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ASYMPTOTIC COMPARISONS OF FUNCTIONALS OF BROWNIAN MOTION AND RANDOM WALK

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In this paper we make comparisons involving stopping times τ of a process X and the maximal function X_τ^* of that process, where X is either Brownian motion or random walk. In particular, we give conditions implying that $P(X_\tau^* > \lambda) \approx P(\tau^{1/2} > \lambda)$ in the sense of a two-sided inequality holding. We show that if, for all large λ there exist constants $\beta > 1$ and $\gamma > 0$ satisfying

$$0 < P(\tau^{1/2} > \lambda) \leq \gamma P(\tau^{1/2} > \beta\lambda),$$

and if X is a one-dimensional Brownian motion, then $P(X_\tau^* > \lambda) \approx P(\tau^{1/2} > \lambda)$. An analogous result is given for n-dimensional Brownian motion ($n \geq 3$). We also consider a similar result for one-sided maximal functions of local martingales. Finally, we look at a random walk X, where $X_n = x_1 + x_2 + \cdots + x_n$, and give two different sets of conditions on τ and the x_i's under which the result $P(\tau^{1/2} > \lambda) \approx P(X_\tau^* > \lambda)$ is true.

1. Introduction and notation. In recent years several interesting results concerning stopping times of martingales have been proved. In particular, through various inequalities it has been possible to relate quantities involving the stopping time τ to quantities involving the values of the martingale itself. The problem that we will consider will be that of comparing the probabilities $P(\tau^{1/2} > \lambda)$ and $P(X_\tau^* > \lambda)$ for large values of λ, where X is a martingale and X_τ^* is the maximal function of X up to time τ. In particular we will be looking at Brownian motion and random walk.

We will work with the probability space $(\Omega, \mathscr{A}, P)$, taking $\mathscr{A}_0, \mathscr{A}_1, \cdots$ to be a nondecreasing sequence of sub-σ-fields of $\mathscr{A}$ and letting $f = (f_1, f_2, \cdots)$ be a martingale relative to $\mathscr{A}_1, \mathscr{A}_2, \cdots$. In the continuous case our martingale will be $\{X_t : t \geq 0\}$ and the sub-σ-fields will be $\{\mathscr{A}_t : t \geq 0\}$ instead. For the discrete case we will define the maximal function f^* by

$$f^* = \sup_n |f_n|$$

and

$$f_n^* = \sup_{1 \leq i \leq n} |f_i|.$$

We will further define the difference sequence $(d_1, d_2, \cdots)$ of f by

$$d_k = f_k - f_{k-1}$$

where we take $f_0 = 0$. For the continuous case we will define X^* by taking the supremum over all nonnegative values of t. Also we shall take

$$\|f_n\|_p = [E|f_n|^p]^{1/p}$$

and

$$\|f\|_p = \sup_n \|f_n\|_p$$

for $0 < p < \infty$, with similar definitions for the continuous case.

If τ is a stopping time of f, we let f^τ be the martingale f stopped at time τ, i.e.,

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$$f_n^\tau = f_{\tau\wedge n},$$

where $\tau \wedge n = \min(\tau, n)$. Then we define

$$f_\tau^* = (f^\tau)^*.$$

Finally, if f and g are functions from some interval (a, ∞) to $\mathbb{R}$, we will use the notation $f(\lambda) \approx g(\lambda)$ to mean that there are positive real numbers c, C, and $\lambda_0 \geq a$ such that

$$cg(\lambda) \leq f(\lambda) \leq Cg(\lambda)$$

for all $\lambda > \lambda_0$.

Now suppose that we look at a random walk X of the form $X_n = x_1 + x_2 + \cdots + x_n$ where the x_i are independent and identically distributed random variables. In this case Greenwood and Monroe [7], [8] have shown that under various sets of conditions it is true that

$$\lim_{y\to\infty} P(X_\tau > y)/P(X_1 > y) = E\tau$$

and

$$\lim_{y\to\infty} P(\sup_n X_{\tau\wedge n} > y)y^P = E\tau.$$

We will be considering a problem rather similar to that which Greenwood and Monroe considered but will approach it somewhat differently. We will try to find conditions upon τ and x_1 that will yield results that will allow us to compare $P(X_\tau^* > \lambda)$ and $P(\tau^{1/2} > \lambda)$ asymptotically, where X is either Brownian motion or its discrete counterpart, random walk.

In Section 2, in which we present some previously unpublished results of Burkholder, we will let $X = \{X_t : t \geq 0\}$ be a one-dimensional Brownian motion. We will prove in Theorem 2.1 that if τ is a stopping time for X such that

$$0 < P(\tau^{1/2} > \lambda) \leq \gamma P(\tau^{1/2} > \beta\lambda)$$

for some $\beta > 1$ and $\gamma > 0$, then

$$P(\tau^{1/2} > \lambda) \approx P(X_\tau^* > \lambda).$$

We will also consider the case of n-dimensional Brownian motion, $n \geq 2$.

In addition we will look at a local martingale $Y = \{Y(t), t \geq 0\}$ with continuous sample functions and constant initial position $Y(0) = 0$. We define

$$M^+ = \sup_{t\geq 0} Y^+(t)$$

and

$$M^- = \sup_{t\geq 0} Y^-(t)$$

where $Y^+(t) = \max(0, Y(t))$ and $Y^-(t) = -\min(0, Y(t))$. We will show in Theorem 2.4 that if

$$0 < P(M^+ > \lambda) \leq \gamma P(M^+ > \beta\lambda),$$

where $\gamma < \beta$, then

$$P(M^+ > \lambda) \approx P(M^- > \lambda).$$

In Section 3 we will consider the case of random walk. In Theorem 3.2 we will see that if

(i) $X_n = x_1 + x_2 + \cdots + x_n$ where the x_i satisfy $E(x_i \mid \mathscr{A}_{i-1}) = 0$, $E(x_i^2 \mid \mathscr{A}_{i-1}) = 1$, and $E(|x_i| \mid \mathscr{A}_{i-1}) \geq c > 0$, and

(ii) τ is a stopping time for X such that for some p, $0 < p < 2$,

$$P(\tau^{1/2} > \lambda) \approx \lambda^{-p},$$

then

$$P(X_\tau^* > \lambda) \approx P(\tau^{1/2} > \lambda).$$

Note in particular that the conditions of (i) are satisfied if the x_i are independent with $Ex_i = 0$, $Ex_i^2 = 1$, and $E|x_i| \geq c$.

