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ORDER CONVERGENCE OF MARTINGALES IN TERMS OF COUNTABLY ADDITIVE AND PURELY FINITELY ADDITIVE MARTINGALES[1]

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Let $(E, \mathscr{B}, \mu)$ be a measure space, let θ be a directed set with a countable cofinal subset, and let $(\mathscr{B}_\tau)_{\tau\in\theta}$ be an increasing family of sub-σ-algebras of $\mathscr{B}$. A martingale $(f_\tau)_{\tau\in\theta}$ is said to be of semibounded variation whenever the set $\{\int_B f_\tau \, d\mu \mid \tau \in \theta, B \in \mathscr{B}_\tau\}$ is bounded either from above or below. Denote conditional expectation by $\mathscr{E}$. We show that if every martingale of the form $\mathscr{E}(f \mid \mathscr{B}_\tau)_{\tau\in\theta}$ for some $\mathscr{B}$-measurable function f with $\int |f| \, d\mu < \infty$ is order convergent, then every martingale of semibounded variation is order convergent. When the family $(\mathscr{B}_\tau)_{\tau\in\theta}$ satisfies a certain refinement condition, we obtain a sufficient condition for order convergence of martingales of semibounded variation in terms of order convergence of martingales which converge stochastically to 0.

1. Introduction. Let $(E, \mathscr{B}, \mu)$ be a σ-finite measure space and let N be the set of positive integers. Let θ be a directed set under the relation $\ll$. (θ is nonempty, and $\ll$ is reflexive, transitive, and has the following property: for each $\tau, \sigma \in \theta$ there exists $\rho \in \theta$ such that $\rho \gg \tau, \rho \gg \sigma$.) A subset Δ of θ is called cofinal whenever for each $\tau \in \theta$ there exists $\sigma \in \Delta$ such that $\tau \ll \sigma$. Let $(\mathscr{B}_\tau)_{\tau\in\theta}$ be a family of sub-σ-algebras of $\mathscr{B}$ such that $(E, \mathscr{B}_\tau, \mu)$ is σ-finite for each $\tau \in \theta$, and $\mathscr{B}_\rho \subseteq \mathscr{B}_\sigma$ for each $\rho \ll \sigma$. By function we mean an extended real valued, β-measurable function. Our setting and notation are virtually the same as those of [5].

The conditional expectation of a function f with respect to the sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}$ is denoted by $\mathscr{E}(f \mid \mathscr{C})$. A family of functions $(f_\tau)_{\tau\in\theta}$ is called a martingale whenever f_τ is $\mathscr{B}_\tau$-measurable for each $\tau \in \theta$ and $\mathscr{E}(f_\sigma \mid \mathscr{B}_\tau) = f_\tau$ for each $\sigma \gg \tau$. A martingale $(f_\tau)_{\tau\in\theta}$ is said to be of bounded (semibounded) variation whenever the set $\{\int_B f_\tau \, d\mu \mid \tau \in \theta, B \in \mathscr{B}_\tau\}$ is bounded from above and below (either from above or below).

In the case that $(\theta, \ll)$ is N with the usual ordering, then, according to Doob's martingale convergence theorem, every martingale of bounded variation is pointwise convergent almost everywhere. However, for an arbitrary countable directed index set, the pointwise convergence of martingales requires additional assumptions. In [6] Krickeberg introduced the notion of order convergence, which coincides with pointwise convergence for countable index sets, and he showed that for an arbitrary directed index set, if the family of σ-algebras $(\mathscr{B}_\tau)_{\tau\in\theta}$ satisfies the Vitali condition, then all martingales of semibounded variation are order convergent. We remark that if θ is totally ordered by $\ll$, then the Vitali condition holds. In [1] we give a condition on the family of σ-algebras $(\mathscr{B}_\tau)_{\tau\in\theta}$ which is weaker than the Vitali condition and which is also sufficient for the order convergence of martingales of semibounded variation. Whether this condition is also necessary is an open problem. The purpose of this paper is to give necessary and sufficient conditions for the order convergence of martingales of semibounded variation in terms of the order convergence of countably additive and purely finitely additive martingales. We consider only the case that θ has a countable cofinal subset; whether our results are valid for arbitrary directed index sets is an open problem.

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Throughout this paper, sets and functions are considered equal if they are equal except on a μ-nullset. Consequently, for $B \in \mathscr{B}$, $\mu(B) = 0$ if and only if $B = \varnothing$. Denote by $\mathscr{B}_\infty$ the sub-σ-algebra of $\mathscr{B}$ which is generated by $\cup_{\tau\in\theta} \mathscr{B}_\tau$. We recall that the essential supremum and essential infimum of a family of functions $(f_\tau)_{\tau\in\theta}$ are the unique functions ess $\sup_{\tau\in\theta} f_\tau$ and ess $\inf_{\tau\in\theta} f_\tau$ such that all functions g satisfy:

(i) $f_\tau \leq g$ for all $\tau \in \theta \Leftrightarrow$ ess $\sup_{\tau\in\theta} f_\tau \leq g$;

(ii) $f_\tau \geq g$ for all $\tau \in \theta \Leftrightarrow$ ess $\inf_{\tau\in\theta} f_\tau \geq g$.

