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LIMITING BEHAVIOR OF A PROCESS OF RUNS[1]

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Let $X_1, X_2, \cdots$ be independent identically distributed (i.i.d.) random variables with a continuous distribution function. Let $R_0 = 0$, $R_k = \min\{j: j > R_{k-1} \text{ and } X_j > X_{j+1}\}$ and $T_k = R_k - R_{k-1}$, $k \geq 1$. We prove that a process $T^{(n)} = \{T_{k+n}\}_{k=1}^{\infty}$ converges, in the sense of distribution functions, exponentially fast to a strongly mixing ergodic process. It is shown that $(\max_{1\leq k\leq n} T_k)/\log n(\log\log n)^{-1} \to 1$ almost surely and in L_p, $p > 0$. Also, the number of runs T_k, $1 \leq k \leq n$, larger than or equal to some m is proven to be Poisson distributed in the limit, if $n/m!$ converges to a positive number.

1. Introduction. Let $\{X_k\}_{k=0}^{\infty}$ be a sequence of i.i.d. random variables with a continuous common distribution F. Consider events $A_k = \{X_k > X_{k+1}\}$ and their set indicators $U_k = \chi_{A_k}$, $k = 0, 1, 2, \cdots$. A sequence $\{U_k\}_{k=0}^{\infty}$ is a stationary process, whose distribution does not depend on F. Moreover, this process is ergodic, as its every tail event is a tail event of $X_0, X_1, \cdots$, and hence, by Kolmogorov's zero-one law, has probability zero or one.

Let $R_1 = \min\{n: U_n = 1, n > 0\}$, $R_k = \min\{n: U_n = 1, n > R_{k-1}\}$, $k \geq 2$, and let $T_1 = R_1$, $T_k = R_k - R_{k-1}$, $k \geq 2$. Following an accepted terminology, R_k's are called the occurrence times, and T_k's are called the recurrence times, of the events A_s. Contrary to intuition, $\{T_k\}_{k=2}^{\infty}$ is not a stationary process. We shall prove, however, that all distribution functions of $\{T_{k+n}\}_{k=1}^{\infty}$ converge, as n $\to +\infty$, exponentially fast to those of a stationary ergodic process $\{\tilde{T}_k\}_{k=1}^{\infty}$. Its distribution is the distribution of the original process $\{T_k\}_{k=1}^{\infty}$ *conditioned* on the event $\{U_0 = 1\}$.

In literature, the process $\{T_k\}_{k=1}^{\infty}$ is known as the runs up process, and was probably first studied by MacMahon (1908), (1915), who obtained a well-known determinantal formula for its distribution functions. Many relevant results and references can be found in Wolfowitz (1944), David and Barton (1962), Barton and Mallows (1965), Knuth (1973).

In Pittel (1980), it was proved that a process $W^{(n)}(t) = (R_{[nt]} - 2[nt])/\sqrt{(2/3)n}$, $t \in [0, 1]$, converges, in the sense of distribution functions, to the standard Brownian motion. In the present paper, we study the limiting behavior of the recurrence times T_k, $\tilde{T}_k$. We show that, for large n, the most probable values m of $M_n = \max_{1\leq k\leq n} T_k$ are such that $m!$ and n are of the same order. More precisely, $\lim_{n\to\infty} M_n/\log n(\log\log n)^{-1} = 1$ almost surely and in L_p, $p > 0$. Also, let $\lim_{n\to\infty} (2n)(m!)^{-1} = \lambda$ and let V_k be the number of the recurrence times T_j, $j \leq k$, larger than or equal to m. We prove that a process $V^{(n)}(t) = V_{[nt]}$, $t \in [0, 1]$, is asymptotically Poisson distributed with parameter λ. This last statement is similar, in essence, to results of Wolfowitz (1944) and David and Barton (1962) concerning runs generated by a finite sequence $X_1, \cdots, X_n$.

2. Preliminaries. Let $l_1, \cdots, l_k$ be positive integers, $k \geq 2$, and $L_s = \sum_{j=1}^{s} l_s$, $1 \leq s \leq k$. Denote $l^{(k)} = (l_1, \cdots, l_k)$, $Q(l^{(k)}) = P(T_j = l_j, 1 \leq j \leq k-1, T_k \geq l_k)$. Since $X_1, X_2, \cdots$ are i.i.d. with a continuous distribution function, we have

$$Q(l^{(k)}) = P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1}) \cdot P(X_{L_{k-1}+1} \leq \cdots \leq X_{L_{k-1}+l_k})$$
$$- P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + l_k),$$

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or

$$\begin{aligned} Q(l^{(k)}) &= Q(l^{(k-1)})/l_k! - Q(l_+^{(k-1)}), \\ l^{(k-1)} &= (l_1, \cdots, l_{k-1}) \\ l_+^{(k-1)} &= (l_1, \cdots, l_{k-2}, l_{k-1} + l_k). \end{aligned} \tag{2.1}$$

Together with an initial condition $Q(l^{(1)}) = P(T_1 \geq l_1) = 1/l_1!$, this relation leads to an explicit formula for $Q(l^{(k)})$, due to MacMahon (1908). Namely, if $\mathscr{P}_k$ is the set of all partitions $p = (I_1, \cdots, I_r)$ of the set $(1, \cdots, k)$ into consecutive "intervals" $I_1 = (1, \cdots, t_1)$, $I_2 = (t_1 + 1, \cdots, t_1 + t_2)$, $\cdots$, $I_r = (t_1 + \cdots + t_{r-1} + 1, \cdots, t_1 + \cdots + t_{r-1} + t_r)$, $(t_1 + \cdots + t_r = k)$, then

$$Q(l^{(k)}) = \sum_{p \in \mathscr{P}_k} \prod_{I \in p} (-1)^{|I|+1}/l(I)!, \tag{2.2}$$

where

$$l(I) = \sum_{i \in I} l_i.$$

LEMMA 1. *Let $\mathscr{F}_k(T)$ be the σ-field generated by $T_1, \cdots, T_k$, $\mathscr{F}_0(T) = (\Omega, \varnothing)$. Then, for all positive l, k,*

$$1/l! \leq P(T_k \geq l \mid \mathscr{F}_{k-1}(T)) \leq 1/(l-1)!. \tag{2.3}$$

PROOF. The statement is obvious for $k = 1$. Let $k \geq 2$ and let $l_1, \cdots, l_{k-1}$ be positive integers. Then

$$\begin{aligned} P(T_j = l_j, 1 \leq j \leq k-1; T_k \geq l) &\leq P(T_j = l_j, 1 \leq j \leq k-1; \\ X_{L_{k-1}+2} \leq \cdots \leq X_{L_{k-1}+l}) &\leq P(T_j = l_j, 1 \leq j \leq k-1)/(l-1)!, \end{aligned}$$

which proves the right-hand side estimate in (2.3).