In Theorem 3.3 we show that the assumption that $0 < p < 2$ in condition (ii) above can be relaxed if some further assumption is made about the distribution of the x_i.

To prove these theorems we will need several inequalities from [2], [3], [4] and [5]. In addition we will use the fact that if f is of the form

$$f_n = \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} v_k x_k$$

for $n \geq 1$, where v_k is $\mathscr{A}_{k-1}$ measurable for $k \geq 1$ and $x = (x_1, x_2, \cdots)$ is a martingale difference sequence relative to $\mathscr{A}_1, \mathscr{A}_2, \cdots$, then

$$\|f\|_2 = \|s(f)\|_2, \tag{1.1}$$

where we define

$$s(f) = (\sum_{k=1}^{\infty} E(d_k^2 \mid \mathscr{A}_{k-1}))^{1/2}.$$

This fact is due to the orthogonality of the difference sequence d of f, giving that

$$\|f\|_2 = \|(\sum_{k=1}^{\infty} (d_k)^2)^{1/2}\|_2 = \|\sum_{k=1}^{\infty} E(d_k^2 \mid \mathscr{A}_{k-1})^{1/2}\|_2$$

as is noted in Lemma 2.2 of [6].

Other necessary facts will be described where needed.

2. Asymptotic results for Brownian motion. In this section, we shall present some heretofore unpublished work of Burkholder on asymptotic comparisons for Brownian motion, work that motivated our study of similar problems for random walk.

In the following we will let Φ be a function satisfying the following conditions:

(a) Φ is a nonnegative, nondecreasing, real-valued function on $(0, \infty)$.

(b) There exist real numbers $\beta > 1$ and γ such that

$$\Phi(\beta\lambda) \leq \gamma\Phi(\lambda)$$

for all $\lambda > 0$. Also suppose that δ and η are positive real numbers such that

(c) $\Phi(\delta^{-1}\lambda) \leq \eta\Phi(\lambda)$, $\lambda > 0$.

Conditions (a) and (b) imply that if $\delta > 0$, then there does exist a real number η such that (c) holds: If the positive integer k satisfies $\delta^{-1} \leq \beta^k$, then

$$\Phi(\delta^{-1}\lambda) \leq \Phi(\beta^k\lambda) \leq \gamma^k\Phi(\lambda).$$

So η may be chosen to be γ^k.

Note that condition (b) with $\beta = 2$ is the familiar Δ_2 condition used in the study of Orlicz spaces. However, it is important to notice that here Φ is not required to be convex.

We now proceed to give a proof of the following lemma.

LEMMA 2.1. *If Φ is a function satisfying* (a)–(c) *above, and if f and g are nonnegative measurable functions on $(\Omega, \mathscr{A}, P)$ satisfying*

$$P(f > \beta\lambda, g \leq \delta\lambda) \leq \epsilon P(f > \lambda) \tag{2.1}$$

for all $\lambda > 0$, where $\gamma\epsilon < 1$, then

(i) $\sup_{\lambda>0} \Phi(\lambda)P(f > \lambda) \leq (\gamma\eta/(1 - \gamma\epsilon)) \sup_{\lambda>0} \Phi(\lambda)P(g > \lambda)$;

(ii) $\limsup_{\lambda\to\infty} \Phi(\lambda)P(f > \lambda) \leq (\gamma\eta/(1 - \gamma\epsilon)) \limsup_{\lambda\to\infty} \Phi(\lambda)P(g > \lambda)$.

PROOF. Part (i) is Lemma 1 of [5]. Inequality (2.1) implies that

$$P(f > \beta\lambda) \leq \epsilon P(f > \lambda) + P(g > \delta\lambda).$$

Multiplying through by $\Phi(\beta\lambda)$, we get

$$\Phi(\beta\lambda)P(f > \beta\lambda)$$

$$\leq \epsilon\Phi(\beta\lambda)P(f>\lambda) + \Phi(\beta\delta^{-1}\delta\lambda)P(g>\delta\lambda) \tag{2.2}$$
$$\leq \epsilon\gamma\Phi(\lambda)P(f>\lambda) + \gamma\eta\Phi(\delta\lambda)P(g>\delta\lambda).$$

We may assume that

$$\limsup_{\lambda\to\infty} \Phi(\lambda)P(g>\lambda) < \infty.$$

If not, the result is trivial.

Since Φ is nondecreasing, it follows that

$$\sup_{\lambda>0} \Phi(\lambda)P(g>\lambda) < \infty,$$

and so, by (i),

$$\sup_{\lambda>0} \Phi(\lambda)P(f>\lambda) < \infty.$$

This means that

$$\limsup_{\lambda\to\infty} \Phi(\lambda)P(f>\lambda) < \infty.$$

Thus if we take the lim sup of both sides in (2.2), we see that

$$\limsup_{\lambda\to\infty} \Phi(\lambda)P(f>\lambda) = \limsup_{\lambda\to\infty} \Phi(\beta\lambda)P(f>\beta\lambda)$$
$$\leq \gamma\epsilon \limsup_{\lambda\to\infty} \Phi(\lambda)P(f>\lambda) + \gamma\eta \limsup_{\lambda\to\infty} \Phi(\lambda)P(g>\lambda).$$

Therefore

$$(1-\gamma\epsilon)\limsup_{\lambda\to\infty} \Phi(\lambda)P(f>\lambda) \leq \gamma\eta \limsup_{\lambda\to\infty} \Phi(\lambda)P(g>\lambda),$$

and the proof is finished.

