \frac{(ux)^2H(ux)}{2\int_0^{ux} yH(y)\,dy}\frac{du}{u}$$

$$\geq \exp[2(b-\varepsilon)\log\lambda] = \lambda^{2(b-\varepsilon)}.$$

Using this result and the definition of b, for $x \geq x_0$

$$(b-\varepsilon)2\int_0^{x\lambda} uH(u)\,du \leq \lambda^2x^2H(x\lambda) \text{ and } 2\lambda^{2(b-\varepsilon)}\int_0^x uH(u)\,du \leq 2\int_0^{x\lambda} uH(u)\,du$$

so by the monotonicity of H,

$$\lambda^{2(b-\varepsilon)}x^2H(x) \leq \lambda^{2(b-\varepsilon)}2\int_0^x uH(u)\,du \leq \lambda^2x^2H(x\lambda)/(b-\varepsilon)$$

and

$$\liminf_{x\to+\infty} H(x\lambda)/H(x) \geq b\lambda^{2b-2}, \text{ for } \lambda > 1,$$

which contradicts $\liminf_{x\to+\infty} H(x\lambda)/H(x) \leq \lambda^{-2}$ for $\lambda \geq 1$. This proves Theorem 1.

REMARKS. (i) If $F \in D_p(2)$ then by Gnedenko and Kolmogorov (1968, page 128) there are sequences $n_i \to +\infty$, $x_i \to +\infty$, for which $n_ix_i^{-2}\{V(x_i\lambda) - [\int_{-\lambda x_i}^{\lambda x_i} u\,dF(u)]^2\} \to 1$ as $i \to +\infty$ for $\lambda > 0$. When $EX^2 = +\infty$ we have $[\int_{-x}^{x} u\,dF(u)]^2 = o[V(x)]$ as $x \to +\infty$ (cf. Lévy (1937, page 111)) so $n_ix_i^{-2}V(x_i\lambda) \to 1$ for $\lambda > 0$, $V(x_i\lambda)/V(x_i) \to 1$ for $\lambda > 0$ and $\liminf_{x\to+\infty} V(x\lambda)/V(x) = 1$ for $\lambda \geq 1$. When $EX^2 < +\infty$ this also holds since then V is slowly varying, i.e., $V(x\lambda)/V(x) \to 1$ for $\lambda > 0$.

(ii) We note that if $F \notin D_p(2)$, Theorem 1 actually gives a result stronger than is required, namely that $\liminf_{x\to+\infty} H(x\lambda)/H(x) \geq b\lambda^{2b-2}$ for $\lambda \geq 1$. This fact is used in the proof of Theorem 3. It is easily seen to be implied by (and hence is equivalent to) condition A1‴ of Simons and Stout (1978). We also mention that Theorem 1 is a special case, greatly simplified by the monotonicity of $H(x)$, of Theorem 2 of Maller (1977).

PROOF OF THEOREM 2. If $F \in D_p$ then by Gnedenko and Kolmogorov (1968, page 116) there are sequences $n_i \to +\infty$, $x_i \to +\infty$ and canonical measures N and M for which $n_i[1 - F(x_i\lambda)] \to N(\lambda)$, $n_iF(-x_i\lambda) \to M(-\lambda)$ for $\lambda > 0$, λ a point of continuity of N and M. Thus if $T(\lambda) = N(\lambda) + M(-\lambda)$, $n_iH(x_i\lambda) \to T(\lambda)$ as $i \to +\infty$ at points of continuity of T. That N and M are canonical measures means

$N(+\infty) = M(-\infty) = 0$, so $T(+\infty) = 0$, and if the limit is not normal or degenerate, there is a $\lambda_0 > 0$ for which $T(\lambda_0) > 0$. This means

$$\liminf_{x\to+\infty} H(x\lambda\lambda_0^{-1})/H(x) \leqslant \liminf_{y\to+\infty} H(y\lambda)/H(y\lambda_0) \leqslant T(\lambda)/T(\lambda_0) \to 0 \text{ as } \lambda \to +\infty.$$

If the limit is normal then $F \in D_p(2)$ and $\lim_{\lambda\to+\infty}\liminf_{x\to+\infty} H(x\lambda)/H(x) = 0$ by Theorem 1. (This argument is essentially due to Doeblin (1940)).

