The Annals of Probability
1980, Vol. 8, No. 3, 475–482

MARTINGALE TRANSFORM AND RANDOM ABEL-DINI SERIES[1]

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Let $X_1, X_2, \cdots$ be identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $E|X_1| < \infty$ and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ be nondecreasing sub-σ-algebras of $\mathcal{F}$ such that X_n is $\mathcal{F}_n$-measurable for $n \geq 1$. Define $S_n = X_1 + \cdots + X_n$ and $\xi_n = E(X_n|\mathcal{F}_{n-1})$. The convergence and divergence of the series $\Sigma_{n=1}^{\infty} \operatorname{sgn}(S_{n+k})|S_{n+k}|^{-\alpha}(X_n - \xi_n)$, where α is a real number and k a nonnegative integer, is considered and related to that of martingale transforms. This paper answers a question raised by Kai Lai Chung.

1. Introduction and notation. The Abel-Dini theorem (see Knopp (1971; page 290)) states that if $\{a_n\}$ is a sequence of positive numbers such that $\Sigma_{n=1}^{\infty} a_n = \infty$, then $\Sigma_{n=1}^{\infty} s_n^{-\alpha} a_n$ converges or diverges according as $\alpha > 1$ or $\alpha \leq 1$, where $s_n = a_1 + \cdots + a_n$. A simple application of this theorem shows that for every real number c there exist many sequences $\{a_n\}$ of positive numbers with $\Sigma_{n=1}^{\infty} a_n = \infty$ such that

(1.1) $\qquad \Sigma_{n=1}^{\infty} s_n^{-\alpha}(a_n - c)$ also converges or diverges

according as $\alpha > 1$ or $\alpha \leq 1$. A random series analogous to that in (1.1) is $\Sigma_{n=1}^{\infty} \operatorname{sgn}(S_n)|S_n|^{-\alpha}(X_n - c)$ where $X_1, X_2, \cdots$ is a sequence of random variables (not necessarily positive or nonnegative), c is a real number and $S_n = X_1 + \cdots + X_n$. Here and throughout the rest of this paper, a random series $\Sigma_{n=1}^{\infty} U_n$ is said to be defined on a probability space $(\Omega, \mathcal{F}, P)$ if for almost all $\omega \in \Omega$ there exists $N(\omega)$ such that $U_n(\omega)$ is defined for all $n \geq N(\omega)$. It is said to converge a.s. on a set A if for almost all $\omega \in A$ there exists $N(\omega)$ such that $\Sigma_{n=N(\omega)}^{\infty} U_n(\omega)$ converges. It is said to diverge a.s. on A if for almost all $\omega \in A$, $\Sigma_{n=N}^{\infty} U_n(\omega)$ diverges for all sufficiently large N.

Kai Lai Chung (personal communication) proved that if $X_1, X_2, \cdots$ are independent and identically distributed random variables with nonzero mean μ such that $E|X_1|\log^+|X_1| < \infty$, then (in the case $\alpha = 1$)

(1.2) $\qquad \Sigma_{n=1}^{\infty} S_n^{-1}(X_n - \mu)$ converges a.s.

Note that since $\mu \neq 0$, $|S_n| \to \infty$ a.s. by the strong law of large numbers. Thus in view of (1.1), the almost sure convergence of $\Sigma_{n=1}^{\infty} S_n^{-1}(X_n - \mu)$ is due to the probablistic structure of $X_1, X_2, \cdots$.

The following question was asked by Chung. Can (1.2) be generalized to a result

Received September 20, 1978; revised February 26, 1979.
[1]This research was partially supported by National Science Foundation Grant MCS 78-01108.
AMS 1970 *subject classifications.* Primary 60F15; secondary 60G45.
Key words and phrases. Martingale transform, random Abel-Dini series, conditional strong law, Burkholder's strong law.