We now go on to state the following result.

THEOREM 2.1. *Suppose that, for some* $\lambda_0 > 0$ *and all* $\lambda \geq \lambda_0$, *there exist constants* $\beta > 1$ *and* $\gamma > 0$ *satisfying*

$$0 < P(\tau^{1/2} > \lambda) \leq \gamma P(\tau^{1/2} > \beta\lambda). \tag{2.3}$$

If $X = \{X_t : t \geq 0\}$ *is a one-dimensional Brownian motion, then*

$$P(X_\tau^* > \lambda) \approx P(\tau^{1/2} > \lambda).$$

It is clear that the assumption (2.3) is satisfied if

$$P(\tau^{1/2} > \lambda) \approx \lambda^{-p}$$

for some $p > 0$.

Burgess Davis, in response to the present work, has noticed the following case in which the condition of Theorem 2.1 is satisfied. We will state it as a theorem and give its proof before we prove Theorem 2.1.

THEOREM 2.2. *Let* X *be a one-dimensional Brownian motion and let* $K > 1$. *If* τ *is a stopping time satisfying*

$$|X_\tau| > K\tau^{1/2} \quad \text{a.s.},$$

then there is a positive constant $\alpha = \alpha(K)$ *such that for all* $\lambda > 0$

$$P(\tau^{1/2} > 2\lambda) \geq P(\tau^{1/2} > \lambda).$$

To prove this theorem we need two lemmas. We will let $P_{x,t}$ stand for the probability measure associated with Brownian motion started at time t and height x and we will let $P_{0,0} = P$.

LEMMA 2.2. *Let $K > 1$. There is a constant $\delta = \delta(K) > 0$ such that if $|x| > Kt^{1/2}$, $0 \leq t < 1$*, then

$$P_{x,t}(|X_1| < \delta) < P(|X_1| < \delta).$$

PROOF. In order to prove the lemma we need only show that, for $0 \leq t < 1$,

$$P_{Kt^{1/2},t}(|X_1| < \delta) < P(|X_1| < \delta). \tag{2.4}$$

First we write the probability on the left-hand side of (2.4) in terms of a standard normal integral and then make the change of variables $s = t/(1-t)$, $0 \leq t < 1$. Calling this probability $f(s)$, we get

$$f'(s) = (2\pi)^{-1/2}\exp(-(\delta(1+s)^{1/2} - Ks^{1/2})^2/2)(\delta/(2(1+s)^{1/2}) - K/(2s^{1/2}))$$
$$+ (2\pi)^{-1/2}\exp(-(\delta(1+s)^{1/2} + Ks^{1/2})^2/2)(\delta/(2(1+s)^{1/2}) + K/(2s^{1/2})).$$

It is then enough to show that $f'(s) < 0$ for $s > 0$. By simplifying and letting $\delta = \alpha K$, it can be shown that $f'(s) < 0$ is equivalent to $g(s) > 0$, where

$$g(s) = \exp(2\alpha K^2(s(1+s))^{1/2})(1 + s(1-\alpha^2)) - ((s+1)^{1/2} + \alpha s^{1/2})^2.$$

Finally we observe that $g(0) = 0$ and that $g'(s) > 0$ providing that $K > 1$ and $K^2 - 1 > \alpha$. Since $\delta = \alpha K$, we have proven that Lemma 2.2 holds with $\delta < K^3 - K$.

LEMMA 2.3. *If τ and X_t are as in Theorem 2.2, then*

$$P(|X_1| < \delta \mid \tau \geq 1) \geq P(|X_1| < \delta).$$

PROOF. Using Lemma 2.2,

$$\begin{aligned} P(|X_1| < \delta) &= P(|X_1| < \delta, \tau \geq 1) + P(|X_1| < \delta, \tau < 1) \\ &= P(|X_1| < \delta, \tau \geq 1) + EP_{X_\tau,\tau}(|X_1| < \delta)I(\tau < 1) \\ &\leq P(|X_1| < \delta, \tau \geq 1) + P(|X_1| < \delta)P(\tau < 1), \end{aligned}$$

which proves Lemma 2.3.

PROOF OF THEOREM 2.2. First we observe that if X_t is a one-dimensional Brownian motion, then so is $\lambda^{-1/2}X_{t/\lambda}$. Applying this fact to the previous two lemmas, the conclusion of Lemma 2.3 becomes

$$P(|X_{\lambda^2}| < \delta\lambda \mid \tau \geq \lambda^2) \geq P(|X_{\lambda^2}| < \delta\lambda).$$

Now assume, without loss of generality, that $\delta(K) \leq \frac{1}{2}$. Then the above facts imply that

$$\begin{aligned} P(\tau^{1/2} \geq 2\lambda)/P(\tau^{1/2} \geq \lambda) &= P(\tau \geq 4\lambda^2 \mid \tau \geq \lambda^2) \\ &= P(\tau \geq 4\lambda^2 \mid \tau \geq \lambda^2, |X_{\lambda^2}| < \delta\lambda)P(|X_{\lambda^2}| < \delta\lambda \mid \tau \geq \lambda^2) \\ &\geq P(\tau \geq 4\lambda^2 \mid \tau \geq \lambda^2, |X_{\lambda^2}| < \delta\lambda)P(|X_{\lambda^2}| < \delta\lambda) \\ &\geq P(\sup_{\lambda^2 \leq t \leq 4\lambda^2} |X_t - X_{\lambda^2}| < \lambda/2)P(|X_{\lambda^2}| < \delta\lambda) \\ &= P(\sup_{1 \leq t \leq 4} |X_t - X_1| < \tfrac{1}{2})P(|X_1| < \delta) \\ &= CP(|X_1| < \delta) = \alpha, \end{aligned}$$

as desired.