By union and intersection of a family of $\mathscr{B}$-members we mean the essential union and essential intersection, which are defined analogously. For a family of functions $(f_\tau)_{\tau\in\theta}$, the extreme order limits are defined by

$$\limsup_{\tau\in\theta} f_\tau = \text{ess}\inf_{\rho\in\theta}(\text{ess}\sup_{\tau\gg\rho} f_\tau)$$

$$\liminf_{\tau\in\theta} f_\tau = \text{ess}\sup_{\rho\in\theta}(\text{ess}\inf_{\tau\gg\rho} f_\tau)$$

[6]. Whenever the extreme order limits are equal, the common function is called the order limit, denoted by $\lim_{\tau\in\theta} f_\tau$, and the family $(f_\tau)_{\tau\in\theta}$ is said to be order convergent (to $\lim_{\tau\in\theta} f_\tau$). Let $B \in \mathscr{B}_\infty$. Whenever both of the extreme order limits coincide with a function f on B, we say that $(f_\tau)_{\tau\in\theta}$ is order convergent (to f) on B, and we write $\lim_{\tau\in\theta} f_\tau = f$ on B.

To every martingale of bounded variation $\Phi = (f_\tau)_{\tau\in\theta}$ there corresponds a finitely additive set function (of bounded variation) $Z(\Phi)$ on $\cup_{\tau\in\theta} \mathscr{B}_\tau$ defined by

$$Z(\Phi)(A) = \int_A f_\tau \, d\mu \qquad \text{for any } \tau \text{ such that} \quad A \in \mathscr{B}_\tau.$$

According to a theorem of Yosida and Hewitt [9], every finitely additive set function can be expressed uniquely as the sum of a countably additive set function and a purely finitely additive set function. A martingale of bounded variation Φ is called countably additive or purely finitely additive whenever $Z(\Phi)$ is countably additive or purely finitely additive, respectively. It follows that every martingale of bounded variation can be expressed uniquely as the sum of a countably additive martingale and a purely finitely additive martingale. A countably additive set function on $\cup_{\tau\in\theta} \mathscr{B}_\tau$ can be extended to a measure on $(E, \mathscr{B}_\infty)$ [3, 4], and by the Radon-Nikodym theorem, it follows that the countably additive martingales are precisely those of the form $(\mathscr{E}(f \mid \mathscr{B}_\tau))_{\tau\in\theta}$ for some function f with $\int |f| \, d\mu < \infty$. For a more complete discussion of the above see [5]. The purely finitely additive martingales can be characterized by a well-known property of purely finitely additive set functions [9]: for each $\epsilon > 0$ there exist $\sigma \in \theta$ and $D \in \mathscr{B}_\sigma$ such that

$$(a) \qquad \int |1_D f_\tau| \, d\mu < \epsilon \qquad \text{for all} \quad \tau \gg \sigma,$$

and

$$(b) \qquad \mu(E \backslash D) < \epsilon.$$

The purely finitely additive martingales also are known to be precisely those martingales of bounded variation which converge stochastically to 0 [8] (see also [5]). Every countably additive martingale $(\mathscr{E}(f \mid \mathscr{B}_\tau))_{\tau\in\theta}$ satisfies

$$\liminf_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau) \leq \mathscr{E}(f \mid \mathscr{B}_\infty) \leq \limsup_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau),$$

and every purely finitely additive martingale $(g_\tau)_{\tau\in\theta}$ satisfies

$$\liminf_{\tau\in\theta} g_\tau \leq 0 \leq \limsup_{\tau\in\theta} g_\tau$$

[7, 8] (see also [5]). Consequently, if the martingale of bounded variation $(f_\tau)_{\tau\in\theta}$ is order convergent on $B \in \mathscr{B}_\infty$, then $\liminf_{\tau\in\theta} f_\tau = \limsup_{\tau\in\theta} f_\tau < \infty$ on B.

We call $S \in \cup_{\tau\in\theta} \mathscr{B}_\tau$ a stable set whenever there exists $\sigma \in \theta$ such that $S \in \mathscr{B}_\sigma$ and for each $\rho \gg \sigma$ the families of sets $\{B \in \mathscr{B}_\sigma \mid B \subseteq S\}$ and $\{B \in \mathscr{B}_\rho \mid B \subseteq S\}$ are identical. We assume for the remainder of this paper that the index set θ has a countable cofinal subset, which we denote $\{\tau_n \in \theta \mid n \in N\}$.

In Section 2 we show that, for $B \in \mathscr{B}_\infty$, if every countably additive martingale is order convergent on B, then every martingale of semibounded variation is order convergent on B. In Section 3 we establish a property of sub-σ-algebras which we use in Section 4. In Section 4 we show that, for $C \in \bigcup_{\tau\in\theta} \mathscr{B}_\tau$, if C contains no stable sets and if every purely finitely additive martingale is order convergent on C, then every martingale of semibounded variation is order convergent on C. We also show, by example, that the preceding statement is not always true for $C \in \mathscr{B}_\infty$.

2. Martingales of semibounded variation and countably additive martingales. We call a function positive whenever it takes only nonnegative functional values, and we call a martingale positive whenever all of its member functions are positive. The following theorem appears in [1].