Further, by (2.1), we have

$$\begin{aligned} &P(T_j = l_j, 1 \leq j \leq k-1; T_k \geq l) \\ &\quad = P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1})/l! \\ &\qquad - P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + l) \\ &\quad = P(T_j = l_j, 1 \leq j \leq k-1)/l! + P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + 1)/l! \\ &\qquad - P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + l). \end{aligned} \tag{2.4}$$

But, again by (2.1),

$$\begin{aligned} &P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + 1)/l! \\ &\quad - P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + l) \\ &= P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + 1)/l! \\ &\quad - P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + 1 + l) \\ &\quad + P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + 1 + l) \\ &\quad - P(T_j = l_j, 1 \leq j \leq k-2; T_{k-1} \geq l_{k-1} + l) \\ &= P(A_1) - P(A_2), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} A_1 &= \{T_j = l_j, 1 \leq j \leq k-2; T_{k-1} = l_{k-1} + 1; T_k \geq l\}, \\ A_2 &= \{T_j = l_j, 1 \leq j \leq k-2; T_{k-1} = l_{k-1} + l\}. \end{aligned}$$

Assuming, without loss of generality, that $X_1, X_2, \cdots$ are uniformly distributed on $[0, 1]$, we get after some simple integrations, $(L_0 = 0)$, that

$$P(A_i \mid \mathscr{F}(X_1, \cdots, X_{L_{k-2}+1})) = f_i(X_{L_{k-2}+1})X_A, \qquad i = 1, 2,$$

where

$$f_1(x) = P(x \le X_1 \le \cdots \le X_{l_{k-1}} > X_{l_{k-1}+1} \le \cdots \le X_{l_{k-1}+l}) = \frac{(1-x)^{l_{k-1}}}{l_{k-1}!\,l!} - \frac{(1-x)^{l_{k-1}+l}}{(l_{k-1}+l)!},$$

$$f_2(x) = P(x \le X_1 \le \cdots \le X_{l_{k-1}+l-1} > X_{l_{k-1}+l}) = \frac{(1-x)^{l_{k-1}+l-1}}{(l_{k-1}+l-1)!} - \frac{(1-x)^{l_{k-1}+l}}{(l_{k-1}+l)!}.$$

Furthermore,

$$f_1(x) - f_2(x) = (1-x)^{l_{k-1}}\left[\frac{1}{l_{k-1}!\,l!} - \frac{(1-x)^{l-1}}{(l_{k-1}+l-1)!}\right]$$

$$\ge (1-x)^{l_{k-1}}\left[\binom{l_{k-1}+l}{l} - (l_{k-1}+l)\right] \Big/ (l_{k-1}+l)! \ge 0, \qquad \forall\, x \in [0, 1].$$

Hence,

$$P(A_1) - P(A_2) = E[(f_1(X_{L_{k-2}+1}) - f_2(X_{L_{k-2}+1}))X_A] \ge 0.$$

So, by (2.4), (2.5),

$$P(T_j = l_j,\ 1 \le j \le k-1;\ T_k \ge l) \ge P(T_j = l_j,\ 1 \le j \le k-1)/l!,$$

which proves the left-hand side estimate in (2.3).

REMARK. Neither of the estimates in (2.3) can be improved, since $P(T_1 \ge l) = 1/l!$ and

$$P(T_k \ge l \mid T_1 = \cdots = T_{k-1} = 1) = \left(\frac{k}{k+l-1}\right) \Big/ (l-1)! \to 1/(l-1)!,$$

as $k \to \infty$. The last fact is really disappointing, because on the other hand $P(T_k \ge l \mid \mathscr{F}_{k-1}(T)) \le 2/l!$, whenever $T_{k-1} \ge 2$.

COROLLARY 1. *Given two finite disjoint subsets A, B of the set of natural numbers, we have*

$$\prod_{a\in A} 1/l_a! \times \prod_{b\in B} (1 - 1/(l_b - 1)!) \le P(T_a \ge l_a,\ a \in A;\ T_b < l_b,\ b \in B) \le \prod_{a\in A} 1/(l_a - 1)! \times \prod_{b\in B} (1 - 1/l_b!), \tag{2.6}$$

where $l_i = 1, 2, \cdots, i \in A \cup B$. (Thus, the behavior of $T_1, T_2, \cdots$ is somewhat close to the one of i.i.d. *random variables $t_1, t_2, \cdots$, having $P(t_i \ge l) = 1/l! = P(T_1 \ge l)$).*

3. Convergence of the process $T^{(n)} = \{T_{k+n}\}_{k=1}^{\infty}$. Since the stationary process $\{U_k\}_{k=0}^{\infty}$ is ergodic, then the process $\{T_k\}_{k=1}^{\infty}$ considered on the sample space $\Omega_0 = \{\omega \in \Omega;\ U_0 = 1\}$ is stationary and ergodic under the probability $P(\cdot \mid U_0 = 1)$, (Breiman (1968)). Denoting this process by $\tilde{T} = \{\tilde{T}_k\}_{k=1}^{\infty}$, we have then: for $B \in \mathscr{B}(N^{(\infty)})$,

$$P(\{\tilde{T}_k\}_{k=1}^{\infty} \in B) = P(\{T_k\}_{k=1}^{\infty} \in B,\ U_0 = 1)\cdot P^{-1}(U_0 = 1) = 2P(\{T_k\}_{k=1}^{\infty} \in B,\ U_0 = 1), \tag{3.1}$$

as $P(U_0 = 1) = P(X_0 > X_1) = ½$. In particular, for positive integers $m, l_1, \cdots, l_m$,

$$P(\tilde{T}_s = l_s,\ 1 \le s \le m-1;\ \tilde{T}_m \ge l_m) = 2P(X_0 > X_1;\ T_s = l_s,\ 1 \le s \le m-1;\ T_m \ge l_m) \tag{3.2}$$

$$= 2P(T_1 = 1;\ T_s = l_{s-1},\ 2 \le s \le m;\ T_{m+1} \ge l_m)$$
$$= 2Q(\tilde{l}^{(m+1)}),$$

where $\tilde{l}^{(m+1)} = (1, l_1, \cdots, l_m) = (1, l^{(m)})$. Moreover, given $m \ge 1$, $B \in \mathscr{B}(N^{(m)})$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P((T_{1+j}, \cdots, T_{m+j}) \in B) = P((\tilde{T}_1, \cdots, \tilde{T}_m) \in B), \tag{3.3}$$

(Breiman (1968)).