in a martingale setting? In this paper, we answer Chung's question by showing that the convergence of $\Sigma_{n=1}^{\infty} \operatorname{sgn}(S_n)|S_n|^{-\alpha}(X_n - \mu)$ is indeed closely related to that of martingale transforms. Using this relationship we generalize (1.2) to a result for identically distributed but arbitrarily dependent random variables $X_1, X_2, \cdots$, and in addition consider analogs of the series in (1.1) for all values of α in this setting. Let $(\Omega, \mathfrak{F}, P)$ be the probability space on which $X_1, X_2, \cdots$ are defined and let $\mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots$ be nondecreasing sub-σ-algebras of $\mathfrak{F}$ such that X_n is $\mathfrak{F}_n$-measurable. Assume $E|X_1| < \infty$ and let $\xi_n = E(X_n|\mathfrak{F}_{n-1})$. For each $\alpha > \frac{1}{2}$, we give a moment condition on X_1 for which the series

$$\Sigma_{n=1}^{\infty} \operatorname{sgn}(S_{n+k})|S_{n+k}|^{-\alpha}(X_n - \xi_n) \tag{1.3}$$

converges a.s. on a set, where k is a nonnegative integer. For $\alpha \leqslant \frac{1}{2}$, it is proved that no moment condition on X_1 is sufficient for the same to hold in general. However, it is shown that under other assumptions (1.3) will converge a.s. on similar sets for all values of α.

A corollary is deduced for independent and identically distributed random variables where a strong law of Burkholder (1962) is applied. In the case $\alpha = 1$ and $k = 0$, this corollary yields a result still more general than that of Chung. A conditional strong law is also deduced as another corollary. Finally it is shown that the proofs of the above main results lead to a different proof of the strong law of Burkholder (1962) used in the corollary.

Let $f = (f_1, f_2, \cdots)$ be a martingale relative to nondecreasing σ-algebras $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ and let $d = (d_1, d_2, \cdots)$ be the difference sequence of f. Also let $v = (v_1, v_2, \cdots)$ be a predictable process relative to $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$, that is v_1 is $\mathfrak{F}_1$-measurable and v_n is $\mathfrak{F}_{n-1}$-measurable for $n \geqslant 2$. We shall adopt the following notation. $f_0 \equiv 0$, $S(f) = (\Sigma_{n=1}^{\infty} d_n^2)^{\frac{1}{2}}$, $d^* = \sup_n|d_n|$, $f^* = \sup_n|f_n|$, $v^* = \sup_n|v_n|$ and $\|f\|_p = \sup_n\|f_n\|_p = \sup_n(E|f_n|^p)^{1/p}$ for $1 \leqslant p < \infty$. The σ-algebra generated by a set of random variables $\{X_\alpha : \alpha \in J\}$ defined on a probability space will be denoted by $\mathfrak{B}(X_\alpha : \alpha \in J)$. All the functions will be assumed to be real-valued, Borel measurable and defined on the real line.

2. Statements of main results. Let $X_1, X_2, \cdots$ be identically distributed random variables defined on the probability space $(\Omega, \mathfrak{F}, P)$ such that $E|X_1| < \infty$ and let $\mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots$ be nondecreasing sub-σ-algebras of $\mathfrak{F}$ such that X_n is $\mathfrak{F}_n$-measurable for $n \geqslant 1$. Define $S_n = X_1 + \cdots + X_n$, $\xi_n = E(X_n|\mathfrak{F}_{n-1})$ and $\eta_n = E(X_n I(|X_n| \leqslant n)|\mathfrak{F}_{n-1})$ for $n \geqslant 1$. Also define

$$\Lambda = \{0 < \liminf_n n^{-1}|\xi_1 + \cdots + \xi_n|\},$$

$$\Lambda^* = \{0 < \liminf_n n^{-1}|\eta_1 + \cdots + \eta_n|\}$$

and

$$\Lambda^\sim = \{\limsup_n n^{-1}|\eta_1 + \cdots + \eta_n| < \infty\}.$$

We shall prove the following results.

THEOREM 2-1. *Consider the series*

$$\Sigma_{n=1}^{\infty} \operatorname{sgn}(S_{n+k})|S_{n+k}|^{-\alpha}(X_n - \xi_n) \tag{2.1}$$

where α is a real number and k a nonnegative integer.

(a) *If $\alpha > 1$, then* (2.1) *converges* a.s. *on* Λ^*.