THEOREM 2.1. *There exist $A, B \in \mathscr{B}_\infty$, $B = E\backslash A$, such that*:
(i) *There exists a positive martingale of bounded variation $(f_\tau)_{\tau\in\theta}$ satisfying* $\limsup_{\tau\in\theta} f_\tau = \infty$ *on A*;
(ii) *Every martingale of semibounded variation is order convergent on B.*

We remark that the extreme order limit inequalities given in Section 1 imply that the decomposition is unique. Furthermore, the decomposition is independent of the base measure μ, which we now show. Let ν be a positive measure on $(E, \mathscr{B})$ which is equivalent to μ and which is σ-finite on $(E, \mathscr{B}_\sigma)$ for some $\sigma \in \theta$; denote by A_ν, B_ν the corresponding decomposition of Theorem 2.1. Let $(g_\tau)_{\tau\in\theta}$ be a martingale of bounded variation in the space $(E, \mathscr{B}, \nu)$. Define the function f to be the Radon-Nikodym derivative of $(E, \mathscr{B}_\infty, \nu)$ with respect to $(E, \mathscr{B}_\infty, \mu)$, and for each $\tau \in \theta$ define the function f_τ to be the Radon-Nikodym derivative of $(E, \mathscr{B}_\tau, \nu)$ with respect to $(E, \mathscr{B}_\tau, \mu)$. Then $(f_\tau)_{\tau\in\theta}$, restricted to an appropriate $\mathscr{B}_\sigma$-member, is a countably additive martingale in $(E, \mathscr{B}, \mu)$ and $\lim_{\tau\in\theta} f_\tau = f > 0$ on B. For each $\sigma \gg \tau$ and for each $C \in \mathscr{B}_\tau$

$$\int_C g_\sigma f_\sigma \, d\mu = \int_C g_\sigma \, d\nu = \int_C g_\tau \, d\nu = \int_C g_\tau f_\tau \, d\mu;$$

hence $(g_\tau f_\tau)_{\tau\in\theta}$ is a martingale of bounded variation in $(E, \mathscr{B}, \mu)$, and $\lim_{\tau\in\theta} g_\tau f_\tau$ exists and is finite on B. Therefore

$$\infty > \frac{\lim_{\tau\in\theta} g_\tau f_\tau}{\lim_{\tau\in\theta} f_\tau} = \lim_{\tau\in\theta} \frac{g_\tau f_\tau}{f_\tau} = \lim_{\tau\in\theta} g_\tau \quad \text{on} \quad B.$$

Hence $A_\nu \cap B = \emptyset$ and $B \subseteq E\backslash A_\nu = B_\nu$. The other direction follows by symmetry.

We now prove an existence theorem for countably additive martingales.

THEOREM 2.2. *Let $A \in \mathscr{B}_\infty$ and let $(f_\tau)_{\tau\in\theta}$ be a positive martingale of bounded variation satisfying* $\limsup_{\tau\in\theta} f_\tau = \infty$ *on A. Then there exists a positive function f with $\int f \, d\mu < \infty$ and satisfying* $\limsup_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau) = \infty$ *on A.*

PROOF. Let $(D_n)_{n\in N}$ be an increasing sequence of $\mathscr{B}_{\tau_1}$-members such that $\mu(D_n) < \infty$ for each $n \in N$, and $E = \bigcup_{n\in N} D_n$. For each $\rho \gg \sigma$ define the function

$$g(\sigma, \rho) = \operatorname{ess\,sup}_{\sigma \ll \tau \ll \rho} f_\tau.$$

Then for each $\sigma \in \theta$ the family of functions $(g(\sigma, \rho))_{\rho\gg\sigma}$ is increasing, and $\lim_{\rho\gg\sigma} g(\sigma, \rho) = \operatorname{ess\,sup}_{\tau\gg\sigma} f_\tau = \infty$ on A. For $n \in N$ we choose inductively $\sigma_n \in \theta$ as follows. Choose $\sigma_1 = \tau_1$; choose $\sigma_n \gg \sigma_{n-1}, \tau_n$ such that

$$g(\sigma_{n-1}, \sigma_n) > n2^n \quad \text{on} \quad A \cap D_n \quad \text{except for a set having } \mu\text{-measure less than } 1/n.$$

Define the positive function $f = \sum_{n\in N} (1/2^n) f_{\sigma_n}$. Then

$$\int f\,d\mu = \sum_{n\in N} \frac{1}{2^n} \int f_{\sigma_n}\,d\mu = \sum_{n\in N} \frac{1}{2^n} \int f_{\sigma_1} d\mu = \int f_{\sigma_1}\,d\mu < \infty.$$

Furthermore, for each $m \in N$,

$$\begin{aligned} \text{ess sup}_{\tau\gg\tau_m} \mathscr{E}(f \mid \mathscr{B}_\tau) &\geq \text{ess sup}_{n>m}\ \text{ess sup}_{\sigma_{n-1}\ll\tau\ll\sigma_n} \mathscr{E}(f \mid \mathscr{B}_\tau) \\ &\geq \text{ess sup}_{n>m}\ \text{ess sup}_{\sigma_{n-1}\ll\tau\ll\sigma_n} \frac{1}{2^n} \mathscr{E}(f_{\sigma_n} \mid \mathscr{B}_\tau) \\ &= \text{ess sup}_{n>m} \frac{1}{2^n} g(\sigma_{n-1}, \sigma_n) \\ &= \infty \quad \text{on} \quad A. \end{aligned}$$

Therefore,

$$\lim \sup_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau) = \text{ess inf}_{m\in N} (\text{ess sup}_{\tau\gg\tau_m} \mathscr{E}(f \mid \mathscr{B}_\tau)) = \infty \quad \text{on} \quad A.$$

We now combine Theorem 2.1 and Theorem 2.2.

THEOREM 2.3. *There exist $A, B \in \mathscr{B}_\infty$, $B = E\backslash A$, such that:*
(i) *There exists a positive function f with $\int f\,d\mu < \infty$ and satisfying* $\lim\sup_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau) = \infty$ *on A;*
(ii) *Every martingale of semibounded variation is order convergent on B.*

A related partition in terms of countably additive martingales is given in [1].

