The Annals of Probability
1981, Vol. 9, No. 4, 699–704

GROWTH OF RANDOM WALKS CONDITIONED TO STAY POSITIVE

BY GRANT A. RITTER

University of Florida

For random walk $S_k = \sum_{i=1}^{k} \xi_i$ let T be the hitting time of the lower half plane. By conditioning the process $\{S_k\}_{k=1}^{n}$ relative to $[T > n]$ we create a "random walk conditioned to stay positive." Sample paths of such processes tend to grow rather quickly. In studying this growth we find that except for a set of probability ϵ all such sample paths have as lower bounds any sequence of the form $\{\delta k^{\eta}\}_{k=1}^{n}$ where $\eta \in (0, ½)$ and $\delta < \delta(\epsilon, \eta)$. Applications of this result to sample path behavior of a random walk as it approaches or leaves the lowest of its first n values are also given.

1. Introduction. Recent work concerning random walks includes the study of such processes when they are conditioned to be positive. From a classical set-up with $S_n = \sum_{i=1}^{n} \xi_i$ where the ξ_i's are i.i.d. r.v.'s with mean μ and finite variance σ^2, one creates these processes by defining the Markov time $T = \min\{k \geq 1 : S_k \leq 0\}$ and conditioning relative sets of the form $[T > n]$. Weak convergence for such processes have been established when the drift is negative (Iglehart (1974a)) and when it is zero (Iglehart (1974b) and Durrett (1978)). In studying the latter Iglehart states "conditioning to stay positive serves to eliminate from the probability space all those smallish sample paths which are 'flirting with the zero level'." Just how flirtatious can a sample path be without facing elimination? In Section 2 of this paper we try to answer this question. We find that sequences of the form $\{\delta k^{\eta}\}_{k=1}^{n}$ for small δ and $\eta \in (0, ½)$ lie under most of the sample paths. In a sense this result is a discrete analogue to Millar's findings concerning Brownian meander (1976). In Section 3 of this paper we give a simple application of our theorem, describing random walk behavior near the index where a maximal or minimal value is obtained.

2. Theorems of lower bounds. Since flirting is not an intrinsic mathematical concept we must somehow translate it into a more quantitative expression. One such possibility is with the use of positive arrays. Define an array $\{a_{k,n}\}_{l \leq k \leq n, l \leq n < \infty}$ as lower bounding if and only if uniformly in n

$$\lim_{(\delta \to 0)} P[\inf_{k \leq n}(S_k - \delta a_{k,n}) > 0 \mid T > n] = 1.$$

With this definition we have

THEOREM 1. *Let* $S_n = \sum_{k=1}^{n} \xi_k$ *be the sum of* i.i.d. *random variables with mean* 0 *and variance* σ^2. *Let* $T = \min(k > 0 : S_k \leq 0)$. *For every* $\gamma > 0$ *the array* $(a_{k,n})_{1 \leq k \leq n, 1 < n < \infty}$ *is lower bounding where*

$$a_{k,n} = \begin{cases} 0 & k < [\gamma n] \\ k^{1/2} & k \geq [\gamma n]. \end{cases}$$

PROOF. Let $X_n(t) = S_{[tn]}/\sigma n^{1/2}$ $0 \leq t \leq 1$. From Durrett (1978) we know that under the conditions the theorem $(X_n(t) \mid T > n)^{w}\ W^{+}$, where the limiting process W^{+} is strictly positive for $t \in (0, 1]$ and has continuous sample paths with probability 1. It follows that for $\gamma > 0$ and $\epsilon > 0$ there exists a $\delta > 0$ such that

---

Received December, 1978; revised April, 1980.
AMS 1970 *subject classifications.* Primary 60G17; secondary 60J15.
Key words and phrases. Random walk, random walk conditioned to stay positive.

$$P[W^+(t) > \delta \quad \text{for all} \quad t \in [\gamma, 1]] \geq 1 - \epsilon.$$

Thus for n sufficiently large,

$$P[X_n(t) > \delta \quad \text{for all} \quad t \in [\gamma, 1]] \geq 1 - 2\epsilon$$

and the theorem readily follows. □

The shortcomings of this result are obvious. The array remains for a time at zero and then takes a jump. Our intuition of the sample path is that it starts to climb immediately. It seems appropriate to seek a result reflecting this.