(b) *If $\alpha \geqslant 1$ and $E|X_1|\log^+|X_1| < \infty$, then* (2.1) *converges* a.s. *on* Λ.

(c) *If $\frac{1}{2} < \alpha < 1$ and $E|X_1|^{1/\alpha} < \infty$, then* (2.1) *converges* a.s. *on* Λ.

(d) *If $\alpha \leqslant \frac{1}{2}$, $X_1, X_2, \cdots$ are independent and nondegenerate with $\mu = EX_1 \neq 0$, and $\mathcal{F}_n = \mathcal{B}(X_1, \cdots, X_n)$, then* (2.1) *diverges* a.s.

The purpose of stating (d) is that it implies that in the case $\alpha \leqslant \frac{1}{2}$, no moment condition on X_1 is sufficient for (2.1) to converge a.s. on Λ^* or Λ in general. However, if we are prepared to make other assumptions, we will have the next theorem.

THEOREM 2-2. *If the martingale $f^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots)$ defined by $f_n^{(\alpha)} = \Sigma_{i=1}^n i^{-\alpha}(X_i - \xi_i)$ is L_1-bounded, where α is a real number, then*

(a) *for $\alpha > 0$,* (2.1) *converges* a.s. *on* Λ^*;

(b) *for $\alpha \leqslant 0$,* (2.1) *converges* a.s. *on* $\Lambda^\sim$.

We give here a nontrivial example where the martingale defined in Theorem 2-2 is L_1-bounded. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables defined on $(\Omega, \mathcal{F}, P)$ such that $E|X_1| < \infty$. Define $Y_n = X_n$ if $|X_n| > n^{\alpha-2}$ and $= \psi_n(X_n)$ if $|X_n| \leqslant n^{\alpha-2}$, where ψ_n is a function bounded by $n^{\alpha-2}$. Let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be independent sub-σ-algebras of $\mathcal{F}$ such that X_n is $\mathcal{G}_n$-measurable and let $\mathcal{B}_n = \mathcal{B}(Y_n)$. Define $\mathcal{F}_0 = \mathcal{B}_1$ and $\mathcal{F}_n = \mathcal{G}_1 \vee \cdots \vee \mathcal{G}_n \vee \mathcal{B}_{n+1}$ for $n \geqslant 1$. Then $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ and X_n is $\mathcal{F}_n$-measurable for $n \geqslant 1$. By independence, $\xi_n = E(X_n|\mathcal{F}_{n-1}) = E(X_n|\mathcal{B}_n)$ and so $|X_n - \xi_n| = |X_n - Y_n + E(Y_n - X_n|\mathcal{B}_n)| \leqslant 4n^{\alpha-2}$ a.s. Therefore $\Sigma_{n=1}^\infty n^{-\alpha}|X_n - \xi_n| \leqslant 4\Sigma_{n=1}^\infty n^{-2} < \infty$ a.s. and this implies that the martingale $f^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots)$ defined by $f_n^{(\alpha)} = \Sigma_{i=1}^n i^{-\alpha}(X_i - \xi_i)$ is L_1-bounded.

The following corollary is an immediate consequence of Theorem 2-1.

COROLLARY 2-1. *Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $\mu = EX_1 \neq 0$; let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be independent sub-σ-algebras of $\mathcal{F}$ such that X_n is $\mathcal{G}_n$-measurable; and let $\mathcal{B}_n$ be a sub-σ-algebra of $\mathcal{G}_n$. Consider the series*

$$\Sigma_{n=1}^{\infty} \operatorname{sgn}(S_{n+k})|S_{n+k}|^{-\alpha}(X_n - E(X_n|\mathcal{B}_n)) \tag{2.2}$$

where α is a real number and k a nonnegative integer.

(a) *If $\alpha > 1$ and $\mathcal{B}_n = \{\phi, \Omega\}$, then* (2.2) *converges* a.s.

(b) *If $\alpha \geqslant 1$ and $E|X_1|\log^+|X_1| < \infty$, then* (2.2) *converges* a.s.