COROLLARY 2.1. *Let $B \in \mathscr{B}_\infty$. Assume every positive function f with $\int f\,d\mu < \infty$ satisfies* $\lim\sup_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau) < \infty$ *everywhere on B. Then every martingale of semibounded variation is order convergent on B.*

3. A property of sub-σ-algebras.

THEOREM 3.1. *Let $\mathscr{C} \subseteq \mathscr{D}$ be sub-σ-algebras of $\mathscr{B}$. Then there exist $A, B \in \mathscr{C}$, $B = E\backslash A$, such that:*
(i) $\{F \in \mathscr{C} \mid F \subseteq A\} = \{F \in \mathscr{D} \mid F \subseteq A\}$;
(ii) *There exists $D \in \mathscr{D}$ with $D \subseteq B$ such that all $C \in \mathscr{C}$ satisfy*

$$C \subseteq D \text{ or } C \subseteq B\backslash D \Rightarrow C = \varnothing.$$

PROOF. Define $\mathscr{L} = \{L \in \mathscr{C} \mid$ there exists $D \in \mathscr{D}$ with $D \subseteq L$ such that all $C \in \mathscr{C}$ satisfy $C \subseteq D$ or $C \subseteq L\backslash D \Rightarrow C = \varnothing\}$. Clearly $\varnothing \in \mathscr{L}$. Define $B = \cup_{L\in\mathscr{L}} L \in \mathscr{C}$. If $L \in \mathscr{L}$ and if $L \supseteq K \in \mathscr{C}$ then $K \in \mathscr{L}$; hence B can be expressed as the countable union of disjoint $\mathscr{L}$-members

$$B = \cup_{n\in N} L_n.$$

For each $n \in N$, let $D_n \in \mathscr{D}$ be such that $D_n \subseteq L_n$ and all $C \in \mathscr{C}$ satisfy

$$C \subseteq D_n \text{ or } C \subseteq L_n\backslash D_n \Rightarrow C = \varnothing.$$

Define $D = \cup_{n\in N} D_n \in \mathscr{D}$. Clearly $D \subseteq B$. Let $C \in \mathscr{C}$. Then

$$C \subseteq D \Rightarrow C = \cup_{n\in N} (C \cap D_n) = \cup_{n\in N} (C \cap L_n)$$

where $C \cap L_n \in \mathscr{C}$ and $C \cap L_n = C \cap D_n \subseteq D_n$ for $n \in N$ by the disjointness of $(L_n)_{n\in N}$, and

$$C \subseteq D \Rightarrow C = \cup_{n\in N} (C \cap L_n) = \cup_{n\in N} \varnothing = \varnothing;$$

analogously

$$C \subseteq B\backslash D \Rightarrow C = \cup_{n\in N} (C \cap (L_n\backslash D_n)) = \cup_{n\in N} (C \cap L_n)$$

where $C \cap L_n \in \mathscr{C}$ and $C \cap L_n = C \cap (L_n\backslash D_n) \subseteq L_n\backslash D_n$ for $n \in N$ by the disjointness of $(L_n)_{n\in N}$, and

$$C \subseteq B\backslash D \Rightarrow C = \cup_{n\in N} (C \cap L_n) = \cup_{n\in N} \varnothing = \varnothing.$$

Therefore (ii) holds. Define $A = E\backslash B \in \mathscr{C}$. Let $F \in \mathscr{D}$ with $F \subseteq A$. Define

$$D = F\backslash \cup_{C\in\mathscr{C};C\subseteq F} C \in \mathscr{D}$$

$$L = \cap_{C\in\mathscr{C};C\supseteq D} C \in \mathscr{C}.$$

Then $D \subseteq L$, and all $C \in \mathscr{C}$ satisfy

$$C \subseteq D \text{ or } C \subseteq L\backslash D \Rightarrow C = \varnothing;$$

hence $L \in \mathscr{L}$ and $L \subseteq B$. However $D \subseteq F \subseteq A \in \mathscr{C}$ and, hence, $L \subseteq A$. Therefore $L = \varnothing$, $D = \varnothing$, and $F = \cup_{C\in\mathscr{C};C\subseteq F} C \in \mathscr{C}$, establishing (i).

As an application of Theorem 3.1 we prove the following lemma, which is used in the construction of purely finitely additive martingales.

LEMMA 3.1. *Let $\sigma \in \theta$, let f be a positive $\mathscr{B}_\sigma$-measurable function with $\int f\, d\mu < \infty$, and let $\{f > 0\}$ contain no stable sets. Then there exists a positive function g such that $\mathscr{E}(g \mid \mathscr{B}_\sigma) = f$, $\mu(\{g > 0\} \le \frac{2}{3}\, \mu(\{f > 0\})$, and g is $\mathscr{B}_\rho$-measurable for some $\rho \in \theta$.*

PROOF. Consider the restriction of $(E, \mathscr{B}_\sigma, \mu)$ to $\{f > 0\}$. By Theorem 3.1, for each $\rho \gg \sigma$ let $A_\rho, B_\rho \in \mathscr{B}_\sigma$ with $A_\rho \subseteq \{f > 0\}$, $B_\rho = \{f > 0\}\backslash A_\rho$ be such that:
(i) $\{F \in \mathscr{B}_\sigma \mid F \subseteq A_\rho\} = \{F \in \mathscr{B}_\rho \mid F \subseteq A_\rho\}$;
(ii) There exists $D_\rho \in \mathscr{B}_\rho$ with $D_\rho \subseteq B_\rho$ such that all $C \in \mathscr{B}_\sigma$ satisfy

$$C \subseteq D_\rho \text{ or } C \subseteq B_\rho\backslash D_\rho \Rightarrow C = \varnothing.$$

