The Annals of Probability
1981, Vol. 9, No. 1, 162–166

MULTIDIMENSIONAL IFRA PROCESSES[1]

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Two types of multidimensional processes are defined. The first of these generalizes a univariate IFRA process due to Ross and relates to a multivariate concept of IFRA due to Esary and Marshall. The second of these relates to a multivariate concept of IFRA due to the present authors.

0. Introduction. Ross [4] has defined a univariate nonincreasing process to be IFRA (increasing failure rate average) if certain lifetimes associated with the process are IFRA. See Barlow and Proschan [1] for a discussion of IFRA lifetimes. Extensions of IFRA to multivariate lifetimes have been proposed by Block and Savits [2] and Esary and Marshall [3]. In this paper the univariate concept of Ross is extended to multidimensional processes and related to IFRA multivariate lifetimes.

In Section 1 a characterization of univariate IFRA processes is given. The Ross concept of IFRA processes is extended to vector processes and an alternate form is derived. A closure theorem and various properties are established for these processes. It is shown in Theorem 2.4 of Section 2 that lifetimes associated with these processes satisfy the condition that any monotone system formed with these lifetimes is IFRA in the univariate sense. Furthermore this property characterizes such processes. This property, called Condition B in Esary and Marshall [3], was one of the definitions of multivariate IFRA discussed by those authors. Another type of multidimensional IFRA process is defined. For this process, the associated lifetimes satsify the MIFRA property of Block and Savits [2].

1. IFRA processes and the IFRA closure theorem. Let $X(t)$ be a nonnegative, nonincreasing right-continuous random process. According to Ross [4], the process $X(t)$ is called an IFRA *process* if and only if the random variable

$$T_a = \inf\{t \geq 0 : X(t) \leq a\} \tag{1.1}$$

is IFRA for every $a \geq 0$. Equivalently, we have the alternate characterization below.

(1.2) THEOREM. *$X(t)$ is an* IFRA *process if and only if*

$$E[h(X(t))] \leq E^{1/\alpha}[h^\alpha(X(\alpha t))] \tag{1.3}$$

for all nonnegative nondecreasing functions h and all $0 < \alpha \leq 1$, $t \geq 0$.

PROOF. First assume that $X(t)$ is an IFRA process and consider h of the form $h(x) = I_{(a,\infty)}(x)$, $a \geq 0$. Since, by right-continuity, $X(t) > a$ if and only if $T_a > t$, we have

$$E[h(X(t))] = P(X(t) > a) = P(T_a > t)$$
$$\leq P^{1/\alpha}(T_a > \alpha t) = P^{1/\alpha}(X(\alpha t) > a) = E^{1/\alpha}[h^\alpha(X(\alpha t))]$$

for all $0 < \alpha \leq 1$, $t \geq 0$. Now consider h of the form $h(x) = I_{[a,\infty)}(x)$, $a > 0$ (the case $a = 0$ is clear). Since $I_{(a-1/n,\infty)}(x) \downarrow h(x)$, the inequality (1.3) is also valid for such h. The general

Received August 27, 1979.

[1] Research has been supported by ONR Contract N00014-76-C-0839 and by NSF Grant MCS77-01458.

AMS 1970 *subject classifications.* Primary 60K10; secondary 62N05.

Key words and phrases. Multivariate IFRA distributions, multidimensional IFRA processes.

result now follows by taking nonnegative linear combinations of such functions and passing to the limit as in Block and Savits [2].

Conversely, if (1.3) is true, then (1.1) follows by taking $h(x) = I_{(a,\infty)}(x)$.

Ross [4] proved the IFRA closure theorem under the assumption of independent components. We obtain the same results without the assumption of independence. First, however, we need some definitions.

(1.4) DEFINITION. An *upper set* $U \subset \mathbb{R}^n$ is a subset having the property that if $\mathbf{x} \in U$ and $\mathbf{y} \geq \mathbf{x}$, then $\mathbf{y} \in U$. If in addition U is an open subset, we call U an *upper domain.*

Now let $\mathbf{X}(t) = (X_1(t), \cdots, X_n(t))$ be a vector-valued stochastic process. We assume that $\mathbf{X}(t)$ is nonnegative, nonincreasing and right-continuous.