Conversely, suppose $\lim_{\lambda\to+\infty}\liminf_{x\to+\infty} H(x\lambda)/H(x) = 0$. If $F \in D_p(2)$ then $F \in D_p$, so suppose $F \notin D_p(2)$. Then $EX^2 = +\infty$ and $\liminf_{x\to+\infty} x^2H(x)/V(x) = a > 0$. Suppose first that we can find a sequence $x_i \to +\infty$ for which $\limsup_{i\to+\infty} H(x_i\lambda)/H(x_i) \to 0$ as $\lambda \to +\infty$, and let n_i be the integer nearest to $1/H(x_i)$, so $n_iH(x_i) \to 1$, and $\limsup_{i\to+\infty} n_iH(x_i\lambda) \to 0$ as $\lambda \to +\infty$. By Helly's selection theorem, we can take further subsequences if necessary to make $n_i[1 - F(x_i\lambda)] \to N(\lambda)$ and $n_iF(-x_i\lambda) \to M(-\lambda)$, at points of continuity of the limits, where N and M are nonincreasing functions on $(0, \infty)$. We have $N(+\infty) = M(-\infty) = 0$, since $\limsup_{i\to+\infty} n_i[1 - F(x_i\lambda) + F(-x_i\lambda)] = \limsup_{i\to+\infty} n_iH(x_i\lambda) \to 0$ as $\lambda \to +\infty$. We can also make $n_ix_i^{-2}V(x_i\lambda)$ converge to a nondecreasing function of λ for $\lambda > 0$, so σ^2 exists, where

$$\sigma^2 = \lim_{\lambda\to0+}\limsup_{i\to+\infty} n_ix_i^{-2}V(x_i\lambda) = \lim_{\lambda\to0+}\liminf_{i\to+\infty} n_ix_i^{-2}V(x_i\lambda),$$

and σ^2 is finite, because V is nondecreasing and $\limsup_{i\to+\infty} n_ix_i^{-2}V(x_i) \leqslant a^{-1}\limsup_{i\to+\infty} n_iH(x_i) = a^{-1}$. We finally want to check that $N(\lambda)$ and $M(-\lambda)$ are finite for $\lambda > 0$. Letting $T(\lambda) = N(\lambda) + M(-\lambda)$, all three being nonincreasing functions taken as continuous at 1 for simplicity, we have $T(\lambda) \leqslant T(1) = \lim_{i\to+\infty} n_iH(x_i) = 1$, when $\lambda \geqslant 1$. Note that, on integrating by parts,

$$\limsup_{i\to+\infty} 2n_ix_i^{-2}\int_0^{x_i} uH(u)\,du \leqslant \limsup_{i\to+\infty} n_ix_i^{-2}V(x_i) + \lim_{i\to+\infty} n_iH(x_i) \leqslant a^{-1} + 1,$$

so by Fatou's lemma, if $0 < \lambda < 1$,

$$1 + a^{-1} \geqslant \liminf_{i\to+\infty} 2n_ix_i^{-2}\int_0^{x_i\lambda} uH(u)\,du \geqslant 2\int_0^{\lambda} u \liminf_{i\to+\infty} n_iH(ux_i)\,du = 2\int_0^{\lambda} uT(u)\,du \geqslant \lambda^2T(\lambda),$$

proving that $T(\lambda)$, and hence $N(\lambda)$ and $M(-\lambda)$, are finite for $0 < \lambda < 1$, and hence for $\lambda > 0$.