If $\tau \gg \rho \gg \sigma$ then the $\mathscr{B}_\tau$-members $A_\tau \cap D_\rho$ and $A_\tau \cap (B_\rho\backslash D_\rho)$ are also $\mathscr{B}_\sigma$-members by (i), $A_\tau \cap B_\rho = (A_\tau \cap D_\rho) \cup (A_\tau \cap (B_\rho\backslash D_\rho)) = \varnothing \cup \varnothing = \varnothing$ by (ii), and $A_\tau \subseteq \{f > 0\}\backslash B_\rho = A_\rho$; hence $(A_\rho)_{\rho\gg\sigma}$ is a decreasing family. Furthermore $\cap_{\rho\gg\sigma} A_\rho \in \mathscr{B}_\sigma$ is a stable set and consequently $\cap_{\rho\gg\sigma} A_\tau = \varnothing$. Let $\rho \gg \sigma$ be such that $\mu(A_\rho) \le \frac{1}{3}\, \mu(\{f > 0\})$. By redefining D_ρ to be $B_\rho\backslash D_\rho$ if necessary, we assume $\mu(D_\rho) \le \frac{1}{2}\, \mu(B_\rho)$. Define $E_\rho = A_\rho \cup D_\rho \in \mathscr{B}_\rho$. Then

$$\mu(E_\rho) = \mu(A_\rho) + \mu(D_\rho) \le \mu(A_\rho) + \tfrac{1}{2}\, \mu(B_\rho)$$
$$= \tfrac{1}{2}\, \mu(A_\rho) + \tfrac{1}{2}\, \mu(\{f > 0\}) \le \tfrac{2}{3}\, \mu(\{f > 0\}).$$

If $C \in \mathscr{B}_\sigma$ and $C \cap E_\rho = \varnothing$, then

$$C \cap \{f > 0\} = C \cap (B_\rho\backslash D_\rho) \subseteq B_\rho\backslash D_\rho,$$

and, by (ii),

$$C \cap \{f > 0\} = \varnothing.$$

Therefore we can define a positive measure ν on

$$(E_\rho, \{C \cap E_\rho \mid C \in \mathscr{B}_\sigma\}) \quad \text{by} \quad \nu(C \cap E_\rho) = \int_C f\, d\mu \quad \text{for} \quad C \in \mathscr{B}_\sigma.$$

Let g be the function defined by

$$g = \begin{cases} \text{the Radon-Nikodym derivative of } (E_\rho, \{C \cap E_\rho \mid C \in \mathscr{B}_\sigma\}, \nu) \text{ with respect to } (E_\rho, \\ \quad \{C \cap E_\rho \mid C \in \mathscr{B}_\sigma\}, \mu) \text{ on } E_\rho, \\ 0 \quad \text{elsewhere.} \end{cases}$$

Then g is positive, $\mathscr{B}_\rho$-measurable, and

$$\mu(\{g>0\}) \le \mu(E_\rho) \le \tfrac{2}{3}\,\mu(\{f>0\}).$$

Finally, for each $C \in \mathscr{B}_\sigma$

$$\int_C g\,d\mu = \int_{C\cap E_\rho} g\,d\mu = \nu(C\cap E_\rho) = \int_C f\,d\mu;$$

hence $\mathscr{E}(g\,|\,\mathscr{B}_\sigma) = f$.

4. Martingales of semibounded variation and purely finitely additive martingales. In this section we make use of the following result, which is a consequence of a corresponding result for purely finitely additive set functions, or which can be established with an elementary argument using the characterization of purely finitely additive martingales given in Section 1. Let $\{(g_{n,\tau})_{\tau\in\theta}\,|\,n\in N\}$ be a family of purely finitely additive martingales satisfying

$$\textstyle\sum_{n\in N} \sup_{\tau\in\theta} \displaystyle\int |g_{n,\tau}|\,d\mu < \infty,$$

and for each $\tau\in\theta$ let $g_\tau = \sum_{n\in N} g_{n,\tau}$; then $(g_\tau)_{\tau\in\theta}$ is a purely finitely additive martingale.

We now prove two existence theorems for purely finitely additive martingales.

THEOREM 4.1. *Let $\sigma\in\theta$, let f be a positive $\mathscr{B}_\sigma$-measurable function with $\int f\,d\mu<\infty$, and let $\{f>0\}$ contain no stable sets. Then there exists a positive purely finitely additive martingale $(g_\tau)_{\tau\in\theta}$ with $g_\sigma = f$.*

PROOF. By the property of purely finitely additive martingales given above, and by the σ-finiteness of $(E, \mathscr{B}_\sigma, \mu)$, it suffices to prove the theorem for the case $\mu(\{f>0\})<\infty$. For $n\in N$ we choose inductively $\sigma_n\in\theta$ and f_n, a positive $\mathscr{B}_{\sigma_n}$-measurable function, as follows: choose $\sigma_1 \gg \sigma, \tau_1$ and define $f_1 = f$; choose f_n according to Lemma 3.1, satisfying $\mathscr{E}(f_n\,|\,\mathscr{B}_{\sigma_{n-1}}) = f_{n-1}$ and $\mu(\{f_n>0\}) \le \tfrac{2}{3}\,\mu(\{f_{n-1}>0\})$, and choose $\sigma_n \gg \sigma_{n-1}, \tau_n$ such that f_n is $\mathscr{B}_{\sigma_n}$-measurable. We remark that the hypotheses of the lemma are satisfied throughout the inductive process because $\int f_n\,d\mu = \int f_{n-1}\,d\mu < \infty$ and $\{f_n>0\}\subseteq\{f_{n-1}>0\}$ contains no stable sets. $(\sigma_n)_{n\in N}$ is a countable cofinal subset of θ, and we define the martingale $(g_\tau)_{\tau\in\theta}$ by

$$g_\tau = \mathscr{E}(f_n\,|\,\mathscr{B}_\tau) \qquad \text{for any} \quad n\in N \quad \text{such that} \quad \sigma_n \gg \tau.$$