Using some special properties of $\{T_k\}_{k=1}^{\infty}$, we prove a stronger result.

NOTATION. Given a sequence $\{s_k\}_{k=1}^{\infty}$ and numbers $a \le b$, denote $s(a, b) = (s_a, s_{a+1}, \cdots, s_b)$.

THEOREM 1. (a) *Given* $m \ge 1$, *we have*

$$P(T(1+n, m+n) \in B) = P(\tilde{T}(1, m) \in B) + O(q^n) \qquad n \to \infty \tag{3.4}$$

uniformly over $B \in \mathscr{B}(N^{(m)})$, *for some* $q \in (0, 1)$.

(b) *More generally, given* $k \ge 1$, $m_1 \ge 1, \cdots, m_k \ge 1$, *we have*

$$P(T(\textstyle\sum_{j=0}^{s-1} (m_j + n_j) + 1, \sum_{j=0}^{s-1} (m_j + n_j) + m_s) \in B_s,\ 1 \le s \le k) \tag{3.5}$$
$$= P(T(1, m_1) \in B_1) \cdot \textstyle\prod_{s=2}^{k} P(\tilde{T}(1, m_s) \in B_s) + O(q^n), \quad n = \min_{1\le s\le k} n_s \to \infty,$$

uniformly over $B_s \in \mathscr{B}(N^{(m_s)})$, $1 \le s \le k$. ($m_0 = n_0 = 0$).

From (3.1), (3.2), and (3.5) follows

COROLLARY 2. *Part* (b) *is valid, if* T *on both sides of* (3.5) *is replaced by* $\tilde{T}$.

REMARKS. (1) From (3.4) follows that

$$\lim_{n\to\infty} P(T_n \ge l) = P(\tilde{T}_1 \ge l) = 2((l!)^{-1} - ((l+1)!)^{-1}),$$

which is a known result. Barton and Mallows (1965) found the generating function of the sequence $\{ET_n\}_{n=1}^{\infty}$, and it was later used (Hooker (1969), Knuth (1973)) to show that $E(T_n) = 2 + O(\rho^n)$, where

$$\rho = \min\{|z|^{-1}: 1 - ze^{1-z} = 0, |z| > 1\} < 0.125.$$

(2) From the proof of the theorem, we shall see that relations (3.4), (3.5) are valid whenever $q \in (\rho, 1)$.

(3) Corollary 2 implies that the process $\{\tilde{T}_k\}_{k=1}^{\infty}$ is not only ergodic, but, moreover, has a strong mixing property, (Billingsley (1965)).

PROOF OF THEOREM 1. It suffices to prove (3.5). To avoid too complicated notations, we confine ourselves to the case $k = 2$.

LEMMA 2. *Let* $\sigma^{(m_1)} = (\sigma_1, \cdots, \sigma_{m_1}) \in C^{m_1}$, $\tau^{(m_2)} = (\tau_1, \cdots, \tau_{m_2}) \in C^{m_2}$. *Then the generating functions* $f_{m_1}(\cdot)$, $\tilde{f}_{m_2}(\cdot)$, $f_{m_1 m_2}(\cdot, \cdot)$ *defined by*

$$f_{m_1}(\sigma^{(m_1)}) = E(\textstyle\prod_{r=1}^{m_1} \sigma_r^{T_r}),$$
$$\tilde{f}_{m_2}(\tau^{(m_2)}) = \textstyle\sum_{l^{(m_2)}>0} \prod_{s=1}^{m_2} \tau_s^{l_s} P(\tilde{T}(1, m_2 - 1)$$
$$= l^{(m_2-1)},\ \tilde{T}_{m_2} \ge l_{m_2}),$$

$$f_{m_1 m_2}^{(h)}(\sigma^{(m_1)}, \tau^{(m_2)}) = \textstyle\sum_{k^{(m_1)}>0, l^{(m_2)}>0} \prod_{r=1}^{m_1} \sigma_r^{k_r} \prod_{s=1}^{m_2} \tau_s^{l_s} P(T(1, m_1) \tag{3.6}$$
$$= k^{(m_1)};\ T(m_1 + h + 1, m_1 + h + m_2 - 1)$$

$$= l^{(m_2-1)},\ T_{m_1+h+m_2} \ge l_{m_2}),$$
$$(l^{(m_2-1)} = (l_1, \cdots, l_{m_2-1})),$$

are analytic everywhere, and

$$f^{(h)}_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}) = f_{m_1}(\sigma^{(m_1)})\tilde{f}_{m_2}(\tau^{(m_2)}) + O((1/8)^h), \qquad h \to \infty, \tag{3.7}$$

uniformly over any bounded domain of variables $\sigma^{(m_1)}, \tau^{(m_2)}$.

Before proving (3.7), we shall show how this leads to (3.5). Let $D^{(m_1)} = \{\sigma^{(m_1)}: |\sigma_r| < 2, 1 \le r \le m_1\}$, $D^{(m_2)} = \{\tau^{(m_2)}: |\tau_s| < 2, 1 \le s \le m_2\}$. According to the Lemma 2 and Cauchy's integral formula in polydiscs (Hörmander (1966)) we have