(c) *If $\frac{1}{2} < \alpha < 1$ and $E|X_1|^{1/\alpha} < \infty$, then* (2.2) *converges* a.s.

The corollary is proved by constructing sub-σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ as in the above example. For part (a), we then apply the strong law of large numbers. For parts (b) and (c), we apply a strong law of Burkholder (1962) which states that $E|X_1|\log^+|X_1| < \infty$ implies $P(n^{-1}\Sigma_{i=1}^n E(X_i|\mathfrak{B}_i) \to \mu) = 1$. Note that in the case $\alpha = 1$ and $k = 0$, we still have a result more general than that of Chung.

The next corollary to Theorem 2-1 is a conditional strong law. Other variants of the conditional strong law can be found in Dubins and Freedman (1965), Brown (1971), Meyer (1972) and Freedman (1973).

COROLLARY 2-2. *Let* $X_1, X_2, \cdots,\ \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots,\ \xi_1, \xi_2, \cdots$ *and* Λ *be as defined at the beginning of this section. Suppose* $X_1, X_2, \cdots$ *are nonnegative and* $EX_1 \log^+ X_1 < \infty$. *Then*

$$\frac{X_1 + \cdots + X_n}{1 + \xi_1 + \cdots + \xi_n} \to 1 \qquad \text{a.s. } on\ \Lambda\ as\ n \to \infty.$$

An example in Burkholder (1962) shows that the condition $EX_1 \log^+ X_1 < \infty$ in Corollary 2-2 cannot be relaxed.

Finally we remark that the above theorems and Corollary 2-1 also apply to the series

$$\Sigma_{n=1}^\infty |S_{n+k}|^{-\alpha}(X_n - \xi_n). \tag{2.3}$$

3. Proofs. This section is devoted to the proof of Theorem 2-1. The most crucial observation and perhaps the crux of the whole matter is the following lemma.

LEMMA 3-1. *Let* $f = (f_1, f_2, \cdots)$ *be an* L_1*-bounded martingale and* $v = (v_1, v_2, \cdots)$ *a predictable process relative to nondecreasing* σ*-algebras* $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$. *Let* $w = (w_1, w_2, \cdots)$ *be a process such that* $\Sigma_{n=1}^\infty w_n^2 < \infty$ a.s. *Suppose* ϕ *is a function which is an indefinite integral of* ϕ' *such that one of the following conditions is satisfied.*

(a) ϕ' *is bounded*;

(b) ϕ' *is bounded on* $(-\infty, x]$ *for every real number* x *and* $|\phi(\infty)| = \infty$;

(c) ϕ' *is bounded on* $[x, \infty)$ *for every real number* x *and* $|\phi(-\infty)| = \infty$;

(d) ϕ' *is bounded on bounded intervals and* $|\phi(\infty)| = |\phi(-\infty)| = \infty$.

Then $\Sigma_{n=1}^\infty d_n\phi(v_n + w_n)$ *converges* a.s. *on* $\{\sup_n|\phi(v_n)| < \infty\}$.

PROOF. We have

$$\begin{aligned} |d_n\phi(v_n + w_n) - d_n\phi(v_n)| &= |d_n \textstyle\int_0^{w_n}\phi'(v_n + t)dt| \\ &\leqslant |d_n w_n|\sup_{|x|\leqslant|w_n|}|\phi'(v_n + x)|. \end{aligned}$$

Therefore summing over n and applying the Cauchy-Schwarz inequality, we obtain

$$\Sigma_{n=1}^\infty |d_n\phi(v_n + w_n) - d_n\phi(v_n)| \leqslant S(f)(\Sigma_{n=1}^\infty w_n^2)^{\frac{1}{2}} U$$

where $U = \sup_n \sup_{|x|\leqslant|w_n|}|\phi'(v_n + x)|$. Since $\Sigma_{n=1}^\infty w_n^2 < \infty$ a.s., we have $\lim_{n\to\infty} w_n = 0$ a.s. and hence $\{U < \infty\} \supset_{\text{a.s.}} \{v^* < \infty\}$. On the other hand, a little reflection shows that $\{U < \infty\} \supset_{\text{a.s.}} \{\sup_n|\phi(v_n)| < \infty, v^* = \infty\}$. Therefore $\{U < \infty\} \supset_{\text{a.s.}} \{\sup_n|\phi(v_n)| < \infty\}$; and by a result of Austin (1966),

$$S(f)(\Sigma_{n=1}^{\infty} w_n^2)^{\frac{1}{2}} U < \infty \text{ a.s. on } \{\sup_n|\phi(v_n)| < \infty\}.$$