Clearly $(g_\tau)_{\tau\in\theta}$ is positive and is of bounded variation. $(g_\tau)_{\tau\in\theta}$ is purely finitely additive because for each $n\in N$

$$\int_{\{f_n=0\}} g_\tau\,d\mu = \int_{\{f_n=0\}} f_n\,d\mu = 0 \qquad \text{for all} \quad \tau \gg \sigma_n,$$

$$\text{and} \quad \mu(E\backslash\{f_n=0\}) = \mu(\{f_n>0\}) \le \left(\frac{2}{3}\right)^{n-1} \mu(\{f>0\}).$$

Finally,

$$g_\sigma = \mathscr{E}(f_1\,|\,\mathscr{B}_\sigma) = \mathscr{E}(f\,|\,\mathscr{B}_\sigma) = f.$$

THEOREM 4.2. *Let $C\in\bigcup_{\tau\in\theta}\mathscr{B}_\tau$ contain no stable sets, let $A\in\mathscr{B}_\infty$ be a subset of C, and let $(f_\tau)_{\tau\in\theta}$ be a positive martingale of bounded variation satisfying* lim $\sup_{\tau\in\theta} f_\tau = \infty$ *on A. Then there exists a positive purely finitely additive martingale $(g_\tau)_{\tau\in\theta}$ satisfying* lim $\sup_{\tau\in\theta} g_\tau = \infty$ *on A.*

PROOF. Our proof is similar to the proof of Theorem 2.2. $C \in \mathscr{B}_{\sigma_0}$ for some $\sigma_o \in \theta$. Denote by l_C the indicator function of C and define the martingale $(h_\tau)_{\tau\in\theta}$ by

$$h_\tau \begin{cases} = l_C f_\tau & \text{for } \tau \gg \sigma_o \\ = \mathscr{E}(l_C f_\rho \mid \mathscr{B}_\tau), \quad \text{any } \rho \gg \sigma_o, \tau & \text{for } \tau \not\gg \sigma_o. \end{cases}$$

Then for $\tau \gg \sigma_o$, the set $\{h_\tau > 0\} \subseteq C$ contains no stable sets. Furthermore

$$\limsup_{\tau\in\theta} h_\tau = \limsup_{\tau\in\theta} l_C f_\tau = l_C \limsup_{\tau\in\theta} f_\tau = \infty \quad \text{on} \quad A.$$

Let $(D_n)_{n\in N}$ be an increasing sequence of $\mathscr{B}_{\sigma_0}$-members such that $\mu(D_n) < \infty$ for each $n \in N$, and $E = \cup_{n\in N} D_n$. For each $\rho \gg \sigma$ define the function

$$g(\sigma, \rho) = \operatorname{ess\,sup}_{\sigma\ll\tau\ll\rho} h_\tau.$$

Then for each $\sigma \in \theta$ the family of functions $(g(\sigma, \rho))_{\rho\gg\sigma}$ is increasing, and $\lim_{\rho\gg\sigma} g(\sigma, \rho) = \operatorname{ess\,sup}_{\tau\gg\sigma} h_\tau = \infty$ on A. For $n \in N$ we choose inductively $\sigma_n \in \theta$ as follows. Choose $\sigma_1 \gg \sigma_o, \tau_1$; choose $\sigma_n \gg \sigma_{n-1}, \tau_n$ such that

$$g(\sigma_{n-1}, \sigma_n) > n2^n \quad \text{on} \quad A \cap D_n \qquad \text{except for a set having } \mu\text{-measure less than } 1/n.$$

Applying Theorem 4.1 for each $n \in N$, let $(g_{n,\tau})_{\tau\in\theta}$ be a positive purely finitely additive martingale such that $g_{n,\sigma_n} = (1/2^n)h_{\sigma_n}$. Then

$$\sum_{n\in N} \sup_{\tau\in\theta} \int |g_{n,\tau}|\, d\mu = \sum_{n\in N} \int g_{n,\sigma_n}\, d\mu = \sum_{n\in N} \frac{1}{2^n} \int h_{\sigma_n}\, d\mu$$

$$= \sum_{n\in N} \frac{1}{2^n} \int h_{\sigma_o}\, d\mu = \int h_{\sigma_o}\, d\mu < \infty.$$

For each $\tau \in \theta$ define $g_\tau = \sum_{n\in N} g_{n,\tau}$, and $(g_\tau)_{\tau\in\theta}$ is a positive purely finitely additive martingale. For each $m \in N$

$$\begin{aligned} \operatorname{ess\,sup}_{\tau\gg\tau_m} g_\tau &\geq \operatorname{ess\,sup}_{n>m} \operatorname{ess\,sup}_{\sigma_{n-1}\ll\tau\ll\sigma_n} \mathscr{E}(g_{\sigma_n} \mid \mathscr{B}_\tau) \\ &\geq \operatorname{ess\,sup}_{n>m} \operatorname{ess\,sup}_{\sigma_{n-1}\ll\tau\ll\sigma_n} \mathscr{E}(g_{n,\sigma_n} \mid \mathscr{B}_\tau) \\ &= \operatorname{ess\,sup}_{n>m} \operatorname{ess\,sup}_{\sigma_{n-1}\ll\tau\ll\sigma_n} \frac{1}{2^n} \mathscr{E}(h_{\sigma_n} \mid \mathscr{B}_\tau) \\ &= \operatorname{ess\,sup}_{n>m} \frac{1}{2^n} g(\sigma_{n-1}, \sigma_n) \\ &= \infty \quad \text{on} \quad A. \end{aligned}$$

Therefore

$$\limsup_{\tau\in\theta} g_\tau = \operatorname{ess\,inf}_{m\in N} (\operatorname{ess\,sup}_{\tau\gg\tau_m} g_\tau) = \infty \quad \text{on} \quad A.$$

We now relate the order convergence of martingales of semibounded variation with that of purely finitely additive martingales.