PROOF OF THEOREM 2.1. We shall need the following inequalities. If $\beta > 1$ and $\delta > 0$, then

$$P(\tau^{1/2} > \beta\lambda, X_\tau^* \leq \delta\lambda) \leq \delta^2(\beta^2 - 1)^{-1}P(\tau^{1/2} > \lambda) \tag{2.5}$$

and

(2.6) $$P(X_\tau^* > \beta\lambda,\ \tau^{1/2} \le \delta\lambda) \le \delta^2(\beta - 1)^{-2} P(X_\tau^* > \lambda).$$

For the proofs see [2].

Next, we take

$$\Phi(\lambda) = \begin{cases} 1/P(\tau^{1/2} > \lambda) & \text{if } \lambda \ge \lambda_0 \\ \Phi(\lambda_0) & \text{if } 0 < \lambda \le \lambda_0. \end{cases}$$

It can be seen that under the assumptions of Theorem 2.1, Φ satisfies (a)–(c); for example, (b) holds since

$$\Phi(\beta\lambda) = 1/P(\tau^{1/2} > \beta\lambda) \le \gamma/P(\tau^{1/2} > \lambda) = \gamma\Phi(\lambda)$$

for all large enough λ. Taking $f = X_\tau^*$ and $g = \tau^{1/2}$, we can apply part (ii) of Lemma 2.1 to obtain

(2.7) $$\limsup_{\lambda\to\infty} \Phi(\lambda)P(X_\tau^* > \lambda) \le (\gamma\eta/(1 - \gamma\epsilon_1))\limsup_{\lambda\to\infty} \Phi(\lambda)P(\tau^{1/2} > \lambda).$$

Here $\epsilon_1 = \delta^2(\beta - 1)^{-2}$ is the constant of (2.6). Note that γ is the constant given by assumption (2.3). Also, η is the constant from (c), which we already have shown to be satisfied. Thus the constants γ and η depend only on Φ. In order to make sure that $\gamma\epsilon_1 < 1$, we fix β and γ in (2.3). Then, since for any $\delta > 0$ there is an η such that (c) is satisfied, we take δ small enough to ensure that $\epsilon_1 < \gamma^{-1}$. Since

$$\Phi(\lambda)P(\tau^{1/2} > \lambda) = 1$$

for large enough λ, (2.7) implies that

$$\limsup_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \le \gamma\eta/(1 - \gamma\epsilon_1) = C.$$

If we can in addition show that

$$\liminf_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \ge c > 0,$$

then we are finished.

By choosing δ small enough, we can choose $\epsilon_2 = \delta^2(\beta^2 - 1)^{-1}$, the constant of (2.5), so that $\gamma\epsilon_2 < 1$. Then we know by (2.5) that

$$P(\tau^{1/2} > \beta\lambda) \le \epsilon_2 P(\tau^{1/2} > \lambda) + P(X_\tau^* > \delta\lambda).$$

Thus

$$P(\tau^{1/2} > \lambda) \le \gamma P(\tau^{1/2} > \beta\lambda) \le \gamma\epsilon_2 P(\tau^{1/2} > \lambda) + \gamma P(X_\tau^* > \delta\lambda).$$

This allows us to conclude, since $\gamma\epsilon_2 < 1$, that

$$P(X_\tau^* > \delta\lambda)/P(\tau^{1/2} > \lambda) \ge (1 - \gamma\epsilon_2)/\gamma$$

for all large enough λ. Replacing λ by $\delta^{-1}\lambda$, we get

$$P(X_\tau^* > \lambda)/P(\tau^{1/2} > \delta^{-1}\lambda) \ge (1 - \gamma\epsilon_2)/\gamma$$

for all large enough λ.

Finally we see that (2.3) implies that

$$P(\tau^{1/2} > \lambda) \le \gamma^n P(\tau^{1/2} > \beta^n\lambda)$$

for positive integers n. If k is the smallest such integer for which $\beta^k \ge \delta^{-1}$, then we can use this fact to see that

$$P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \ge \gamma^{-k}(1 - \gamma\epsilon_2)/\gamma = c.$$

Thus we have shown that

$$\liminf_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} \geq \lambda) \geq c > 0,$$

and the proof is finished.

Note that if, instead of assumption (2.3), we required that

$$0 < P(X_\tau^* > \lambda) \leq \gamma P(X_\tau^* > \beta\lambda)$$

the same proof would suffice. We would only need to take $\Phi(\lambda) = 1/P(X_\tau^* > \lambda)$ instead.

Results analogous to Theorem 2.1 can also be shown to hold for n-dimensional Brownian motion, $n \geq 2$. In fact if $n \geq 3$, $0 < \delta_0 < 1$, and $P(\tau < \infty) = 1$, then, as shown in [4],

$$P(X_\tau^* > \lambda) \leq (1 - \delta_0^{n-2})P(|X_\tau| > \delta_0\lambda),$$

for all $\lambda > 0$. This gives us the following theorem.

THEOREM 2.3. *If X is Brownian motion in $\mathbb{R}^n$ $(n \geq 3)$ starting at* 0 *and τ is a stopping time of X satisfying*

$$P(\tau < \infty) = 1$$

and

$$0 < P(\tau^{1/2} > \lambda) \leq \gamma P(\tau^{1/2} > \beta\lambda)$$

for all large λ, then

$$P(|X_\tau| > \lambda) \approx P(X_\tau^* > \lambda) \approx P(\tau^{1/2} > \lambda).$$

Finally, we consider a somewhat different application of the method used to prove Theorem 2.1. Here we will let $Y = \{Y_t : 0 \leq t < \infty\}$ be a local martingale with continuous sample functions and $Y(0) = 0$. Also we will define M^+ and M^- as in Section 1.