$$P(T(1, m_1) = k^{(m_1)};\ T(m_1 + h + 1, m_1 + h + m_2 - 1) = l^{(m_2-1)},\ T_{m_1+h+m_2} \ge l_{m_2})$$
$$= (2\pi i)^{-(m_1+m_2)} \cdot \int_{\partial D^{(m_1)} \times \partial D^{(m_2)}} f^{(h)}_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}) \cdot \textstyle\prod_{r=1}^{m_1} d\sigma_r/\sigma_r^{k_r+1}$$
$$\cdot \textstyle\prod_{s=1}^{m_2} d\tau_s/\tau_s^{l_s+1}$$
$$= \left[(2\pi i)^{-m_1} \cdot \int_{\partial D^{(m_1)}} f_{m_1}(\sigma^{(m_1)}) \textstyle\prod_{r=1}^{m_1} d\sigma_r/\sigma_r^{k_r+1}\right]$$
$$\times \left[(2\pi i)^{-m_2} \int_{\partial D^{(m_2)}} \cdot \tilde{f}_{m_2}(\tau^{(m_2)}) \textstyle\prod_{s=1}^{m_2} d\tau_s/\tau_s^{l_s+1}\right] \tag{3.8}$$
$$+ O((1/8)^h \cdot 2^{-(k+l)})$$
$$= P(T(1, m_1) = k^{(m_1)}) \cdot P(\tilde{T}(m_1 + h + 1, m_1 + h + m_2 - 1)$$
$$= l^{(m_2-1)},\ \tilde{T}_{m_1+h+m_2} \ge l_{m_2})$$
$$+ O((1/8)^h \cdot 2^{-(k+l)}), \qquad h \to \infty,$$

where

$$k = \textstyle\sum_{r=1}^{m_1} k_r, \qquad l = \sum_{s=1}^{m_2} l_s.$$

Subtracting from both sides of (3.8) the similar expressions obtained by writing in (3.8) $l_{m_2} + 1$ instead of l_{m_2}, we get

$$P(T(1, m_1) = k^{(m_1)};\ T(m_1 + h + 1, m_1 + h + m_2) = l^{(m_2)})$$
$$= P(T(1, m_1) = k^{(m_1)}) \cdot P(\tilde{T}(1, m_2) = l^{(m_2)}) + O((1/8)^h \cdot 2^{-(k+l)}),\ h \to \infty. \tag{3.9}$$

From (3.9) follows easily that

$$P(T(1, m_1) \in B_1;\ T(m_1 + h + 1, m_1 + h + m_2) \in B_2)$$
$$= P(T(1, m_1) \in B_1) \cdot P(\tilde{T}(1, m_2) \in B_2) + O((1/8)^h), \qquad h \to \infty,$$

uniformly over $B_1 \in \mathscr{B}(N^{(m_1)})$, $B_2 \in \mathscr{B}(N^{(m_2)})$.

PROOF OF LEMMA 2. Let $\nu \ge 1$, $\eta^{(\nu)} = (\eta_1, \cdots, \eta_\nu) \in C^\nu$. Given positive integers $n_1, \cdots, n_\nu$, let

$$A_\alpha = \{j: \textstyle\sum_{s=0}^{\alpha-1} n_s + 1 \le j \le \sum_{s=0}^{\alpha} n_s\}, \qquad 1 \le \alpha \le \nu,$$

$n_0 = 0$, and

$$Q(\eta^{(\nu)}, n^{(\nu)}) = \sum_{l^{(n)}} (\prod_{\alpha=1}^{\nu} \eta_\alpha^{L_\alpha}) Q(l^{(n)}), \tag{3.10}$$

where $n^{(\nu)} = (n_1, \cdots, n_\nu)$, $n = n_1 + \cdots, + n_\nu$, and $L_\alpha = \sum_{j \in A_\alpha} l_j$.

According to (2.6),

$$\sum_{l^{(n)}} | \prod_{\alpha=1}^{\nu} \eta_\alpha^{L_\alpha} | Q(l^{(n)}) \le \sum_{l^{(n)}} \prod_{j=1}^{n} \eta_*^{l_j}/(l_j - 1)!$$
$$= (\sum_{l \ge 1} \eta_*^l/(l-1)!)^n = (\eta_* e^{\eta_*})^n < \infty,$$

where $\eta_* = \max_{1 \le j \le \nu} |\eta_j|$. Hence, (3.10) defines a function of $\eta^{(\nu)}$, which is analytic everywhere. But then, so are the functions $f_{m_1}(\cdot)$, $\tilde{f}_{m_2}(\cdot)$, $f_{m_1 m_2}^{(h)}(\cdot, \cdot)$, because:

$$f_{m_1 m_2}^{(h)}(\sigma^{(m_1)}, \tau^{(m_2)}) = Q(\eta^{(\nu)}, n^{(\nu)}), \tag{3.11}$$

for $\nu = m_1 + 1 + m_2$, $\eta^{(\nu)} = (\sigma^{(m_1)}, 1, \tau^{(m_2)})$, $n_{m_1+1} = h$, $n_s \equiv 1$ if $s \ne m_1 + 1$;

$$\begin{aligned} f_{m_1}(\sigma^{(m_1)}) &= \sum_{k^{(m_1)}} \prod_{r=1}^{m_1} \sigma_r^{k_r} Q((k^{(m_1)}, 1)) \\ &= \lim_{\sigma_{m_1+1} \to 0} (\sigma_{m_1+1})^{-1} Q(\sigma^{(m_1+1)}, n^{(m_1+1)}), \end{aligned} \tag{3.12}$$

for $n_1 = \cdots = n_{m_1+1} = 1$, $\sigma^{(m_1+1)} = (\sigma^{(m_1)}, \sigma_{m_1+1})$; and (see (3.2), (3.6))

$$\tilde{f}_{m_2}(\tau^{(m_2)}) = 2 \cdot \lim_{\tau_0 \to 0} (\tau_0)^{-1} Q(\tau^{(m_2+1)}, n^{(m_2+1)}), \tag{3.13}$$

for $n_1 = \cdots = n_{m_2+1} = 1$, $\tau^{(m_2+1)} = (\tau_0, \tau^{(m_2)})$.