In a second attempt at finding a mathematical equivalent of flirting we use positive sequences instead of arrays. Define a sequence $\{a_k\}_{k=1}^{\infty}$ as *lower bounding* iff uniformly in n

$$\lim_{(\delta \to 0)} P[\inf_{k \leq n}(S_k - \delta a_k) > 0 \mid T > n] = 1.$$

In studying Brownian meander, Millar (1976) has proved that

$$P[W^+(t) > t^{1/2} f(t) \quad \text{for sufficiently small } t]$$

is 0 or 1 according as $\int_0^1 f(t)t^{-1}dt$ is infinite or finite. From this result our intuition is that for a_k's of the form k^{η}, the sequence $\{a_k\}_{k=1}^{\infty}$ will be lower bounding if $\eta < ½$ and will not be lower bounding if $\eta \geq ½$. This proves to be the case.

THEOREM 2. *Let* $S_n = \sum_{i=1}^{n} \xi_i$ *be the sum of* i.i.d. r.v.'s *with* $E\xi_i = 0$ *and* $E\xi_i = \sigma^2 < \infty$. *The sequences* $\{k^{\eta}\}_{k=1}^{\infty}$ *are lower bounding for* $\eta < ½$ *and are not lower bounding for* $\eta \geq ½$.

PROOF. Taking the second part of the theorem first, let $\eta = ½$. From results of Durrett (1978) and Millar (1976) we kno that for any $\delta < 0$

$$\begin{aligned} &\overline{\lim}_{n\to\infty} P[\inf_{1\leq k\leq n} (S_k - \delta k^{1/2}) > 0 \mid T > n] \\ &\quad = \overline{\lim}_{n\to\infty} P\left[\left(\frac{S_{[tn]}}{\sigma n^{1/2}} - \frac{\delta}{\sigma}\frac{[tn]^{1/2}}{n^{1/2}}\right) > 0 \quad \text{for some} \quad t \in \left[\frac{1}{n}, 1\right]\right] \\ &\quad = P[W^+(t) - \frac{\delta}{\sigma} t^{1/2} > 0 \text{ for } t \in (0, 1]] \\ &\quad = 0 \quad \text{since} \quad \int_0^1 \frac{\delta}{\sigma} t^{-1}\, dt = \infty. \end{aligned}$$

This confirms that the sequence $\{k^{1/2}\}_{k=1}^{\infty}$ is not lower bounding and thus neither will be $\{k^{\eta}\}_{k=1}^{\infty}$ for any η greater than ½.

To prove the first half of the theorem we need to start with several lemmas.

LEMMA 1. *Under the conditions of the theorem there exists a constant* c_1 *such that*

$$P[T > n] \sim \frac{c_1}{n^{1/2}}$$

PROOF. This is a result first proved by Spitzer (1960).

LEMMA 2. *Let* $\bar{S}_n = \max_{0\leq i\leq n}(-S_i)$. *Under the conditions of the theorem there exists a constant* $M_1 > 0$ *such that for all* $x \geq 0$:

$$P[\bar{S}_n < x] \leq M_1 \frac{x + 1}{n^{1/2}}.$$

PROOF. This is Theorem A of Kozlov's (1976).

LEMMA 3. *Under the conditions of the theorem, for all integers $1 \le a < b \le n$ and for all nondecreasing sequences* $\{x_k\}_{k=a+1}^{b}$:

$$P[\cup_{k=a+1}^{b} (S_k < x_k, T > (1+\epsilon)n)] \le M_1 \frac{x_b+1}{[\epsilon n]^{1/2}} P[T > a].$$

PROOF. Let $A_{a+1} = [S_{a+1} < x_{a+1}, T > a+1]$ and for $k \in \{a+2, a+3, \cdots b\}$ let $A_k = [S_k < x_k, T > k] \cap (\cup_{j=a+1}^{k-1} A_j)^c$. Let θ_k be the shift operator $\xi_i(\theta_k\omega) = \xi_{i+k}(\omega)$. Since each A_k is σ_k-measurable ($\sigma_k = \sigma(S_1, S_2, \cdots S_k)$)

$$\begin{aligned} P[A_k, T > (1+\epsilon)n] &= P[A_k, \max\nolimits_{k<i\le[(1+\epsilon)n]}(S_k - S_i) < S_k] \\ &\le P[A_k, \max\nolimits_{k<i\le[(1+\epsilon)n]}(S_k - S_i) < x_k] \\ &= P[A_k, \bar{S}_{[(1+\epsilon)n]-k}\circ\theta_k < x_k] \\ &= E[E[\bar{S}_{[(1+\epsilon)n]-k}\circ\theta_k < x_k \mid \sigma_k]; A_k] \\ &\le E[P[\bar{S}_{[(1+\epsilon)n]-k} < x_k]; A_k] \\ &\le M_1 \frac{x_b+1}{[\epsilon n]^{1/2}} P[A_k]. \end{aligned}$$

Since the A_k's are disjoint and contained in $[T > a]$, the lemma follows by summing.