It is shown in [3] that if $\beta > 1$ and $\delta > 0$, then

$$P(M^+ > \beta\lambda,\ M^- \leq \delta\lambda) \leq (1 + \delta)(\beta + \delta)^{-1}P(M^+ > \lambda). \tag{2.8}$$

Now since Y is a local martingale with $Y(0) = 0$, so is $-Y$. Then, since $M^+(Y) = M^-(-Y)$ and $M^-(Y) = M^+(-Y)$, we can apply (2.8) to $-Y$ to show that

$$P(M^- > \beta\lambda,\ M^+ \leq \delta\lambda) \leq (1 + \delta)(\beta + \delta)^{-1}P(M^- > \lambda). \tag{2.9}$$

This allows us to prove the following result.

THEOREM 2.4. *Let Y be a local martingale starting at* 0 *with continuous sample functions. Let $1 < \gamma < \beta$ and suppose that for all large $\lambda > 0$,*

$$0 < P(M^+ > \lambda) \leq \gamma P(M^+ > \beta\lambda). \tag{2.10}$$

Then

$$P(M^+ > \lambda) \approx P(M^- > \lambda). \tag{2.11}$$

PROOF. We proceed as in Theorem 2.1; however, here we replace (2.5) and (2.6) by (2.8) and (2.9). We then take $\Phi(\lambda) = 1/P(M^+ > \lambda)$, and we see exactly as in Theorem 2.1 that Φ satisfies (a)–(c). We need only check that we can make $\epsilon = \epsilon_1 = \epsilon_2 = (1 + \delta)/(\beta + \delta)$ small enough so that $\gamma\epsilon < 1$. Here we note that $\gamma\epsilon = (\gamma + \gamma\delta)/(\beta + \delta)$. Then if we choose δ small, subject to the requirement that $\gamma < \beta$, we can ensure that $\gamma\epsilon < 1$. Therefore we can replace $\tau^{1/2}$ by M^+ and X_τ^* by M^- in the proof of Theorem 2.1, and the result is proven. It is interesting to note that examples can be constructed in which $\gamma \geq \beta$ in (2.10) and in which (2.11) does not hold (see [10]).

3. Asymptotic results for random walk. The results for Brownian motion proved in Chapter 2 might lead us to wonder whether similar results can be proved for random walk. We might guess that a few more conditions would be necessary for the theorems involving random walk since often the size of the jumps can make things a bit more difficult. For example, if τ is a hitting time of a given closed interval, then a Brownian motion starting outside the interval will always hit the interval right on its boundary whereas a random walk can jump into the interior of the interval or possibly even jump over the interval altogether. In this case we might guess that it might be harder to relate the hitting time τ to the hitting position X_τ than it would be in the case of Brownian motion.

We first prove a theorem with a conclusion similar to that of Lemma 2.1 but with a somewhat different kind of assumption. We will let $f = (f_1, f_2, \cdots)$ be a martingale adapted to the sequence $(\mathscr{A}_1, \mathscr{A}_2, \cdots)$ of σ-fields. Then we obtain the following result.

THEOREM 3.1. *If f is a martingale, then there exists a constant c_p, depending only on p, such that*

(i) $\sup_{\lambda>0}\lambda^p P(f^* > \lambda) \le c_p \sup_{\lambda>0} \lambda^p P(s(f) > \lambda)$ *for* $0 < p < 2$,

(ii) $\limsup_{\lambda\to\infty} \lambda^p P(f^* > \lambda) \le c_p \limsup_{\lambda\to\infty} \lambda^p P(s(f) > \lambda)$ *for* $0 < p < 2$.

PROOF OF (i). In the following proof we let

$$s_n(f) = [\textstyle\sum_{k=1}^n E(d_k^2 \mid \mathscr{A}_{k-1})]^{1/2}.$$

Also, we let τ be the stopping time defined by

$$\tau = \inf\{n : s_{n+1}(f) > \lambda\}.$$

Then τ has the property that $s_\tau(f) \le \lambda$. Also

$$P(f^* > \lambda) \le P(f^* > \lambda,\quad s(f) \le \lambda) + P(s(f) > \lambda). \tag{3.1}$$

It can be shown (see the proof of Lemma 2.2 of [6]) that (3.1) implies

$$P(f^* > \lambda) \le 2P(s(f) > \lambda) + (1/\lambda^2) \int_{\{s(f)\le\lambda\}} s^2(f)\, dP. \tag{3.2}$$

If we let

$$M = \sup_{\lambda>0} \lambda^p P(s(f) > \lambda),$$

then

$$\begin{aligned}
\int_{\{s(f)\le\lambda\}} s^2(f)\, dP &= \int_{\{s(f)\le\lambda\}} \int_0^{s(f)} 2t\, dt\, dP \\
&= \int_0^\lambda 2t \int_{\{t<s(f)\le\lambda\}} dP\, dt \\
&= \int_0^\lambda 2tP(t < s(f) \le \lambda)\, dt \\
&\le \int_0^\lambda 2tP(s(f) > t)\, dt \\
&\le \int_0^\lambda 2t(M/t^p)\, dt \\
&= 2M\lambda^{2-p}/(2-p).
\end{aligned}$$

Therefore (3.2) becomes

$$\lambda^p P(f^* > \lambda) \le 2\lambda^p P(s(f) > \lambda) + 2M/(2-p) \le 2M(1 + 1/(2-p)).$$

This completes the proof of (i) and gives that $c_p = 2(1 + 1/(2-p))$.

