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A NOTE ON AN INEQUALITY INVOLVING THE NORMAL DISTRIBUTION[1]

BY HERMAN CHERNOFF

Massachusetts Institute of Technology

The following inequality is useful in studying a variation of the classical isoperimetric problem. Let X be normally distributed with mean 0 and variance 1. If g is absolutely continuous and $g(X)$ has finite variance, then

$$E\{[g'(X)]^2\} \geq \mathrm{Var}[g(X)]$$

with equality if and only if $g(X)$ is linear in X. The proof involves expanding $g(X)$ in Hermite polynomials.

1. Introduction. The following inequality is useful in studying a variation of the classical isoperimetric problem. Let X be normally distributed with density $\varphi(x)$ and mean 0 and variance 1. If g is absolutely continuous and $g(X)$ has finite variance, then

$$E\{[g'(X)]^2\} \geq \mathrm{Var}[g(X)]$$

with equality if and only if $g(X)$ is linear in X.

2. Proof. A completeness theorem for Hermite polynomials ([3], page 355) is equivalent to the following statement. If $\int h^2(x)dx < \infty$ and $\int h(x)x^n \exp(-ax^2)dx = 0$ for some $a > 0$ and $n = 0, 1, 2, \cdots$ then $h(x) = 0$ almost everywhere. It follows that if $\int h^2(x)dx < \infty$, $h(x)$ may be expanded in polynomials which are orthogonal with respect to the measure μ where $d\mu = \exp(-ax^2)dx$ for $a > 0$. Let $h(x) = g(x)\exp(-x^2/4)$. Then $\int h^2(x)dx < \infty$ and $h(x)x^n \exp(-x^2/4) = g(x)x^n \exp(-x^2/2)$. It follows that $g(x)$ can be expanded in orthonormalized Hermite polynomials to give

$$g(X) = a_0 + a_1H_1(X) + a_2H_2(X) + \cdots + a_nH_n(X) + \cdots \qquad \text{w.p. } 1$$

where $E[H_i(X)] = 0$, $E[H_i(X)H_j(X)] = \delta_{ij}$, $H_i'(X) = \sqrt{i}\,H_{i-1}(X)$ and $a_i = E[g(X)H_i(X)]$. Then

$$\mathrm{Var}[g(X)] = a_1^2 + a_2^2 + \cdots + a_n^2 \cdots.$$

Let

$$R_n(X) = g(X) - [a_0 + a_1H_1(X) + \cdots + a_nH_n(X)]$$

which is orthogonal to $H_i(X)$ (i.e., $E[R_n(X)H_i(X)] = 0$) for $i \leq n$. Then $R_n(X)$ has finite variance and is orthogonal to $1, X, \cdots, X^n$.

Since g is absolutely continuous, so is R_n, and R_n' exists a.e. and is integrable on a finite interval. Consequently, ([2], page 332), $R_n'(w)w^i\phi(w)$ can be integrated by parts to yield

$$\int_a^b R_n'(w)w^i\phi(w)\,dw = R_n(w)w^i\phi(w)\Big|_a^b - \int_a^b R_n(w)[iw^{i-1} - w^{i+1}]\phi(w)\,dw.$$

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Since $E|g(X)X^i| < \infty$ and g is continuous, $E|R_n(X)X^i| < \infty$ and there exist sequences $\{w_j\}$ approaching $+\infty$ and $-\infty$ so that $R_n(w_j)w_j^i\phi(w_j) \to 0$. Moreover, to establish our inequality we need only consider the case where $E\{[g'(X)]^2\} < \infty$, in which case $E[R_n'(X)X^i]$ exists finite and

$$\begin{aligned} E[R_n'(X)X^i] &= -E[R_n(X)(iX^{i-1} - X^{i+1})] \\ &= 0 \qquad \text{for} \quad i \leq n-1. \end{aligned}$$

Hence

$$g'(X) = a_1 + \sqrt{2}\, a_2 H_1(X) + \cdots + \sqrt{n}\, a_n H_{n-1}(X) + R_n'(X)$$

and

$$E[g'(X)]^2 \geq a_1^2 + 2a_2^2 + \cdots + na_n^2$$

and finally

$$E[g'(X)]^2 \geq \textstyle\sum_{i=1}^{\infty} ia_i^2 \geq \mathrm{Var}[g(X)]$$

with equality only if $a_2 = a_3 = \cdots = 0$ in which case $g(X)$ is linear and equality obtains.

3. An application. This inequality is relevant for determining stationary solutions of the following variational problem. Find a region R with boundary B to minimize

$$L = \int_B \phi(x)\phi(y)\sqrt{dx^2 + dy^2}$$

subject to a fixed value between 0 and 1 of

$$A = \int_R \phi(x)\phi(y)\, dxdy.$$

Note that if the normal densities were replaced by 1, L and A would be the perimeter and area of R and we would have the classical isoperimetric problem. As it stands $A = P\{(X, Y) \in R\}$ where X and Y are independent $N(0, 1)$ random variables and L is the integrated density along the boundary B.

The Euler equation of the calculus of variations applied to $L + \lambda A$ and a proposed solution $B = \{(x, y): y = y(x)\}$ of the variational problem leads to a differential equation

$$y'' = (1 + y'^2)[\lambda\sqrt{1 + y'^2} + xy' - y]$$

of which $y = \lambda$ is a solution. If we expand A and L in powers of h with $y = \lambda + hg(x)$ we have

$$A = \Phi(\lambda) + h\phi(\lambda)Eg(X) + \frac{h^2}{2}\lambda\phi(\lambda)Eg^2(X) + \cdots$$

and

$$L + \lambda A = \phi(\lambda) + \lambda\Phi(\lambda) + \frac{h^2}{2}\phi(\lambda)[E[g'(X)]^2 - Eg^2(X)] + \cdots$$

where Φ is the standard normal cdf. Thus $y = \lambda$ minimizes L among local variations subject to $A = \phi(\lambda)$ and g is absolutely continuous. This is not enough by itself to establish that $y = \lambda$ is the solution of the variational problem.

CONJECTURE. A solution of the variational problem of selecting R to minimize L subject to a specified value of A between 0 and 1 is given by a half plane.

Two remarks may be of interest. First, even though $y = \lambda$ provides a local minimum of $L + \lambda A$ for $A = \Phi(\lambda)$ it does not minimize $L + \lambda A$ for unrestricted A. Indeed, if we take $y = c$, $L + \lambda A = \phi(c) + \lambda\Phi(c)$ which is actually a local maximum at $c = \lambda$.

Second, our conjecture is relevant to another conjecture, on the efficient storage of information, which appears in a paper on information retrieval [1]. We shall not elaborate here on that conjecture except to mention obliquely that it suggests an apparently paradoxical result about the uselessness of additional information when the storage capacity is limited to one bit.

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MASSACHUSETTS INSTITUTE TECHNOLOGY
DEPARTMENT OF MATHEMATICS
ROOM 2-381
77 MASSACHUSETTS AVE
CAMBRIDGE, MASSACHUSETTS 02139