(1.5) DEFINITION. $\mathbf{X}(t)$ is said to be a (*vector-valued*) IFRA *process* if and only if for every upper domain U, the random variable

$$T_U = \inf\{t \geq 0 : \mathbf{X}(t) \notin U\}$$

is IFRA.

Clearly this includes the IFRA class considered by Ross [4] in the case $n = 1$. Again, as in (1.2), we have the alternate characterization given below.

(1.6) THEOREM. *$\mathbf{X}(t)$ is a (vector-valued)* IFRA *process if and only if*

$$E[h(\mathbf{X}(t))] \leq E^{1/\alpha}[h^{\alpha}(\mathbf{X}(\alpha t))] \tag{1.7}$$

for all Borel measurable nonnegative nondecreasing functions h and all $0 < \alpha \leq 1$, $t \geq 0$.

PROOF. The proof is very similar to (1.2): first show that (1.7) is true if $h(\mathbf{x}) = I_U(\mathbf{x})$ for U an upper domain and then use the argument in Block and Savits [2] for general h.

(1.8) REMARK. If (1.7) is valid for the subclass of continuous nonnegative nondecreasing functions h, then it is necessarily valid for all Borel measurable nonnegative nondecreasing functions h (cf., Block and Savits [2]).

The next theorem describes some properties of the class of IFRA processes. We henceforth dispense with the adjective vector-valued.

(1.9) THEOREM.

(i) *If $\mathbf{X}(t)$ is an* IFRA *process and $\psi_1, \cdots, \psi_k$ are left-continuous nonnegative nondecreasing functions, then $(\psi_1(\mathbf{X}(t)), \cdots, \psi_k(\mathbf{X}(t)))$ is an* IFRA *process.*

(ii) *If $(X_1(t), \cdots, X_n(t))$ and $(Y_1(t), \cdots, Y_m(t))$ are* IFRA *processes which are independent at each time t, then $(X_1(t), \cdots, X_n(t), Y_1(t), \cdots, Y_m(t))$ is an* IFRA *process.*

(iii) *If $\mathbf{X}_n(t)$, $n = 1, 2, \cdots$, are* IFRA *processes and $\mathbf{X}_n(t) \to \mathbf{X}(t)$ weakly for each t, then $\mathbf{X}(t)$ is an* IFRA *process provided it is also nonnegative, nondecreasing and right-continuous.*

PROOF. The proofs are clear and are left to the reader.

We say Φ is a multistate monotone structure function of n-components if $\Phi(\mathbf{x}) = \Phi(x_1, \cdots, x_n)$ is nonnegative and nondecreasing in each argument.

(1.10) COROLLARY.

(i) (IFRA closure theorem). *If Φ is a left-continuous multistate monotone structure function and $\mathbf{X}(t)$ is an* IFRA *process, then $\Phi(\mathbf{X}(t))$ is an* IFRA *process.*

(ii) (Convolution theorem). *If* $(X_1(t), \cdots, X_n(t))$ *and* $(Y_1(t), \cdots, Y_n(t))$ *are* IFRA *processes which are independent at each time t, then* $(X_1(t) + Y_1(t), \cdots, X_n(t) + Y_n(t))$ *is an* IFRA *process.*

(iii) *If* $(X_1(t), \cdots, X_n(t))$ *is an* IFRA *process and* $J \subset \{1, \cdots, n\}$, *then* $(X_j(t): j \in J)$ *is an* IFRA *process.*

2. IFRA processes and multivariate IFRA concepts. Let T be a nonnegative random variable and $X(t)$ its indicator process, i.e., $X(t) = I_{(t,\infty)}(T)$. Then clearly $X(t)$ is an IFRA process if and only if T is an IFRA random variable since

$$\begin{aligned} T_a = \inf\{t \geq 0 : X(t) \leq a\} &= 0 && \text{if } a \geq 1 \\ &= T && \text{if } 0 \leq a < 1 \\ &= +\infty && \text{if } a < 0. \end{aligned}$$