PROOF OF (ii). Here instead of defining the quantity M as previously, we define

$$M(0, b) = \sup_{0<\lambda\le b} \lambda^p P(s(f) > \lambda)$$

and

$$M(b, \infty) = \sup_{\lambda > b} \lambda^p P(s(f) > \lambda).$$

Then proceeding as in the proof of (i), we have instead (for $\lambda > b$)

$$\int_0^\lambda 2tP(s(f) > t)\, dt$$

$$\le 2M(0, b)\int_0^b t^{1-p}\, dt + 2M(b, \infty)\int_b^\lambda t^{1-p}\, dt$$

$$\le (2M(0, b)/(2-p))b^{2-p} + 2M(b, \infty)\lambda^{2-p}/(2-p).$$

This means that

$$\lambda^p P(f^* > \lambda) \le 2\lambda^p P(s(f) > \lambda) + (2M(0, b)/(2-p))(b^{2-p}/\lambda^{2-p}) + 2M(b, \infty)/(2-p).$$

For fixed b, the second term of the right-hand side goes to 0 as $\lambda \to \infty$. Now note that as b gets large, $M(b, \infty)$ approaches $\limsup_{\lambda\to\infty} \lambda^p P(s(f) > \lambda)$.

Thus we see that

$$\limsup_{\lambda\to\infty} \lambda^p P(f^* > \lambda) \le 2(1 + 1/(2-p))\limsup_{\lambda\to\infty} \lambda^p P(s(f) > \lambda).$$

Note also that the constant c_p in (ii) is the same as that in (i). It is possible to show that (i) does not hold for $p = 2$ (see [10]).

We can now use Theorem 3.1 to prove a result corresponding to Theorem 2.1, but for random walk. We will consider a martingale $X = (X_1, X_2, \cdots)$, where $X_n = x_1 + x_2 + \cdots + x_n$ and the x_i satisfy the conditional Marcinkiewicz-Zygmund conditions

(i) $E(x_k^2 \mid \mathscr{A}_{k-1}) = 1$

(ii) $E(|x_k| \mid \mathscr{A}_{k-1}) \ge c$

as introduced by Gundy in [9]. Then if we define $f_n = X_{\tau\wedge n}$, we have that $f^* = X_\tau^*$ and $s(f) = \tau^{1/2}$. The latter equality follows from the fact that

$$s_n(f) = [\textstyle\sum_{k=1}^n E(d_k^2 \mid \mathscr{A}_{k-1})]^{1/2}$$

$$= [\textstyle\sum_{k=1}^n I(\tau \ge k)E(x_k^2 \mid \mathscr{A}_{k-1})]^{1/2}$$

$$= [\textstyle\sum_{k=1}^n I(\tau \ge k)\cdot 1]^{1/2} = (\tau \wedge n)^{1/2}.$$

Thus Theorem 3.1 gives us a means for comparing asymptotically the quantities $P(X_\tau^* > \lambda)$ and $P(\tau^{1/2} > \lambda)$, which we state in following theorem.

THEOREM 3.2. *Let $X_n = x_1 + x_2 + \cdots + x_n$ be a martingale, where the x_k satisfy $E(x_k^2 \mid \mathscr{A}_{k-1}) = 1$ and $E(|x_k| \mid \mathscr{A}_{k-1}) \ge c > 0$. Let τ be a stopping time for X such that for some p, $0 < p < 2$,*

$$P(\tau^{1/2} > \lambda) \approx \lambda^{-p}. \tag{3.3}$$

Then

$$P(\tau^{1/2} > \lambda) \approx P(X_\tau^* > \lambda). \tag{3.4}$$

The proof of this theorem is somewhat similar to that of Theorem 3.3 and is thus omitted.

Interesting examples of stopping times satisfying the conditions of this theorem are certain exit times of random walk at a square root boundary, i.e.,

$$\tau = \inf\{n : |X_n| > cn^{1/2}\},$$

as discussed in [1].

Condition (3.3) implies that $P(\tau^{1/2} > \lambda)$ must behave like λ^{-p} for some p, $0 < p < 2$. It might be more natural to ask what would happen if $P(\tau^{1/2} > \lambda)$ instead behaved like λ^{-p} for $0 < p < \infty$. We will show in the next theorem that if, in addition, we require that the x_i are independent and identically distributed and we place a restriction on the behavior of $P(|x_1| > \lambda)$, then we get a result like that of Theorem 3.2, but for all $p > 0$.

THEOREM 3.3. *Let $X_n = x_1 + x_2 + \cdots + x_n$ where the x_i are independent and identically distributed, $Ex_i = 0$, $Ex_i^2 = 1$. Further let τ be a stopping time for X such that*

$$P(\tau^{1/2} > \lambda) \approx P(\tau^{1/2} > \beta\lambda)$$

for some $\beta > 1$. If in addition for some $p > 0$ and $c_1 > 0$,

$$\lambda^p P(\tau^{1/2} > \lambda) \geq c_1 \tag{3.5}$$

for all sufficiently large λ and

$$\lambda^{p+2} P(|x_1| > \lambda) = O(1) \tag{3.6}$$

as $\lambda \to \infty$, then

$$P(X_\tau^* > \lambda) \approx P(\tau^{1/2} > \lambda).$$

Note that (3.6) holds if the x_i are bounded.

PROOF. First we show that

$$\limsup_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \leq C. \tag{3.7}$$

To prove this we use the following inequality [2].

If f is a discrete martingale with difference sequence d, then

$$P(f^* > \beta\lambda,\quad s(f) \vee d^* \leq \delta\lambda) \leq (\delta^2/(\beta - \delta - 1)^2) P(f^* > \lambda)$$

for $\beta > 1$, $0 < \delta < \beta - 1$, and $\lambda > 0$. Rephrased in terms of our random walk X this becomes

$$P(X_\tau^* > \beta\lambda,\quad \tau^{1/2} \vee x^* \leq \delta\lambda) \leq (\delta^2/(\beta - \delta - 1)^2) P(X_\tau^* > \lambda). \tag{3.8}$$

In order to be able to use the techniques of the proof of Theorem 2.1, we need to eliminate the x^* term.