LEMMA 4. *For $\eta \in (0, ½)$ there is an $M_2 > 0$ such that for sufficiently large k_0 and any $n > k_0$,*

$$\begin{aligned} P[S_k < k^\eta \quad \textit{for some} \quad k \in \{k_0+1, k_0+2, \cdots n\} \mid T > (1+\epsilon)n] \\ \le M_2 \left(\frac{1+\epsilon}{\epsilon}\right)^{1/2} k_0^{\eta-1/2} \end{aligned}$$

PROOF. Letting $x_k = k^\eta$ and applying Lemmas 1, 3, and 4 we have for sufficiently large a and $n \ge b > a$:

$$\begin{aligned} P[\cup_{k=a+1}^{b} (S_k < x_k, T > (1+\epsilon)n)] &\le M_1 \frac{b^\eta+1}{[\epsilon n]^{1/2}} P[T > a] \\ &\le M_1 \frac{b^\eta+1}{[\epsilon n]^{1/2}} 2\, c_1\, a^{-1/2}. \end{aligned}$$

Let c be any number greater than 1 and let s be such that for any $a > c^s$ the above inequality holds. For any choice of $n > c^s$ and any $r \epsilon \{s, s+1, s+2, \cdots [\ln n/\ln c]\}$ We know:

$$\begin{aligned} P[\cup_{k=c^r}^{c^{r+1}} (S_k < k^\eta, T > (1+\epsilon)n)] &\le M_1 \frac{c^{(r+1)\eta}+1}{[\epsilon n]^{1/2}} \cdot 2c_1 c^{-r/2} \\ &\le 2\, c_1 M_1 \cdot \frac{c^\eta+1}{[\epsilon n]^{1/2}} \cdot (c^{\eta-1/2})^r \end{aligned}$$

Summing r from s to $[(\ln n)/(\ln c)]$ gives

$$\begin{aligned} &P[\cup_{k=c^s+1}^{n} (S_k < k^\eta, T > (1+\epsilon)n)] \\ &\le 2c_1 M_1 \cdot \frac{c^\eta+1}{[\epsilon n]^{1/2}} \cdot \frac{(c^{\eta-1/2})^s}{1-c^{\eta-1/2}} \\ &\le \frac{1}{2} \cdot \tilde{M}_2 \cdot \frac{c_1}{[\epsilon n]^{1/2}} \cdot (c^s)^{\eta-1/2} \end{aligned}$$

where $\tilde{M}_2 = M_1/1 - c^{\eta-1/2} \cdot (c^\eta + 1)$.

The rest of the proof is fairly straightforward. Note

$$P[S_k < k^\eta \quad \text{for some} \quad k \in \{c^s + 1, c^s + 2, \cdots n\} \mid T > (1 + \epsilon)n] = \frac{P[\cup_{k=c^s+1}^{n}(S_k < k^\eta, T > (1 + \epsilon)n)]}{P[T > (1 + \epsilon)n]}.$$

We have found an upper bound for the numerator of this fraction and by Lemma 1,

$$P[T > (1 + \epsilon)n] \geq \frac{\frac{1}{2}c_1}{[(1 + \epsilon)n]^{1/2}}.$$

Therefore the fraction is bounded by $\tilde{M}_2\left(\frac{1 + \epsilon}{\epsilon}\right)^{1/2}(c^s)^{\eta-1/2}$. This gives the needed result when k_0 is of the form c^s, ($\tilde{M}_2$ depending on c but not s). To obtain the lemma in the more general case note that for k_0 between c^r and c^{r+1}:

$$\begin{aligned} P[\cup_{k=k_0+1}^{n} (S_k < k^\eta, T > (1 + \epsilon)n)] &\subseteq P[\cup_{k=c^r+1}^{n} (S_k < k^\eta, T > (1 + \epsilon)n)] \\ &\leq \tilde{M}_2\left(\frac{1 + \epsilon}{\epsilon}\right)^{1/2}(c^r)^{\eta-1/2} \\ &\leq \tilde{M}_2 \cdot c^{1/2-\eta}\frac{(1 + \epsilon)^{1/2}}{\epsilon}(c^{r+1})^{\eta-1/2} \\ &\leq M_2\left(\frac{1 + \epsilon}{\epsilon}\right)^{1/2} k_0^{\eta-1/2} \end{aligned}$$

where $M_2 = \tilde{M}_2 \cdot c^{1/2-\eta}$.