Now let $(T_1, \cdots, T_n)$ be a nonnegative random vector. If we assume that $(T_1, \cdots, T_n)$ is MIFRA in the sense of Block and Savits [2], then there are many ways of constructing IFRA processes. For example, suppose that $\phi(t; x_1, \cdots, x_n)$, $t, x_1, \cdots, x_n \geq 0$, is nonnegative, Borel measurable and nondecreasing in $\mathbf{x}$ for fixed t, right-continuous and nonincreasing in t for fixed $\mathbf{x}$, and satisfies

$$\phi(t; x_1/\alpha, \cdots, x_n/\alpha) \leq \phi(\alpha t; x_1, \cdots, x_n)$$

for all $0 < \alpha \leq 1$, $t \geq 0$, $\mathbf{x} \in \mathbb{R}_+^n$. Then $X(t) = \phi(t; T_1, \cdots, T_n)$ is an IFRA process. Indeed, let h be any Borel measurable nonnegative nondecreasing function. Then

$$\begin{aligned} E[h(X(t))] = E[h(\phi(t; T_1, \cdots, T_n))] &\leq E^{1/\alpha}[h^\alpha(\phi(t; T_1/\alpha, \cdots, T_n/\alpha))] \\ &\leq E^{1/\alpha}[h^\alpha(\phi(\alpha t; T_1, \cdots, T_n))] = E^{1/\alpha}[h^\alpha(X(\alpha t))]. \end{aligned}$$

In particular, if $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$ is a reordering of $x_1, \cdots, x_n$, then

$$\begin{aligned} \phi(t; x_1, \cdots, x_n) &= n && \text{if } 0 \leq t < x_{1:n} \\ &= n - k && \text{if } x_{k:n} \leq t < x_{k+1:n},\ k = 1, \cdots, n-1 \\ &= 0 && \text{if } t \geq x_{n:n} \end{aligned}$$

has the desired properties.

(2.1) EXAMPLE. Let (S, T) be MIFRA and set

$$\begin{aligned} X(t) &= 2 && \text{if } 0 \leq t < \min(S, T) \\ &= 1 && \text{if } \min(S, T) \leq t < \max(S, T) \\ &= 0 && \text{if } t \geq \max(S, T). \end{aligned}$$

Then $X(t)$ is an IFRA process.

(2.2) EXAMPLE. Let (S, T) have the distribution with joint density

$$\begin{aligned} f(s, t) &= 30 && \text{if } 3/8 < s < 1/2,\ 3/4 < t < 1 \\ &= 1 && \text{if } 1/8 < s < 3/8,\ 1/2 < t < 2/3 \\ &= 2 && \text{if } 0 < s < 1/8,\ 2/3 < t < 3/4, \end{aligned}$$

Then S and T are IFRA and $S \leq T$ with probability one. Consequently,

$$\begin{aligned} X(t) &= 2 && \text{if } 0 \leq t < S \\ &= 1 && \text{if } S \leq t < T \\ &= 0 && \text{if } T \leq t \end{aligned}$$

is an IFRA process. But (S, T) is not MIFRA since $P(S > t, T > 4t)$ has support that is not an interval and we know that if (S, T) was MIFRA, then $\min(S, T/4)$ would be IFRA.

Recall that from Esary and Marshall [3], a nonnegative random vector $(T_1, \cdots, T_n)$ satisfies condition B if and only if $\tau(T_1, \cdots, T_n)$ is IFRA for every life function τ corresponding to a monotone binary structure function ϕ. This condition can be characterized in terms of IFRA processes as follows.

(2.3) THEOREM. *Let* $\mathbf{T} = (T_1, \cdots, T_n)$ *be a nonnegative random vector. Then* $\mathbf{T}$ *satisfies condition* B *if and only if the indicator process* $\mathbf{X}(t) = (X_1(t), \cdots, X_n(t))$, *where* $X_i(t) = I_{(t,\infty)}(T_i)$, *is an* IFRA *process.*

PROOF. Suppose that the indicator process $\mathbf{X}(t)$ is an IFRA process. If ϕ is a binary monotone structure function with corresponding life function τ, then

$$\begin{aligned} P(\tau > t) &= E[\phi(X_1(t), \cdots, X_n(t))] \\ &\le E^{1/\alpha}[\phi^\alpha(\mathbf{X}(\alpha t))] = P^{1/\alpha}(\tau > \alpha t) \end{aligned}$$

and so τ is IFRA.