But, by Burkholder's (1966) convergence theorem for martingale transforms,

$$\Sigma_{n=1}^{\infty} d_n \phi(v_n) \qquad \text{also converges a.s. on} \quad \{\sup_n|\phi(v_n)| < \infty\}.$$

It follows that

$$\Sigma_{n=1}^{\infty} d_n \phi(v_n + w_n) \qquad \text{converges a.s. on} \quad \{\sup_n|\phi(v_n)| < \infty\},$$

and this proves the lemma.

Note that Lemma 3-1 is more general than Burkholder's (1966) convergence theorem for martingale transforms. To deduce the latter, we let $\phi(x) = x$ and $w_n \equiv 0$. The lemma also implies the following. If $f = (f_1, f_2, \cdots)$ is an L_1-bounded martingale with difference sequence $d = (d_1, d_2, \cdots)$ and k is a non-negative integer, then $\Sigma_{n=1}^{\infty}|f_{n+k}|^{-\alpha} d_n$ converges a.s. if $\alpha \leqslant 0$ and converges a.s. on $\{\lim_{n\to\infty} f_n \neq 0\}$ if $\alpha > 0$.

For the next two lemmas, let $X_1, X_2, \cdots$ be identically distributed random variables defined on $(\Omega, \mathfrak{F}, P)$ such that $E|X_1| < \infty$ and let $\mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots$ be nondecreasing sub-σ-algebras of $\mathfrak{F}$ such that X_n is $\mathfrak{F}_n$-measurable. Define

$$Y_{\alpha n} = X_n I(|X_n| \leqslant n^{\alpha}), \qquad Z_{\alpha n} = X_n I(|X_n| > n^{\alpha}), \tag{3.1}$$

$$\eta_{\alpha n} = E(Y_{\alpha n}|\mathfrak{F}_{n-1}), \qquad \zeta_{\alpha n} = E(Z_{\alpha n}|\mathfrak{F}_{n-1}). \tag{3.2}$$

LEMMA 3-2. *Let $f^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots)$ be a martingale relative to $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ such that*

$$f_n^{(\alpha)} = \Sigma_{i=1}^{n} i^{-\alpha}(Y_{\alpha i} - \eta_{\alpha i}) \tag{3.3}$$

where $\alpha > \frac{1}{2}$. If $E|X_1|^{1/\alpha} < \infty$, then $f^{(\alpha)}$ is L_2-bounded.

PROOF. We have

$$\begin{aligned} \|f^{(\alpha)}\|_2^2 &\leqslant \Sigma_{n=1}^{\infty} n^{-2\alpha} E(Y_{\alpha n} - \eta_{\alpha n})^2 \\ &\leqslant \Sigma_{n=1}^{\infty} n^{-2\alpha} E Y_{\alpha n}^2. \end{aligned}$$

The proof is then completed by generalizing the intermediate steps in the proof of the strong law of large numbers.

LEMMA 3-3. *In each of the cases* (a), (b) *and* (c) *of Theorem* 2-1, *the series*

$$\Sigma_{n=1}^{\infty} n^{-\alpha} \zeta_{\alpha n} \qquad \textit{converges absolutely a.s.} \tag{3.4}$$

Furthermore, (3.4) *also holds if $E|X_1|^{1/\alpha} < \infty$ for $0 < \alpha \leqslant \frac{1}{2}$.*

PROOF. We have

$$E\, \Sigma_{n=1}^{\infty} n^{-\alpha} |\zeta_{\alpha n}| \leqslant E\, \Sigma_{n=1}^{\infty} n^{-\alpha} |Z_{\alpha n}|$$

which converges if and only if $E|X_1| \int_1^{|X_1|^{1/\alpha}} t^{-\alpha} dt < \infty$. This proves the lemma.