The conditions of Gnedenko and Kolmogorov (1968, page 116) are now fulfilled, since we can ignore the centering terms $[\int_{-x_i}^{x_i} u\,dF(u)]^2$ when $EX^2 = +\infty$, as we saw in the remark following Theorem 1. Hence there are constants A_i for which $(S_{n_i}/x_i) - A_i$ converges to a random variable, which is nondegenerate since $N(1) + M(-1) = \lim_{i\to+\infty} n_iH(x_i) = 1$. (This argument is closely related to Theorem VIII of Doeblin (1940)).

To complete the proof of Theorem 2 we have to find a sequence $x_i \to +\infty$ for which $\lim_{\lambda\to+\infty}\limsup_{i\to+\infty} H(x_i\lambda)/H(x_i) = 0$. This is where we use Drasin and Shea's result on Pólya peaks. Let $\lambda_n \uparrow +\infty$, so for each n there is a sequence

$y_i(n) \to +\infty$ for which $H(y_i(n)\lambda_n)/H(y_i(n)) \to Q(\lambda_n)$ as $i \to +\infty$. Thus there is a subsequence i_n for which, putting $x_n = y_{i_n}(n)$, $H(x_n\lambda_n)/H(x_n) \sim Q(\lambda_n)$ as $n \to +\infty$. (We are still assuming $F \notin D_p(2)$, so $Q(\lambda) > 0$ for $\lambda \geqslant 1$.) Let $g(x) = 1/H(x)$ and define $p_n > 0$ by $g(x_n\lambda_n)/g(x_n) = \lambda_n^{p_n}$. Then by Drasin and Shea (1972, Theorem 1a(1.5)) (their assumption of the continuity of g being removable just as occurs later in their paper) there is a sequence $s_n \to +\infty$ for which $g(\lambda s_n)/g(s_n) \geqslant (1-\delta_n)\lambda^{p_n(1-\epsilon_n)}$, equivalently, $H(\lambda s_n)/H(s_n) \leqslant \lambda^{-p_n(1-\epsilon_n)}/(1-\delta_n)$, when $1 \leqslant \lambda \leqslant a_n$, for some $\delta_n \to 0$, $\epsilon_n \to 0$, $a_n \to +\infty$. Since under the condition of Theorem 2 $p_n = [-\log H(x_n\lambda_n)/H(x_n)]/\log \lambda_n = -\log Q(\lambda_n)/\log \lambda_n + o(1) \to \rho$, say, where $\rho > 0$, we have $\limsup_{n\to+\infty} H(\lambda s_n)/H(s_n) \leqslant \lambda^{-\rho}$ when $\lambda \geqslant 1$. Thus s_n is a sequence of the type required.

PROOF OF THEOREM 3. Clearly $F \in D_p(2)$ implies (ii) and (iii), and to prove the converses we continue the argument of the second part of Theorem 2. Then if $F \in D_p$ but $F \notin D_p(2)$, we have from Theorem 1 that $\liminf_{x\to+\infty} H(x\lambda)/H(x) \geqslant b\lambda^{2b-2}$ for $\lambda \geqslant 1$, where $b = a/(a+1) > 0$, so if $T(\lambda)$ is as in Theorem 2, then $T(\lambda) \geqslant b\lambda^{2b-2}$ and $-\int_0^\infty \lambda^2\, dT(\lambda) = +\infty$, and the limit distribution has infinite variance. Thus (ii) implies (i). Again the condition on H clearly means $\limsup_{x\to+\infty} H(x\lambda)/H(x) \leqslant b^{-1}\lambda^{2b-2}$ for $\lambda \leqslant 1$, so for $\lambda < 1$,

$$\limsup_{i\to+\infty} n_i x_i^{-2} V(x_i\lambda) \leqslant a^{-1} \limsup_{i\to+\infty} \lambda^2 H(x_i\lambda)/H(x_i) \leqslant (ab)^{-1}\lambda^{2b} \to 0 \text{ as } \lambda \to 0+,$$

so $\sigma^2 = 0$ in Theorem 2, i.e., the limit law has zero normal component. Thus (iii) implies (i).

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