Furthermore, as the series (3.10) converges absolutely, by (2.2) we have, for $n_1 = \cdots = n_\nu = 1$,

$$\begin{aligned} Q(\eta^{(\nu)}, n^{(\nu)}) &= \sum_{p \in \mathscr{P}_\nu} \prod_{I \in p} \omega_I, \\ \omega_I = \omega_I(\{\eta_j\}_{j \in I}) &= (-1)^{|I|+1} \cdot \sum_{l_j \ge 1, j \in I} \prod_{\alpha \in I} \eta_\alpha^{l_\alpha}/l(I)!. \end{aligned} \tag{3.14}$$

Also, we shall need the following formula (Pittel (1980)):

$$\begin{aligned} Q(\eta^{(\nu)}, y^{(\nu)}) &= \sum_{n^{(\nu)}} (\prod_{\alpha=1}^{\nu} y_\alpha^{n_\alpha}) Q(\eta^{(\nu)}, n^{(\nu)}) \\ &= \sum_{q \in \mathscr{T}_\nu} R_q(\eta^{(\nu)}, y^{(\nu)}) \cdot \prod_{j \in (S \cup B)(q)} A(y_j, \eta_j). \end{aligned} \tag{3.15}$$

Here $|y_\alpha| < 1$, $|\eta_\alpha| \le 1$, $1 \le \alpha \le \nu$, and $\mathscr{T}_\nu$ is the set of all "quasi-partitions" $q = (I_1, \ldots, I_\mu)$ of the set $(1, 2, \ldots, \nu)$, $\cup_{t=1}^{\mu} I_t = (1, 2, \ldots, \nu)$, where subintervals $I_1, \ldots, I_\mu$ are not necessarily disjoint, but such that $1 = \min\{j: j \in I_1\} < \cdots < \min\{j: j \in I_\mu\}$ and $|I_t \cap I_{t+1}| = 0$ or 1, $1 \le t < \mu = \mu(q)$. $S(q)$ is the set of elements of $(1, \ldots, \nu)$ belonging to one-element sets I_t, and $B(q)$ is the set of all endpoints of two or more-element sets I_t, $1 \le t \le \mu$. Also

$$A(y, \eta) = (1 - y)(1 - y \cdot \exp(\eta(1 - y)))^{-1}, \tag{3.16}$$

and

$$\begin{aligned} R_q = \prod_{I \in q} \varphi_I, \qquad \varphi_I &= (-1)^{|I|+1} (\prod_{j \in I} \eta_j y_j) \psi_I(\{\eta_j(1 - y_j)\}_{j \in I}), \\ \psi_I(\{x_j\}_{j \in I}) &= \sum_{l_j \ge 1, j \in I} \prod_{j \in I} x_j^{l_j - 1}/l(I)!. \end{aligned} \tag{3.17}$$

Now, according to (3.11) and definition of $Q(\eta^{(\nu)}, y^{(\nu)})$, the function $\sum_{h \ge 1} y^h f_{m_1 m_2}^{(h)}(\sigma^{(m_1)}, \tau^{(m_2)})$ is the coefficient of $(\prod_{s=1}^{m_1} \alpha_s) y^h (\prod_{r=1}^{m_2} \beta_r)$ in the Taylor series for $Q(\eta^{(\nu)}, y^{(\nu)})$, if $\nu = m_1 + 1 + m_2$, $\eta^{(m_1+1+m_2)} = (\sigma^{(m_1)}, 1, \tau^{(m_2)})$, $y^{(m_1+1+m_2)} = (\alpha^{(m_1)}, y, \beta^{(m_2)})$, ($\alpha^{(m_1)} = (\alpha_1, \ldots, \alpha_{m_1})$, $\beta^{(m_2)} = (\beta_1, \ldots, \beta_{m_2})$). By (3.15), this coefficient is a sum, over all quasipartitions q, of the coefficients of $(\prod_{s=1}^{m_1} \alpha_s) y^h (\prod_{r=1}^{m_2} \beta_r)$ in the Taylor series of the functions $R_q(\eta^{(\nu)}, y^{(\nu)}) \cdot \prod_{j \in (S \cup B)(q)} A(y_j, \eta_j)$. Looking closer at relations (3.16), (3.17), we conclude that nonzero contributions to this sum can be made only by quasipartitions q having the following property: each element of the set $(1, 2, \ldots, \nu)$ different from $(m_1 + 1)$ belongs to just one of the sets I_t, $t = 1, \ldots, \mu(q)$. It is also clear that those nonzero contributions are, as functions $\sigma^{(m_1)}$, $\tau^{(m_2)}$, y, analytic everywhere, provided that $(m_1 + 1) \notin (S \cup B)(q)$.

Let $\mathscr{T}^*_{m_1+1+m_2}$ be the set of all quasipartitions q, for which $(m_1 + 1) \in (S \cup B)(q)$ and the sets $I_1, \ldots, I_{\mu(q)}$ are disjoint, except possibly two neighbors I_t, I_{t+1} with $I_t \cap I_{t+1} = (m_1 + 1)$. Then, by previous argument,

$$\sum_{h\geq 1} y^h f^{(h)}_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}) = A(y, 1)\cdot\sum_{q\in\mathscr{T}^*_{m_1+1+m_2}} \prod_{I\in q} \omega^*_I + F(\sigma^{(m_1)}, \tau^{(m_2)}, y). \tag{3.18}$$

Here $F(\sigma^{(m_1)}, \tau^{(m_2)}, y)$ is analytic everywhere and

$$\begin{aligned} \omega^*_I &= \omega_I(\{\sigma_j\}_{j\in I}), && \text{for } I \subset (1, \ldots, m_1), \\ &= \omega_I(\{\tau_{j-m_1-1}\}_{j\in I}), && \text{for } I \subset (m_1 + 2, \ldots, m_1 + 1 + m_2), \\ &= y(\sigma_{m_1+1})^{-1}\omega_I(\{\sigma_j\}_{j\in I}), && \text{for } (m_1 + 1) \in I \subset (1, \ldots, m_1 + 1), \\ &= y(\tau_0)^{-1}\omega_I(\{\tau_{j-m_1-1}\}_{j\in I}), && \text{for } (m_1 + 1) \in I \subset (m_1 + 1, \ldots, m_1 + 1 + m_2). \end{aligned} \tag{3.19}$$

$\sigma_{m_1+1} = \tau_0 = 1 - y$. (See (3.14) concerning ω_I.) In particular,

$$\begin{aligned} \omega^*_{(m_1+1)} &= y(\sigma_{m_1+1})^{-1}\omega_{(m_1+1)}(\sigma_{m_1+1}) \\ &= y(\tau_0)^{-1}\omega_{(m_1+1)}(\tau_0) = y(1 - y)^{-1}(e^{1-y} - 1) = \chi(y). \end{aligned} \tag{3.20}$$

In view of (3.14), we have after simple transformations

$$\begin{aligned} \sum_{h\geq 1} y^h f^{(h)}_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}) &= A(y, 1)(y(\sigma_{m_1+1})^{-1}Q(\sigma^{(m_1+1)}, a^{(m_1+1)})) \\ &\quad \times (y(\tau_0)^{-1}Q(\tau^{(m_2+1)}, b^{(m_2+1)})) + F^*(\sigma^{(m_1)}, \tau^{(m_2)}, y). \end{aligned} \tag{3.21}$$