PROOF OF THEOREM 2-1. We use the notation of theorem 2-1. Let $\phi_{\alpha\varepsilon}$ be a continuously differentiable function such that $\phi_{\alpha\varepsilon}(x) = \operatorname{sgn}(x)|x|^{-\alpha}$ for $|x| > \varepsilon$. Define $Y_{\alpha n}, Z_{\alpha n}, \eta_{\alpha n}, \zeta_{\alpha n}$ as in (3.1) and (3.2), and define the martingale $f^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots)$ relative to $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ by (3.3). Note that in the case $\alpha = 1$, $\eta_{1n} = \eta_n$. Also define the predictable process $v = (v_1, v_2, \cdots)$ relative to $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ by $v_n = n^{-1}S_{n-1}$, where $S_0 \equiv 0$, and define the process $w = (w_1, w_2, \cdots)$ by $w_n = n^{-1}\Sigma_{i=0}^k Y_{1,n+i}$. Let $d^{(\alpha)} = (d_1^{(\alpha)}, d_2^{(\alpha)}, \cdots)$ be the difference sequence of $f^{(\alpha)}$ and let

$$\Lambda_\varepsilon^* = \{2\varepsilon < \liminf_n n^{-1}|\eta_{11} + \cdots + \eta_{1n}|\}.$$

Following the proof of Lemma 3-2, we see that $\Sigma_{n=1}^\infty w_n^2 < \infty$ a.s. By Lemma 3-2 and Kronecker's lemma, we have $n^{-1}\Sigma_{i=1}^n(Y_{1i} - \eta_{1i}) \to 0$ a.s. Combining this with the fact that

$$P(X_n = Y_n \text{ for all but a finite number of } n) = 1 \tag{3.5}$$

(which follows from $E|X_1| < \infty$), we have

$$\{2\varepsilon < \liminf_n |n^{-1}S_{n+k}|\} \supset_{\text{a.s.}} \Lambda_\varepsilon^* \text{ for every } \varepsilon > 0. \tag{3.6}$$

We now consider cases (a), (b) and (c) together. In these cases, Lemmas 3-1 and 3-2 imply that

$$\Sigma_{n=1}^\infty \phi_{\alpha\varepsilon}(v_n + w_n)d_n^{(\alpha)} \tag{3.7}$$

converges a.s. for every $\varepsilon > 0$, which by (3.5) and Lemma 3-3 in turn implies that

$$\Sigma_{n=1}^\infty \phi_{\alpha\varepsilon}(n^{-1}S_{n+k})n^{-\alpha}(X_n - \xi_n) \tag{3.8}$$

converges a.s. for every $\varepsilon > 0$. But $\phi_{\alpha\varepsilon}(n^{-1}S_{n+k}) = \operatorname{sgn}(S_{n+k})n^\alpha|S_{n+k}|^{-\alpha}$ for sufficiently large n a.s. on Λ_ε^*. Letting $\varepsilon\downarrow 0$ and using (3.8), we have

$$\Sigma_{n=1}^\infty \operatorname{sgn}(S_{n+k})|S_{n+k}|^{-\alpha}(X_n - \xi_n) \quad \text{converges a.s. on } \Lambda^*. \tag{3.9}$$

In the cases (b) and (c), Lemma 3-3 and Kronecker's lemma imply that $n^{-1}(\zeta_{11} + \cdots + \zeta_{1n}) \to 0$ a.s. and hence $\Lambda^* =_{\text{a.s.}} \Lambda$. This proves (a), (b) and (c) of the theorem.

For the case (d), we may take $\mathfrak{F}_0 = \{\phi, \Omega\}$. Since the X_i in this case are independent, the strong law of large numbers implies that $n^{-1}S_{n+k} \to \mu \neq 0$ a.s. Thus it suffices to prove the a.s. divergence of

$$\Sigma_{n=1}^\infty \phi_{\alpha\varepsilon_0}(n^{-1}S_{n+k})n^{-\alpha}(X_n - \mu) \tag{3.10}$$

where $\varepsilon_0 = \frac{1}{2}|\mu|$. We consider two subcases.