Theorem 2 will easily follow once a few last details are handled.

LEMMA 5. *Under the conditions of the theorem for $\epsilon > 0$ and $\eta \in (0, 1/2)$ there exists a $k_1 = k_1(\epsilon, \eta)$ such that for all $n \geq k_1$,*

$$P[S_k < k^\eta \text{ for some } k \in \{k_1 + 1, k + 2, \cdots n\} \mid T > n] < 2\epsilon.$$

PROOF. From simple inclusion considerations we know

$$P[S_k < k^\eta \text{ for some } k \in \{k_1 + 1, k_1 + 2, \cdots n\} \mid T > n] = \frac{P[S_k < k^\eta \text{ for some } k \in \{k_1 + 1, k_1 + 2, \cdots n\} \mid T > (1 + \epsilon)n]}{P[T > n \mid T > (1 + \epsilon)n].}$$

From Lemmas 1 and 4 we know that we can choose k_1 sufficiently large to assure that for all $n \geq k_1$ the numerator is bounded above by ϵ and the denominator is bounded below by $1 - \epsilon$. This proves the lemma since for $\epsilon \leq 1/2$, $\frac{\epsilon}{1 - \epsilon} \leq 2\epsilon$, while for $\epsilon > 1/2$ the claim is vaccauously true.

We've taken care of the indices above k_1, so the theorem follows if we can show that some $M_3 > 0$ and for all $\epsilon > 0$ there is a $\delta = \delta(\epsilon)$ such that $\forall \delta \in (0, \delta_\epsilon)$:

$$P[S_k < \delta k^\eta \text{ for some } k \in \{1, 2, 3, \cdots k_1\} \mid T > n] < M_3\epsilon.$$

We begin by noting that for all $k > 0$, $P[S_k \leq 0, T > k] = 0$. Since $\cap_{\delta>0} [S_k < \delta k^\eta, T > k] = P[S_k \leq 0, T > k]$, $\exists \delta_k \in (0, 1)$ such that $\forall \delta \in (0, \delta_k)$

$$P[S_k < \delta_k k^\eta, T > k] \leq \left(\frac{1}{k_1 + 1}\right)^{1/2} \frac{\epsilon}{2^k}.$$

If $\delta(\epsilon) = \min\{\delta_k : 1 \leq k \leq k_1\}$, then for $k \leq k_1 < n$ and $\delta \in (0, \delta(\epsilon))$,

$$P[S_k < \delta k^\eta, T > n] = P[S_k < \delta k^\eta, \bar{S}_{n-k} \circ \theta_k < S_k]$$

$$= E[E[\bar{S}_{n-k}\circ\theta_k < S_k \mid S_k];\ S_k < \delta k^{\eta},\ T > k]$$
$$\le E[P[\bar{S}_{n-k} < \delta k^{-\eta}];\ S_k < \delta k^{-\eta},\ T > k]$$
$$\le M_1 \cdot \frac{\delta k^{-\eta} + 1}{(n-k)^{1/2}} \cdot P[S_k < \delta k^{-\eta},\ T > k]$$
$$\le M_1 \cdot \frac{\delta k^{-\eta} + 1}{n^{1/2}} \cdot \frac{n^{1/2}}{(n-k)^{1/2}} \cdot \frac{1^{1/2}}{k_1 + 1} \cdot \frac{\epsilon}{2^k}$$
$$\le M_1 \cdot \frac{\delta k^{-\eta} + 1}{n^{1/2}} \cdot \frac{\epsilon}{2^k}$$

Together with Lemma 1 this implies

$$P[S_k < \delta k^{-\eta} \mid T > n] \le \frac{2M_1}{c_1} \cdot \frac{\delta k^{-\eta} + 1}{2^k} \cdot \epsilon.$$

Letting $M_3 = 4M_1/c_1$ and summing over $k \in \{1, 2, \cdots k_1\}$ gives the needed result and the theorem follows.