Here $\sigma^{(m_1+1)} = (\sigma^{(m_1)}, \sigma_{m_1+1})$, $a_1 = \cdots = a_{m_1+1} = 1$, $\tau^{(m_2+1)} = (\tau_0, \tau^{(m_2)})$, $b_1 = \cdots = b_{m_2+1} = 1$ and

$$\begin{aligned} F^*(\sigma^{(m_1)}, \tau^{(m_2)}, y) &= F(\sigma^{(m_1)}, \tau^{(m_2)}, y) + A(y, 1)(1 - \chi(y))\cdot\sum_{q\in\mathscr{T}^{**}_{m_1+1+m_2}} \prod_{I\in q} \omega^*_I, \\ \mathscr{T}^{**}_{m_1+1+m_2} &= \{q \in \mathscr{T}^*_{m_1+1+m_2}: I_1, \ldots, I_{\mu(q)} \text{ are disjoint}\}. \end{aligned} \tag{3.22}$$

Now, by (3.16), poles of $A(y, 1)$ coincide with roots of the equation $1 - ye^{1-y} = 0$. An obvious root $y = 1$ has multiplicity 2, and all other roots have their absolute values exceeding some $y_* > 1$. (According to Knuth (1973), $y_* > 8.07$.) From (3.16), (3.20) follows that $y = 1$ is the first order pole of $A(y, 1)$, that $\chi(y)$ is analytic everywhere and that $\lim_{y\to 1} \chi(y) = 1$. Hence, the function $F^*(\sigma^{(m_1)}, \tau^{(m_2)}, y)$ in (3.22) is analytic with respect to all its arguments in the domain $|y| < y_*$.

Denoting the right-hand expression in (3.21) by $f_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}, y)$, we have therefore

$$f^{(h)}_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}) = (2\pi i)^{-1} \int_{\partial D} f_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}, y)/y^{h+1}\, dy,$$

where

$$D = \{y: |y| < ½\}.$$

Using the explicit formula for $f_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}, y)$ and applying the residue theorem to the domain $D^* = \{y: ½ < |y| < 8\}$, we find that

$$\begin{aligned} f^{(h)}_{m_1m_2}(\sigma^{(m_1)}, \tau^{(m_2)}) &= [\lim_{y\to 1} A(y, 1)(1 - y)]\cdot[\lim_{\sigma_{m_1+1}\to 0}(\sigma_{m_1+1})^{-1} \\ &\quad \times Q(\sigma^{(m_1+1)}, a^{(m_1+1)})] \times [\lim_{\tau_0\to 0}(\tau_0)^{-1}Q(\tau^{(m_2+1)}, \\ &\quad b^{(m_2+1)})] + O((⅛)^h), \qquad h \to \infty, \end{aligned} \tag{3.23}$$

uniformly over any bounded domain of $\sigma^{(m_1)}$, $\tau^{(m_2)}$. An observation that $\lim_{y\to 1} A(y, 1)(1 - y) = 2$ together with (3.12), (3.13) lead to (3.7), which completes the proof of Lemma 2 and Theorem 1 is proved.

4. Limiting behavior of $\max(T_1, \ldots, T_n)$, $\max(\tilde{T}_1, \ldots, \tilde{T}_n)$. By Theorem 1, (Part (a)), the double inequality (2.6) still holds true, if all T's are replaced by $\tilde{T}$'s. Since all the statements below are valid for both $T = \{T_k\}_{k=1}^{\infty}$ and $\tilde{T} = \{\tilde{T}_k\}_{k=1}^{\infty}$, we shall use, if convenient, a common notation $\underset{\sim}{\mathscr{T}} = \{\underset{\sim}{\mathscr{T}_k}\}_{k=1}^{\infty}$, so that $\underset{\sim}{\mathscr{T}} = T$ or $\tilde{T}$.

THEOREM 2. *If* $M_n = \max_{1 \le k \le n} \underset{\sim}{\mathscr{T}_k}$ *and* $c_n = \log n / \log(\log n)$, *then*

$$\lim_{n\to\infty} M_n / c_n = 1 \tag{4.1}$$

with probability one, and in L_p, $\forall\, p > 0$.

PROOF. By (2.6), for a positive integer m, we have

$$(1 - 1/(m-1)!)^n \le P(M_n < m) \le (1 - 1/m!)^n. \tag{4.2}$$

Let $\epsilon \in (0, 1)$. To estimate $P(M_n < c_n(1-\epsilon))$ from above, denote $m_n(\epsilon) = [c_n(1 - \epsilon/2)]$. Clearly, $m_n(\epsilon) \ge c_n(1-\epsilon)$ for all $n \ge n_0(\epsilon)$. By the right-hand side of (4.2), for those n's, we have

$$P(M_n < c_n(1-\epsilon)) \le (1 - 1/m_n(\epsilon)!)^n \le \exp(-n/m_n(\epsilon)!). \tag{4.3}$$

Now, by the Stirling's formula,

$$\begin{aligned} \log(n/m_n(\epsilon)!) &= \log n - m_n(\epsilon)\log m_n(\epsilon) + O(m_n(\epsilon)) \\ &= \log n - c_n(1-\epsilon/2)\log c_n + O(c_n) \\ &= \log n - (1-\epsilon/2)\log n(\log\log n)^{-1}[\log\log n - \log\log\log n] + O(c_n) \\ &= \frac{\epsilon}{2}\log n + o(\log n) \ge \frac{\epsilon}{3}\log n, \end{aligned}$$

for $n \ge n_1(\epsilon) \ge n_0(\epsilon)$. Hence,

$$P(M_n < c_n(1-\epsilon)) \le \exp(-\exp(\epsilon/3 \log n)) = \exp(-n^{\epsilon/3}), \qquad n \ge n_1(\epsilon). \tag{4.4}$$

Almost similarly, using the left-hand side of (4.2), one can show that given $\epsilon > 0$ we have

$$P(M_n < c_n(1+\epsilon')) \ge \exp(-n^{-\epsilon'/3}) \ge 1 - n^{-\epsilon'/3}, \qquad n \ge n_2(\epsilon), \tag{4.5}$$

for all $\epsilon' \in [\epsilon, \infty)$.