CASE (i): $0 < \alpha \leq \frac{1}{2}$. Suppose $E|X_1|^{1/\alpha} < \infty$. Then $\Sigma_{n=1}^\infty P(|X_n| > n^\alpha) < \infty$ and by the Borel-Cantelli lemma, $P(|X_n| > n^\alpha \text{ i.o.}) = 0$. This together with Lemma 3-3 imply that it suffices to prove the a.s. divergence of

$$\Sigma_{n=1}^\infty \phi_{\alpha\varepsilon_0}(n^{-1}S_{n+k})d_n^{(\alpha)} \tag{3.11}$$

where as before $d_n^{(\alpha)} = n^{-\alpha}(Y_{\alpha n} - \eta_{\alpha n})$ and in this case $= n^{-\alpha}(Y_{\alpha n} - \mu_{\alpha n})$ with $\mu_{\alpha n} = EY_{\alpha n}$. Now

$$(3.12) \quad |\phi_{\alpha\varepsilon_0}(n^{-1}S_{n+k})d_n^{(\alpha)} - \phi_{\alpha\varepsilon_0}(n^{-1}S_{n-1})d_n^{(\alpha)}| \leq Cn^{-1-\alpha}|\Sigma_{i=0}^k X_{n+i}(Y_{\alpha n} - \mu_{\alpha n})|$$

for some constant C which depends on $\phi_{\alpha\varepsilon_0}$. Since $0 < \alpha \leq \frac{1}{2}$, we have $1/\alpha \geq 2$. Therefore

$$E\Sigma_{n=1}^{\infty} n^{-1-\alpha}|\Sigma_{i=0}^k X_{n+i}(Y_{\alpha n} - \mu_{\alpha n})| \leq 2(k+1)\Sigma_{n=1}^{\infty} n^{-1-\alpha}EX_1^2 < \infty.$$

This implies that

$$(3.13) \qquad \Sigma_{n=1}^{\infty} n^{-1-\alpha}|\Sigma_{i=0}^k X_{n+i}(Y_{\alpha n} - \mu_{\alpha n})| \qquad \text{converges a.s.}$$

Next $g^{(\alpha)} = (g_1^{(\alpha)}, g_2^{(\alpha)}, \cdots)$ with $g_n^{(\alpha)} = \Sigma_{i=1}^n \phi_{\alpha\varepsilon_0}(i^{-1}S_{i-1})d_i^{(\alpha)}$ is a martingale relative to $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. Its difference sequence $e^{(\alpha)} = (e_1^{(\alpha)}, e_2^{(\alpha)}, \cdots)$ is such that $Ee^{(\alpha)2*} < \infty$. Therefore by a theorem of Burkholder (1966),

$$\{g^{(\alpha)} \text{ diverges}\} =_{\text{a.s.}} \{S(g^{(\alpha)}) = \infty\}.$$

Also by the conditional Borel-Cantelli lemma (see, for example, Chen (1978)),

$$\{S(g^{(\alpha)}) = \infty\} =_{\text{a.s.}} \{\Sigma_{n=1}^{\infty} E(e_n^{(\alpha)2}|\mathcal{F}_{n-1}) = \infty\}.$$

Since $\phi_{\alpha\varepsilon_0}(n^{-1}S_{n-1}) \to \phi_{\alpha\varepsilon_0}(\mu) \neq 0$ a.s., it follows that

$$\Sigma_{n=1}^{\infty} E(e_n^{(\alpha)2}|\mathcal{F}_{n-1}) = \Sigma_{n=1}^{\infty} \phi_{\alpha\varepsilon_0}^2(n^{-1}S_{n-1})n^{-2\alpha}E(Y_{\alpha n} - \mu_{\alpha n})^2$$
$$= \infty \text{ a.s.}$$