Proceeding as in part (ii) of the proof of Lemma 2.1, it is easy to show using (3.8) that

$$\limsup_{\lambda\to\infty} \Phi(\lambda) P(X_\tau^* > \lambda) \leq (\gamma\eta/(1 - \gamma\epsilon)) \limsup_{\lambda\to\infty} \Phi(\lambda) P(\tau^{1/2} \vee x^* > \lambda),$$

if Φ satisfies the assumptions (a)–(c) preceding Lemma 2.1 and $\epsilon = \delta^2/(\beta - \delta - 1)^2$. It is therefore only necessary to find conditions on x and τ that will guarantee a constant c for which

$$P(\tau^{1/2} \vee x^* > \lambda) \leq cP(\tau^{1/2} > \lambda). \tag{3.9}$$

Then we will be able to prove (taking $\Phi(\lambda) = 1/P(\tau^{1/2} > \lambda)$) that

$$\limsup_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \leq \gamma\eta c/(1 - \gamma\epsilon) = C,$$

and we will be finished.

Now it is clear that

$$P(\tau^{1/2} \vee x^* > \lambda) \leq P(\tau^{1/2} > \lambda) + P(x^* > \lambda), \tag{3.10}$$

thus we need only examine the quantity $P(x^* > \lambda)$. Rewriting this quantity as a sum we obtain

$$P(x^* > \lambda) = \sum_{n=1}^{[\lambda^2]} P(x_n^* > \lambda, \tau = n) + \sum_{n=[\lambda^2]+1}^{\infty} P(x_n^* > \lambda, \tau = n).$$

However,

$$\sum_{n=[\lambda^2]+1}^{\infty} P(x_n^* > \lambda, \tau = n) \leq \sum_{n>\lambda^2} P(\tau = n) = P(\tau > \lambda^2) = P(\tau^{1/2} > \lambda).$$

Thus our problem is reduced still further and we need only consider how to make

$$\sum_{n\leq[\lambda^2]} P(x_n^* > \lambda, \tau = n) \leq cP(\tau^{1/2} > \lambda). \tag{3.11}$$

Then using the immediately preceding argument and (3.10) we would have

$$P(\tau^{1/2} \vee x^* > \lambda) \leq (2 + c)P(\tau^{1/2} > \lambda)$$

and (3.9) would be verified.

One way of obtaining (3.11) is to write

$$\sum_{n\leq[\lambda^2]} P(x_n^* > \lambda, \tau = n) \leq \sum_{n\leq[\lambda^2]} P(x_{[\lambda^2]}^* > \lambda, \tau = n) \leq P(x_{[\lambda^2]}^* > \lambda).$$

Then, using the fact that the x_i are independent and identically distributed, we get

$$\begin{aligned} P(x_{[\lambda^2]}^* > \lambda) &= 1 - P(x_{[\lambda^2]}^* \leq \lambda) \\ &= 1 - (1 - P(|x_1| > \lambda))^{[\lambda^2]} \\ &= 1 - \exp([\lambda^2]\log(1 - P(|x_1| > \lambda))). \end{aligned}$$

Since we know by (3.6) that

$$\log(1 - P(|x_1| > \lambda)) = -P(|x_1| > \lambda) + o(P(|x_1| > \lambda)) = -O(\lambda^{-p-2}),$$

we have that

$$\begin{aligned} P(x_{[\lambda^2]}^* > \lambda) &= 1 - \exp(-[\lambda^2]O(\lambda^{-p-2})) \\ &= 1 - \exp(-O(\lambda^{-p})) \\ &= O(\lambda^{-p}). \end{aligned}$$

Therefore if we use assumption (3.5) we see that there exists a constant c such that

$$P(x_{[\lambda^2]}^* > \lambda) \leq cP(\tau^{1/2} > \lambda).$$

This verifies (3.11) and finishes the proof of (3.7).

In order to prove Theorem 3.3, we must also show that

$$\liminf_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \geq c > 0. \tag{3.12}$$

We will need a lemma.

LEMMA 3.1. *Suppose that X is a martingale with difference sequence $x = (x_1, x_2, \cdots)$ satisfying*

$$E(x_k^2 \mid \mathscr{A}_{k-1}) = 1 \tag{3.13}$$

and

$$E(|x_k| \mid \mathscr{A}_{k-1}) \geq c_0 > 0 \tag{3.14}$$

for all $k \geq 1$. Let $f = (f_1, f_2, \cdots)$ where $f_n = \sum_{k=1}^n v_k x_k$, $n \geq 1$, and v_k is a real $\mathscr{A}_{k-1}$-measurable function, $k \geq 1$. Choose $\beta > 1$ and $0 < \delta < (\beta^2 - 1)^{1/2}$. Then there exists a constant ϵ such that

$$P(s(f) > \beta\lambda,\ f^* \vee v^* \leq \delta\lambda) \leq \epsilon P(s(f) > \lambda),$$

where ϵ *may be taken as* $c\delta^2/(\beta^2 - \delta^2 - 1)$ *and the choice of c depends only on the constant of* (3.14).