Let $p > 0$. Given $\epsilon \in (0, 1)$, for $n \ge \max(n_1(\epsilon), n_2(\epsilon))$, we get

$$\begin{aligned} E\left(\left|\frac{M_n}{c_n} - 1\right|^p\right) &= p\int_0^{\infty} x^{p-1} P\left(\left|\frac{M_n}{c_n} - 1\right| \ge x\right) dx \le \epsilon^p \\ &\quad + p\int_{\epsilon}^{\infty} x^{p-1} n^{-x/3}\, dx + \exp(-n^{\epsilon/3}). \end{aligned}$$

So $\limsup_{n\to\infty} E\left(\left|\frac{M_n}{c_n} - 1\right|^p\right) \le \epsilon^p$ and, consequently, $\frac{M_n}{c_n} \to 1$ in L_p.

Furthermore, by (4.4),

$$\sum_{n=1}^{\infty} P(M_n < c_n(1-\epsilon)) < \infty, \qquad \forall\, \epsilon \in (0, 1).$$

By Borel-Cantelli lemma, we can conclude that $\liminf_{n\to\infty} M_n/c_n \ge 1$ with probability one. Thus we have only to show that $\limsup_{n\to\infty} M_n/c_n \le 1$ with probability one, too. In this case, the estimate (4.5) does not guarantee that $\sum_{n=1}^{\infty} P(M_n \ge c_n(1+\epsilon)) < \infty$, because the series $\sum_{n=1}^{\infty} n^{-\epsilon/3}$ is divergent for $\epsilon \le 3$. Fortunately, we can avoid this difficulty, using a method suggested by Kingman (1973) in connection with the longest ascending subsequence in a random permutation. Namely, choose a positive integer l such that $l\epsilon/3 > 1$. Since

(4.6) $$\sum_{k=1}^{\infty} P(M_{k^l} \geq c_{k^l}(1+\epsilon)) \leq \sum_{k=1}^{\infty} k^{-l\epsilon/3} < +\infty,$$

$P(M_{k^l} \geq c_{k^l}(1+\epsilon)$ infinitely often$) = 0$. Define $k(n)$ by conditions: $(k(n)-1)^l < n \leq (k(n))^l$. Clearly, $\lim_{n\to\infty} k(n) = \infty$, and one can see that $c_{k(n)^l}/c_n \to 1$, as $n \to \infty$. Since $M_n \leq M_{k(n)^l}$, we obtain

$$P(M_n \geq c_n(1+2\epsilon)\text{i.o.}) \leq P(M_{k(n)^l} \geq c_{k(n)^l}(1+\epsilon)\text{i.o.}) = 0.$$

Therefore, $\limsup_{n\to\infty} M_n/c_n \leq 1$ with probability one, and Theorem 2 is proved.

REMARK. $T_k - 1$, $k = 1, 2, \ldots$, can be considered as the lengths of consecutive zero-runs in the stationary sequence $\{U_k\}_{k=0}^{\infty}$ with $P(U_k = 0) = P(U_k = 1) = ½$, see Introduction. It is worth mentioning that if $L_1, L_2, \cdots$ are the lengths of consecutive head-runs in a *fair* coin-tossing game and $M_n = \max(L_1, \ldots, L_n)$, then $M_n \log 2/\log n \to 1$ with probability one, see Erdös, Révész (1975), Komlós, Tusnády (1975). Thus the longest head-run grows somewhat faster with n than the longest run-up.

Denote by $\mathscr{M}_n$ the length of the longest run (up) generated by a finite sequence $X_1, \ldots, X_n$. It is known, Dixon (1975), that $\mathscr{M}_n/c_n \to 1$ in probability. Theorem 2 enables us to prove a stronger statement.

COROLLARY 3. $\lim_{n\to\infty} \mathscr{M}_n/c_n = 1$ *with probability one.*

PROOF. By ergodicity of $\{U_k\}_{k=0}^{\infty}$, we have (Breiman (1968))

$$\lim_{s\to\infty} 1/s \sum_{j=1}^{s} T_j = E(\tilde{T}_1) = 2 \qquad \text{a.s.}$$

Let $s_n = s_n(\omega) = \min\{s: s \geq 1 \text{ and } T_1 + \cdots + T_s \geq n\}$. Clearly, $\lim_{n\to\infty} s_n = \infty$ a.s. and, by Theorem 2,

$$\limsup_{n\to\infty} T_{s_n}/s_n \leq \limsup_{n\to\infty} M_{s_n}/s_n = 0 \qquad \text{a.s.}$$

As

$$1/s_n \sum_{j=1}^{s_n} T_j = n/s_n + O(T_{s_n}/s_n),$$

we have therefore that $\lim_{n\to\infty} s_n/n = 1/2$ a.s. It shows that

$$\lim_{n\to\infty} c_{s_n-1}/c_n = \lim_{n\to\infty} c_{s_n}/c_n = 1 \qquad \text{a.s.,}$$

which together with an obvious relation $M_{s_n-1} \leq \mathscr{M}_n \leq M_{s_n}$ and (4.1) enable us to conclude that $\lim_{n\to\infty} \mathscr{M}_n/c_n = 1$ a.s.

REMARK. It is interesting that the length $\mathscr{L}_n$ of the longest ascending subsequence in $X_1, \ldots, X_n$ grows much faster. Namely, according to Hammersley (1972), Kingman (1973) (see Section (2.4)) and Kesten (1973) (commentary on the Kingman's paper), $\lim_{n\to\infty} \mathscr{L}_n/n^{1/2} = c$ a.s., and it has been found recently that $c = 2$, (Logan and Shepp (1977), Veršik and Kerov (1977)).