THEOREM 4.3. *Let $B \in \mathscr{B}_\infty$ and let $\mathscr{R} = \{C \in \cup_{\tau\in\theta} \mathscr{B}_\tau \mid$ either C is a stable set or C contains no stable sets$\}$. Assume $B \subseteq \cup_{C\in\mathscr{R}} C$ and assume every positive purely finitely additive martingale $(g_\tau)_{\tau\in\theta}$ satisfies $\limsup_{\tau\in\theta} g_\tau < \infty$ everywhere on B. Then every martingale of semibounded variation is order convergent on B.*

PROOF. On a stable set C, every martingale of semibounded variation is eventually a family of identical functions, and hence, is order convergent on $B \cap C$. For $C \in \cup_{\tau\in\theta} \mathscr{B}_\tau$ which contains no stable sets, every martingale of semibounded variation is order convergent on $B \cap C$ by Theorem 2.1 and Theorem 4.2. Finally $B = \cup_{C\in\mathscr{R}} B \cap C$.

We remark that E can be partitioned according to the following theorem [1] without any assumptions concerning stable sets.

THEOREM 4.4. *There exist* $A, B \in \mathscr{B}_\infty$, $B = E \setminus A$, *such that*:
(i) *There exists a positive purely finitely additive martingale* $(g_\tau)_{\tau\in\theta}$ *satisfying* $\limsup_{\tau\in\theta} g_\tau = \infty$ *on* A;
(ii) *Every purely finitely additive martingale is order convergent to* 0 *on* B.

We conclude with an example which shows that Theorem 4.3 without the assumption concerning stable sets is false. In fact we construct a space on which every purely finitely additive martingale is order convergent (to 0) and we produce a positive function f with $\int f\,d\mu < \infty$ and satisfying $\limsup_{\tau\in\theta} \mathscr{E}(f \mid \mathscr{B}_\tau) = \infty$ everywhere except on stable sets.

EXAMPLE 4.1. For $k \in N$ we define

$$[k] = \{n \in N \mid n \le 2^k\},$$

$$\mathscr{P}_k = \text{the set of all subsets of } [k],$$

$$P_k = \text{the uniform probability measure on } ([k], \mathscr{P}_k).$$

Let

$$(E, \mathscr{B}, \mu) = \prod_{k=2}^{\infty} ([k], \mathscr{P}_k, P_k),$$

$\theta = \{(i, j) \mid i, j \in N, i \ge 2, \text{ and } 2 \le j \le 2^i\}$ with the ordering $(i, j) \gg (m, n)$ if and only if $i > m$ or $(i, j) = (m, n)$.

For $2 \le m \in N$ let $\mathscr{S}(m)$ be the subset of $\mathscr{B}$ whose elements are

$$\left(\prod_{k=2}^{m-1} \{n_k\}\right) \times \{1\} \times \left(\prod_{k=m+1}^{\infty} [k]\right) \qquad \text{where} \quad 2 \le n_k \le 2^k \quad \text{for} \quad k = 1, 2, \ldots, m-1.$$

For $\tau = (i, j) \in \theta$ let $\mathscr{T}(\tau)$ be the subset of $\mathscr{B}$ whose elements are

$$\left(\prod_{k=2}^{i-1} \{n_k\}\right) \times \{1, j\} \times \left(\prod_{k=i+1}^{\infty} [k]\right) \qquad \text{where} \quad 2 \le n_k \le 2^k \quad \text{for} \quad k = 1, 2, \ldots, i-1,$$

let $\mathscr{U}(\tau)$ be the subset of $\mathscr{B}$ whose elements are

$$\left(\prod_{k=2}^{i} \{n_k\}\right) \times \left(\prod_{k=i+1}^{\infty} [k]\right) \qquad \text{where} \quad 2 \le n_k \le 2^k \quad \text{for} \quad k = 1, 2, \ldots, i, \quad \text{and} \quad n_i \ne j,$$

and let

$$\mathscr{W}(\tau) = \left(\bigcup_{m=2}^{i-1} \mathscr{S}(m)\right) \cup \mathscr{T}(\tau) \cup \mathscr{U}(\tau).$$

For $\tau \in \theta$ let $\mathscr{B}_\tau$ be the sub-σ-algebra of $\mathscr{B}$ generated by $\mathscr{W}(\tau)$. It is easy to check that $\mathscr{B}_\tau \subseteq \mathscr{B}_\sigma$ for $\sigma \gg \tau$. For each $\tau \in \theta$, $\mathscr{W}(\tau)$ is a partition of E; hence, the atoms of $\mathscr{B}_\tau$ are precisely the elements of $\mathscr{W}(\tau)$. Therefore every element of $\bigcup_{m=2}^{\infty} \mathscr{S}(m)$ is a stable set. For $S \in \mathscr{B}_\infty$ denote by l_S the indicator function of S. Define the positive function $f = \sum_{m=2}^{\infty} \sum_{S\in\mathscr{S}(m)} m l_S$. Then $\int f\,d\mu = \sum_{m=2}^{\infty} m\mu\left(\bigcup_{S\in\mathscr{S}(m)} S\right) \le \sum_{m=2}^{\infty} m(1/2^m) < \infty$. For $2 \le n \in N$ we define