PROOF. We first define the following stopping times:

$$\mu = \inf\{n : s_n(f) > \lambda\}$$
$$\xi = \inf\{n : s_n(f) > \beta\lambda\}$$
$$\sigma = \inf\{n \geq 0 : |f_n| > \delta\lambda \quad \text{or} \quad |v_{n+1}| > \delta\lambda\}.$$

We next define the process h by

$$h_n = \sum_{k=1}^n I(\mu < k \leq \xi \wedge \sigma) v_k x_k = \sum_{k=1}^n w_k x_k,$$

i.e., h is f started at μ and stopped at $\xi \wedge \sigma$. It is clear that h satisfies the conditions of Theorem 2.1 of [6] and that $w^* \leq \delta\lambda$. Letting

$$\tau = \inf\{n : |h_n| > 2\delta\lambda\}$$

and applying Theorem 2.1 of [6] to h, we see that

$$\| h_\tau^* \|_2^2 \leq c\delta^2\lambda^2 P(w^* > 0) \leq c\delta^2\lambda^2 P(\mu < \infty) = c\delta^2\lambda^2 P(s(f) > \lambda). \tag{3.15}$$

Now suppose that $s(f) > \beta\lambda$ and $f^* \vee v^* \leq \delta\lambda$. Then, since $f^* \vee v^* \leq \delta\lambda$ implies that $\sigma = \infty$, we have

$$\begin{aligned} s^2(h) &= \sum_{k=1}^\infty I(\mu < k \leq \xi \vee \sigma) v_k^2 \\ &= \sum_{k=1}^\infty I(k \leq \xi \vee \sigma) v_k^2 - \sum_{k=1}^\infty I(\mu \geq k) v_k^2 \\ &= \sum_{k=1}^\infty I(k \leq \xi) v_k^2 - \sum_{k=1}^\infty I(\mu \geq k) v_k^2. \end{aligned}$$

We note that

$$\sum_{k=1}^\infty I(\mu \geq k) v_k^2 = s_\mu^2(f) \leq \lambda^2 + \delta^2\lambda^2$$

and

$$\sum_{k=1}^\infty I(k \leq \xi) v_k^2 = s_\xi^2(f) \geq \beta^2\lambda^2.$$

Thus

$$s^2(h) \geq \lambda^2(\beta^2 - \delta^2 - 1).$$

Also since $\sigma = \infty$ we know that $f^* \leq \delta\lambda$, and so

$$\begin{aligned} |h_n| &= |\sum_{k=1}^n I(\mu < k \leq \xi) v_k x_k| \\ &= |f_{\xi \wedge n} - f_{\mu \wedge n}| \\ &\leq 2f^* \leq 2\delta\lambda \end{aligned}$$

for all n, which implies that $\tau = \infty$. This shows that

$$P(s(f) > \beta\lambda, f^* \vee v^* \leq \delta\lambda) \leq P(s^2(h) \geq (\beta^2 - \delta^2 - 1)\lambda^2, \tau = \infty). \tag{3.16}$$

Furthermore, by Chebyshev's inequality,

$$\begin{aligned} P(s^2(h) \geq (\beta^2 - \delta^2 - 1)\lambda^2, \tau = \infty) &\leq P(s_\tau^2(h) \geq (\beta^2 - \delta^2 - 1)\lambda^2) \\ &\leq \| s_\tau(h) \|_2^2 / ((\beta^2 - \delta^2 - 1)\lambda^2). \end{aligned} \tag{3.17}$$

However, we know by (1.1) and (3.15) that

$$\| s_\tau(h) \|_2^2 = \| h^\tau \|_2^2 \leq \| h_\tau^* \|_2^2 \leq c\delta^2\lambda^2 P(s(f) > \lambda). \tag{3.18}$$

Thus, putting together (3.16), (3.17), and (3.18), we have shown that

$$\begin{aligned} P(s(f) > \beta\lambda, f^* \vee v^* \le \delta\lambda) & \\ &\le c\delta^2\lambda^2 P(s(f) > \lambda)/((\beta^2 - \delta^2 - 1)\lambda^2) \\ &= (c\delta^2/(\beta^2 - \delta^2 - 1))P(s(f) > \lambda), \end{aligned}$$

which is the desired result.

We now return to the proof of (3.12). Given our stopping time τ for X, we let $v_k = I(\tau \ge k)$. Then $v^* \le 1$, and so for large enough values of λ (greater than δ^{-1}) we use Lemma 3.1 to show

$$P(s(f) > \beta\lambda, f^* \vee v^* \le \delta\lambda) = P(s(f) > \beta\lambda, f^* \le \delta\lambda) \le \epsilon_2 P(s(f) > \lambda),$$

where $\epsilon^2 = c\delta^2/(\beta^2 - \delta^2 - 1)$. Putting this in terms of X, this means that for large enough λ,

$$P(\tau^{1/2} > \beta\lambda,\ X_\tau^* \le \delta\lambda) \le \epsilon_2 P(\tau^{1/2} > \lambda).$$

We notice immediately that this corresponds to (2.5). We also see that (3.3) implies that for large enough λ,

$$k\lambda^{-p} \le P(\tau^{1/2} > \lambda) \le K\lambda^{-p}.$$

Thus for large enough λ,

$$0 < P(\tau^{1/2} > \lambda) \le (K\beta^p/k)P(\tau^{1/2} > \beta\lambda).$$

This corresponds to (2.3) of Theorem 2.1, with $\gamma = (K\beta^p/k)$ and so we can proceed as in the second part of the proof of that theorem to get the result

$$\liminf_{\lambda\to\infty} P(X_\tau^* > \lambda)/P(\tau^{1/2} > \lambda) \ge c_1 > 0.$$

Again we need to choose ϵ_2 small enough so that $\gamma\epsilon_2 < 1$, but it is clear that this can be done by choosing δ small. This finishes the proof of Theorem 3.3.

We might ask whether the condition $Ex_1^2 = 1$ can be removed. Of course, if we took away this assumption then we could no longer use the fact that $s(f) = \tau^{1/2}$. Examples can in fact be produced in which Ex_1^2 is not finite and the result (3.4) no longer holds (see [10]).

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