THEOREM 3. *Let m and n tend to infinity in such a way that* $\lim_{n\to\infty} n(m!)^{-1} = \gamma$, $\gamma \in (0, \infty)$. *This implies that $m/c_n \to 1$, $n \to \infty$. Let $V_k^{(n)}$ be the number of $\tilde{\mathscr{T}}_j$'s larger than or equal to m, for $1 \leq j \leq k$. Then the process $V^{(n)}(t) = V_{[nt]}^{(n)}$, $t \in [0, 1]$, converges, in the sense of distribution functions, to the Poisson process $V(t)$ with parameter $\lambda = 2\gamma$.*

PROOF. We shall show first that $V_n^{(n)}$ converges in distribution to the Poisson distributed random variable V with parameter $\lambda = 2\gamma$. For that, it suffices to show that the factorial moments of $V_n^{(n)}$ approach those of V, or explicitly, that

(4.7) $$\lim_{n\to\infty} E(V_n^{(n)}(V_n^{(n)}-1)\cdots(V_n^{(n)}-(s-1))) = \lambda^s, \qquad s = 0, 1, 2, \cdots.$$

Let $W_j^{(n)}$ be the set indicator of the event $A_j^{(n)} = \{\tilde{\mathscr{T}}_j \geq m\}$, $1 \leq j \leq n$. Then $V_n^{(n)} = W_1^{(n)} + \cdots + W_n^{(n)}$ and, by Frechet's formula,

$$\mu_{[s]}(V_n^{(n)}) = E(V_n^{(n)}(V_n^{(n)}-1)\cdots(V_n^{(n)}-(s-1))) \tag{4.8}$$
$$= s!\sum_{1\le j_1<\cdots<j_s\le n} P(A_{j_1}^{(n)}\cdots A_{j_s}^{(n)}).$$

For brevity, consider only a case $\mathcal{T} = \tilde{T}$. Choose a positive integer $h < n$ and let

$$D^1 = \{(j_1,\dots,j_s)\colon 1\le j_1<\cdots<j_s\le n,\quad \min_{1\le\alpha\le s-1}(j_{\alpha+1}-j_\alpha)\ge h\},$$
$$D^2 = \{(j_1,\dots,j_s)\colon 1\le j_1<\cdots<j_s\le n,\quad \min_{1\le\alpha\le s-1}(j_{\alpha+1}-j_\alpha)< h\}.$$

Obviously,

$$|D^1|+|D^2| = \binom{n}{s} = \frac{n^s}{s!}\left(1+O\left(\frac{1}{n}\right)\right),\ |D^2| = O\left(s\binom{n}{s-1}h\right) = O(n^{s-1}h). \tag{4.9}$$

By Corollary 2, stationarity of $\tilde{T}$ and (3.2), we have

$$\textstyle\sum^1 = \sum_{(j_1,\dots,j_s)\in D^1} P(A_{j_1}^{(n)}\cdots A_{j_s}^{(n)}) = \sum_{(j_1,\dots,j_s)\in D^1}[P^s(\tilde{T}_1\ge m)+O(q^h)]$$
$$= \frac{1}{s!}(nP(\tilde{T}_1\ge m))^s\left(1+O\left(\frac{1}{n}\right)\right)+O(n^sq^h) \tag{4.10}$$
$$= \frac{1}{s!}\left(\frac{2n}{m!}\right)^s\left(1+O\left(\frac{1}{m}\right)\right)+O(n^sq^h),\qquad n\to\infty.$$

Also, since (2.6) holds true for $\tilde{T}_j$, by (4.9) we have

$$\textstyle\sum^2 = \sum_{(j_1,\dots,j_s)\in D^2} P(A_{j_1}^{(n)}\cdots A_{j_s}^{(n)}) = O(|D^2|((m-1)!)^{-s}) \tag{4.11}$$
$$= O\left(\left(\frac{n}{m!}\right)^s\frac{m^sh}{n}\right) = O\left(\frac{m^sh}{n}\right) = O\left(\frac{\log^s n\cdot h}{n}\right).$$

Choosing $h = [\log^2 n]$ we find that

$$\textstyle\lim_{n\to\infty}\mu_{[s]}(V_n^{(n)}) = s!\,(\lim_{n\to\infty}\sum^1+\lim_{n\to\infty}\sum^2) = (2\gamma)^s = \lambda^s,$$

(see (4.8), (4.10), (4.11)).

The assertion of Theorem 3 follows from the following more general statement. Let r be a positive integer and let $I_1^{(n)},\dots,I_r^{(n)}$ be disjoint subintervals of the set $(1,\dots,n)$. Denote $v^{(n)}(I_\alpha^{(n)})$ the number of $\tilde{T}_j$'s, $j\in I_\alpha^{(n)}$, larger than or equal to m. If $\lim_{n\to\infty}|I_\alpha^{(n)}|/n = p_\alpha > 0$, $1\le\alpha\le r$, then the random vector $V^{(n)} = (v^{(n)}(I_\alpha^{(n)}))_{\alpha=1}^r$ converges in distribution to a vector $Y = (Y_\alpha)_{\alpha=1}^r$ with independent components, Y_α being Poisson distributed with parameter λp_α, $1\le\alpha\le r$.

A way to prove this is to show that

$$\lim_{n\to\infty}\mu_{[s]}(V^{(n)}) = \textstyle\prod_{\alpha=1}^r(\lambda p_\alpha)^{s_\alpha}, \tag{4.12}$$

where $s_1,\dots,s_r$ are given positive integers, $s = (s_1,\dots,s_r)$ and $\mu_{[s]}(V^{(n)})$ is the multivariate sth order factorial moment of $V^{(n)}$,

$$\mu_{[s]}(v^{(n)}) = E(\textstyle\prod_{\alpha=1}^r v^{(n)}(I_\alpha^{(n)})\cdots(v^{(n)}(I_\alpha^{(n)})-(s_\alpha-1))).$$

A multivariate analogue of the formula (4.8) is

$$\mu_{[s]}(V^{(n)}) = \left(\textstyle\prod_{\alpha=1}^r s_\alpha!\right)\cdot\sum_{\substack{j(1,\alpha)<\cdots<j(s_\alpha,\alpha)\\ j(1,\alpha),\dots,j(s_\alpha,\alpha)\in I_\alpha^{(n)}\\ 1\le\alpha\le r}} P(\cap_{\alpha=1}^r A_{j(1,\alpha)}^{(n)}\cdots A_{j(s_\alpha,\alpha)}^{(n)}).$$

In view of the last formula, the relation (4.12) can be proved along the same lines as in case of (4.7). We omit corresponding details.

Theorem 3 is proved.

REMARK. Other related problems are known where Poisson distribution occurs naturally as a limiting distribution. For instance, Wolfowitz (1944) has proved that the number of runs up *and* down of the length m generated by a *finite* sequence $X_1, \ldots, X_n$, provided $n/m! \equiv \gamma$, has in the limit Poisson distribution with parameter 2γ. A similar result for this scheme was obtained by David and Barton (1962) for the number of runs larger than or equal to some m.

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