$$\theta(n) = \{(i, j) \in \theta \mid i = n\},$$

$$\theta[n] = \bigcup_{k=2}^{n} \theta(n).$$

Then for each $2 \le n \in N$, each $\tau \in \theta(n)$, and each $T \in \mathscr{T}(\tau)$

$$\int_T f\,d\mu \ge \int_T \sum_{S\in\mathscr{S}(n)} n l_S\,d\mu = n\mu\left(T \cap \left(\bigcup_{S\in\mathscr{S}(n)} S\right)\right)$$

$$= \frac{n}{2}\mu(T) = \int_T \frac{n}{2}\,d\mu,$$

and $\mathscr{E}(f \mid \mathscr{B}_\tau) \geq n/2$ on T; hence

$$\text{ess sup}_{\tau\in\theta(n)}\ \mathscr{E}(f \mid \mathscr{B}_\tau) \geq \frac{n}{2} \qquad \text{on} \quad \cup_{\tau\in\theta(n)} \cup_{T\in\mathscr{T}(\tau)} T$$

$$= E\backslash(\cup_{m=2}^{n-1} \cup_{S\in\mathscr{S}(m)} S).$$

Consequently, for each $\rho \in \theta$

$$\text{ess sup}_{\tau\gg\rho}\ \mathscr{E}(f \mid \mathscr{B}_\tau) = \infty \qquad \text{on} \quad E\backslash(\cup_{m=2}^{\infty} \cup_{S\in\mathscr{S}(m)} S),$$

and

$$\lim\sup_{\tau\in\theta}\ \mathscr{E}(f \mid \mathscr{B}_\tau) = \infty \qquad \text{on} \quad E\backslash(\cup_{m=2}^{\infty} \cup_{S\in\mathscr{S}(m)} S).$$

Furthermore $\mu[E\backslash(\cup_{m=2}^{\infty} \cup_{S\in\mathscr{S}(m)} S)] = 1 - \sum_{m=2}^{\infty} \mu(\cup_{S\in\mathscr{S}(m)} S) \geq 1 - \sum_{m=2}^{\infty}(1/2^n) = ½ > 0$. Let$(g_\tau)_{\tau\in\theta}$ be a positive purely finitely additive martingale and let $C = \{\lim\sup_{\tau\in\theta} g_\tau = \infty\} \in \mathscr{B}_\infty$. Let $n \in N$. Then there exists $i \in N$ such that

$$\mu(\cup_{\tau\in\theta[i]}\{g_\tau > n\}) \geq \frac{\mu(C)}{2}.$$

Because $\mathscr{B}_\tau \subseteq \mathscr{B}_\sigma$ for $\tau \ll \sigma$, we can find a family of $\mathscr{B}$-members $(M_\tau)_{\tau\in\theta[i]}$ satisfying

$$M_\tau \in \mathscr{B}_\tau, \quad M_\tau \subseteq \{g_\tau > n\} \qquad \text{for} \quad \tau \in \theta[i],$$

$$(\cup_{\tau\in\theta(h)} M_\tau) \cap (\cup_{\tau\in\theta(k)} M_\tau) = \varnothing \qquad \text{for} \quad h \neq k,$$

$$\text{and} \qquad \cup_{\tau\in\theta[i]} M_\tau = \cup_{\tau\in\theta[i]} \{g_\tau > n\}.$$

For $\tau \in \theta[i]$, by subtracting from M_τ an appropriate collection of $\mathscr{U}_\tau$-members, we can obtain a family of $\mathscr{B}$-members $(L_\tau)_{\tau\in\theta[i]}$ satisfying

$$L_\tau \in \mathscr{B}_\tau, \quad L_\tau \subseteq \{g_\tau > n\} \qquad \text{for} \quad \tau \in \theta[i],$$

$$L_\tau \cap L_\sigma \subseteq \cup_{m=2}^{i} \cup_{S\in\mathscr{S}(m)} S \qquad \text{for} \quad \tau \neq \sigma,$$

$$\text{and} \qquad \cup_{\tau\in\theta[i]} L_\tau = \cup_{\tau\in\theta[i]} \{g_\tau > n\}.$$

Let $\rho = (i + 1, 2) \in \theta$. Then $g_\rho = 0$ on the stable set $\cup_{m=2}^{i} \cup_{S\in\mathscr{S}(m)} S \in \mathscr{B}_\rho$ because $(g_\tau)_{\tau\in\theta}$ is a purely finitely additive martingale. Consequently

$$\frac{n\mu(C)}{2} \leq n\mu(\cup_{\tau\in\theta[i]} \{g_\tau > n\}) = n\mu(\cup_{\tau\in\theta[i]} L_\tau)$$

$$\leq n \textstyle\sum_{\tau\in\theta[i]} \mu(L_\tau) \leq \sum_{\tau\in\theta[i]} \displaystyle\int_{L_\tau} g_\tau\, d\mu$$

$$= \textstyle\sum_{\tau\in\theta[i]} \displaystyle\int_{L_\tau} g_\rho\, d\mu = \int_{\cup_{\tau\in\theta[i]}L_\tau} g_\rho\, d\mu \leq \int g_\rho\, d\mu = \int g_{(2,\,2)}\, d\mu.$$

Hence for all $n \in N$,

$$\mu(C) \leq \frac{2}{n} \int g_{(2,\,2)}\, d\mu,$$

and $\mu(C) = 0$. By Theorem 4.4, all purely finitely additive martingales are order convergent to 0.

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