3. An application. As mentioned in the introduction, theorems concerning random walks conditioned to stay positive can be useful in describing the behavior of partial sums near their maximal and minimal values. Here as an example we show how Theorem 2 can be applied to describe a minimum rate of climb (or descent) for a sequence of partial sums as it leaves the lowest (or highest) of its first n values. By considering the sequence of random walks $\tilde{S}_{k,n} = S_n - S_{n-k}$ and noting that $(\tilde{S}_{1,n}, \tilde{S}_{2,n}, \cdots \tilde{S}_{n,n})$ is distributed the same as $(S_1, S_2, \cdots S_n)$, one may use the same argument to describe how a sequence arrives at its lowest (or highest) value.

THEOREM 3. *Let* $\nu(n) = \max\{k \ge 0: S_i \ge S_k$ *for all* $i \le n\}$. *If* $\eta \in (0, 1/2)$, *then uniformly in* n

$$\lim_{\delta\to 0} P[S_{k+\nu(n)} - S_{\nu(n)} > \delta k^{\eta} \quad \textit{for all} \quad k \le n - \nu(n)] = 1.$$

PROOF. The proof takes advantage of the following representation of the set $[\nu(n) = j]$:

$$[\nu(n) = j] = [S_i \ge S_j \quad \forall i \le j][S_i \ge S_j \quad \forall i, j \le i \le n]$$
$$= [S_i \ge S_j \quad \forall i \le j][T\circ\theta_j > n - j].$$

To simplify notation define

$$A_\delta = [S\nu_{(n)+k} - S\nu_{(n)} > \delta k^{\eta} \quad \text{for all} \quad k \le n - \nu(n)]$$

and

$$A_{\delta,j} = A_\delta[\nu(n) = j].$$

This leads to

$$P[A_\delta] = \textstyle\sum_{j=1}^{n} P[A_{\delta,j}]$$
$$= \textstyle\sum_{j=1}^{n} P[A_{\delta,j} \mid \nu(n) = j] P[\nu(n) = j].$$

Both sets $A_{\delta,j}$ and $[T\circ\theta_j > n - j]$ belong to $\sigma(X_{j+1}, X_{j+2}, \cdots X_n)$ and thus are independent of $[S_i \le S_j \quad \forall i \le j]$ which is in $\sigma(X_1, X_2, \cdots X_j)$. It follows that

$$P[A_{\delta,j} \mid \nu(n) = j] = P[A_{\delta,j} \mid T\circ\theta_j > n - j]$$

and

$$P[A_\delta] = \sum_{j=1}^{n} P[A_{\delta,j} \mid T\circ\theta_j > n - j]P[\nu(n) = j].$$

By the Markov property

$$P[A_{\delta,j} \mid T\circ\theta_j > n - j] = P[S_k > \delta k^\eta \quad \text{for all} \quad k \le n - j \mid T > n - j].$$

For $\epsilon \in (0, 1/2)$ use Theorem 2 to determine $a\delta_\epsilon > 0$ and $a_n n_1(\epsilon)$ so that $\forall\delta \in (0, \delta_\epsilon)$, $\forall m \geq \frac{n_1(\epsilon)}{\epsilon}$

$$P[S_k > \delta k^\eta \; \forall \; k \le m \mid T > m] \geq 1 - \epsilon.$$

By letting $m = n - j$ and summing over $j \in \{1, 2, \cdots [(1 - \epsilon)n]\}$ we obtain

$$P[A_\delta] \geq (1 - \epsilon)P[\nu(n) < (1 - \epsilon)n].$$

By a simple application of the invariance principle (Donsker (1951)), for sufficiently large n

$$P[\nu(n) < (1 - \epsilon)n] > 1 - \epsilon,$$

and thus

$$P[A_\delta] > 1 - 2\epsilon.$$

From here the corollary follows by a standard argument.

REFERENCES

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

DURRETT, R. (1978). Conditioned limit theorems for some null-recurrent Markov processes. *Ann. Probability* **6** 798–828.

IGLEHART, D. L. (1974a). Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probability* **2** 608–619.

IGLEHART, D. L. (1974b). Random walks with negative drift conditioned to stay positive. *Journal of Applied Probability.* **11** 742–751.

KOZLOV, M. V. (1976). On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment. *Theor. Probability Appl.* **21** 791–804.

MILLAR, P. W. (1976). Sample functions at a last exit time. *Z. Wahrscheinlichkeitstheorie und verw Gebiete.* **34** 91–111.

SPITZER, F. (1960). A Tauberian theorem and its probability interpretation, *Tran. Amer. Math. Soc.* **94** 150–179.

DEPARTMENT OF MATHEMATICS
201 WALKER HALL
UNIVERSITY OF FLORIDA
GAINESVILLE, FLORIDA 32611