and hence $S(g^{(\alpha)}) = \infty$ a.s. This in turn implies that $g^{(\alpha)}$ diverges a.s. Combining this with (3.12) and (3.13), we prove the a.s. divergence of (3.11) and hence that of (3.10). Next suppose $E|X_1|^{1/\alpha} = \infty$. Then $\Sigma_{n=1}^{\infty} P(|X_n| > n^{\alpha}) = \infty$ and by the Borel-Cantelli lemma again $P(|X_n| > n^{\alpha} \text{ i.o.}) = 1$. This together with the fact that $\phi_{\alpha\varepsilon_0}(n^{-1}S_{n+k}) \to \phi_{\alpha\varepsilon_0}(\mu) \neq 0$ a.s. imply that $P(\phi_{\alpha\varepsilon_0}(n^{-1}S_{n+k})n^{-\alpha}(X_n - \mu) \nrightarrow 0) = 1$ and hence the a.s. divergence of (3.10). This proves case (i).

Case (ii): $\alpha \leq 0$. It is not difficult to see that there exist $c > 0$ and $\delta > 0$ such that $P(|X_n - \mu| > cn^{\alpha}) \geq \delta$ for sufficiently large n. Then $\Sigma_{n=1}^{\infty} P(|X_n - \mu| > cn^{\alpha}) = \infty$, and arguing as above we prove the a.s. divergence of (3.10). This completes the proof of Theorem 2-1.

The proof of Theorem 2-2 is a more straightforward application of Lemma 3-1 and is therefore omitted. We also omit the proofs of Corollaries 2-1 and 2-2 as the former has been sketched and the latter is easy.

4. Burkholder's strong law. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables defined on $(\Omega, \mathcal{F}, P)$ such that $E|X_1|\log^+|X_1| < \infty$ and let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be independent sub-σ-algebras of $\mathcal{F}$ such that X_n is $\mathcal{G}_n$-measurable. Let $\mathcal{B}_n$ be a sub-σ-algebra of $\mathcal{G}_n$ for $n \geq 1$. Burkholder (1966) proved that

$$(4.1) \qquad \lim_{n\to\infty} n^{-1}\Sigma_{i=1}^n E(X_i|\mathcal{B}_i) = EX_1 \text{ a.s.}$$

We have used this result in deducing Corollary 2-1. Here we give a different proof of (4.1) by showing that it is an easy consequence of Lemmas 3-2 and 3-3. Indeed, let $\mathcal{F}_0 = \mathcal{B}_1$ and $\mathcal{F}_n = \mathcal{G}_1 \vee \cdots \vee \mathcal{G}_n \vee \mathcal{B}_{n+1}$ for $n \geqslant 1$ as in the example in Section 2. Then Lemmas 3-2 and 3-3 imply that

$$\Sigma_{n=1}^{\infty} n^{-1}(X_n - E(X_n|\mathcal{B}_n)) \qquad \text{converges a.s.}$$

which by the strong law of large numbers and Kronecker's lemma in turn implies (4.1).

Acknowledgment. Part of this research was done when the author was at Stanford University during the Summer of 1978. The author is thankful to Professor Kai Lai Chung for some helpful discussions.

REFERENCES

[1] Austin, D. G. (1966). A sample function property of martingales. *Ann. Math. Statist.* **37** 1396–1397.
[2] Brown, B. M. (1971). A conditional setting for some theorems associated with the strong law. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **19** 274–280.
[3] Burkholder, D. L. (1962). Successive conditional expectations of an integrable function. *Ann. Math. Statist.* **33** 887–893.
[4] Burkholder, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.
[5] Chen, Louis H. Y. (1978). A short note on the conditional Borel-Cantelli lemma. *Ann. Probability* **6** 699–700.
[6] Dubins, Lester E. and Freedman, David A. (1965). A sharper form of the Borel-Cantelli lemmas and the strong law. *Ann. Math. Statist.* **36** 800–807.
[7] Freedman, David (1973). Another note on the Borel-Cantelli lemma and the strong law, with the Poisson approximation as a by-product. *Ann. Probability* **1** 910–925.
[8] Knopp, Konrad (1971). *Theory and Application of Infinite Series.* Hafner, New York.
[9] Meyer, P. A. (1972). Martingales and stochastic integrals 1. Lecture Notes in Math. 284. Springer, Berlin.

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