Now suppose that $\mathbf{T} = (T_1, \cdots, T_n)$ satisfies condition B and let U be any upper domain in $\mathbb{R}^n$. If $\mathbf{x} = (x_1, \cdots, x_n)$ is any binary vector of ones and zeros, set

$$\begin{aligned} \phi_U(\mathbf{x}) &= 1 \quad \text{if} \quad \mathbf{x} \in U \\ &= 0 \quad \text{if} \quad \text{otherwise.} \end{aligned}$$

Then ϕ_U is a binary monotone structure function. Furthermore, if τ_U is its corresponding life function and $T_U = \inf\{t : \mathbf{X}(t) \notin U\}$, then $T_U = \tau_U$. But by assumption, τ_U is IFRA and so T_U is also IFRA. Consequently, $\mathbf{X}(t)$ is an IFRA process.

Theorem (2.3) extends to the general case as follows. Let $\mathbf{X}(t)$ be an IFRA process and let U be an upper domain in $\mathbb{R}_n$. Then, according to Block and Savits [2], there exist fundamental upper domains U_l such that $\cup_{l=1}^{\infty} U_l = U$ and $T_{U_l} \uparrow T_U$. Since the IFRA class is closed under weak limits, it suffices to show that T_U is IFRA for every fundamental upper domain U. But by definition, U is a fundamental upper domain if and only if $U = \cup_{l=1}^{p} U_l$, where $U_l = \{\mathbf{x} \in \mathbb{R}^n : x_i > z_{il}\}$ and x_{il} are real numbers for $1 \le i \le n$, $1 \le l \le p$. Clearly $T_U = \max_{1 \le l \le p} T_{U_l}$, and if we set $T_{iz} = \inf\{t \ge 0 : X_i(t) \le z_{il}\}$, then

$$T_U = \max_{1 \le l \le p} \min_{1 \le i \le n} T_{iz_{il}}.$$

Consequently we may state the following result.

(2.4) THEOREM. $\mathbf{X}(t)$ *is an* IFRA *process if and only if every finite collection of* $\{T_{iz} : 1 \le i \le n, z \in \mathbb{R}\}$ *satisfies condition* B *of Esary and Marshall.*

(2.5) COROLLARY. *In the finite state case,* i.e., $X_i(t) \in \{0, 1, \cdots, M\}$ *for all* $t \ge 0$, $i = 1, \cdots, n$, $\mathbf{X}(t)$ *is an* IFRA *process if and only if* $\{T_{ij} : 1 \le i \le n, 0 \le j < M\}$ *satisfies the condition* B, *where*

$$T_{ij} = \inf\{t \ge 0 : X_i(t) \le j\}.$$

Clearly, in the finite state case, if the finite collection $\{T_{ij} : 1 \le i \le n, 0 \le j < M\}$ are MIFRA, then they satisfy condition B. This leads to the following definition.

(2.6) DEFINITION. Let $\mathbf{X}(t)$ be a nonnegative nondecreasing right-continuous process. Then we say that $\mathbf{X}(t)$ is an MIFRA *process* if and only if for every finite collection $U_1, \cdots, U_m$ of upper domains in $\mathbb{R}^n$, the random vector $(T_{U_1}, \cdots, T_{U_m})$ is MIFRA.

As Example (2.2) shows, there exist IFRA processes which are not MIFRA. The analogous result to Theorem (2.4) is stated below.

(2.7) THEOREM. $\mathbf{X}(t)$ *is a* MIFRA *process if and only if every finite collection of* $\{T_{iz}: 1 \le i \le n, z \in \mathbb{R}\}$ *is* MIFRA.

(2.8) REMARK. Note that for IFRA processes, the upper domains are defined with respect to the state space, whereas for MIFRA vectors, the upper domains are defined with respect to the time space.

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