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DISTRIBUTION OF SYMMETRIC STABLE LAWS OF INDEX 2^{-n}

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Let $X_1, X_2 \cdots, X_n (n \geq 2)$ be independent standard normal. Then the random variable $U = X_1/V_n$ where

$$V_n = \exp_2[2^{n-2}-1]X_2(X_3)^2 \cdots (X_n^2)^{2^{n-3}} \qquad n \geq 3$$
$$= X_2 \text{ for } n = 2$$

has a symmetric stable distribution with index 2^{2-n}.

1. Introduction and results. The characteristic function of a symmetric stable law is given by

(1) $$f(t) = \exp[-c|t|^{\alpha}] \qquad 0 < \alpha \leq 2, \quad c > 0.$$

The distributions corresponding to the characteristic functions given by (1) are known only in the cases of normal and Cauchy distributions. In the present paper we consider (1) with $c = 1$ and $\alpha = 2^{-n}(n \geq 0)$. Our result is contained in the following:

Theorem 1. *Let $X_1, X_2, \cdots, X_n$ be independent standard normal for $n \geq 2$. Then the random variable*

(2) $$U = X_1/V_n$$

where

(3) $$V_n = X_2 \text{ for } n = 2$$
$$= \exp_2[2^{n-2}-1]X_2(X_3^2)(X_4^2)^2 \cdots (X_n^2)^{2^{n-3}} \text{ for } n \geq 3$$

has a symmetric stable distribution with index 2^{2-n}, where $\exp_2 b$ represents 2^b.

2. Proof of Theorem 1. The proof of Theorem 1 depends upon the following:

Lemma 1. *The value of the integral*

(4) $$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-t^2/(2x^2) - x^2/2]\, dx = e^{-|t|}.$$

Proof. The proof follows from the simple observation that the left side of (4) is the characteristic function of the ratio of two independent and identically distributed standard normal variables and hence must be the characteristic function of the symmetric Cauchy distribution. A direct proof can be easily given.

We now turn to prove Theorem 1 by mathematical induction. For $n = 2$, $U = X_1/X_2$ has a Cauchy distribution which is a symmetric stable law of index 1. Assume next that U/V_k has a symmetric distribution with index 2^{2-k} and consider

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(5) $$Z = X_1/V_{k+1}.$$

Since $\exp_2[2^{(k+1)-2} - 1] = \exp_2[2^{k-2} - 1]\exp_2[2^{k-2}]$, we can write (5) in the form

(6) $$Z = X_1/[V_k(2X_{k+1}^2)^{2^{(k+1)-3}}].$$

Next, since X_1/V_k has symmetric distribution with index 2^{2-k}, we obtain

(7) $$Ee^{itZ} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-x_{k+1/2}^2 - |t/(2x_{k+1}^2)^{2^{k-2}}|^{2^{2-k}}] \, dx_{k+1}$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp[-(x_{k+1})^2/2 - |t|^{2^{2-k}}/[2(x_{k+1})^2] \, dx_{k+1}.$$

From (4), the above integral is

$$\exp[-|t|^{2^{2-(k+1)}}].$$

Thus the proof is complete by induction.

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