The Annals of Probability
1981, Vol. 9, No. 4, 693–698

WEIGHTED SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

BY RALF ULBRICHT

Jena, GDR

We characterize the sequences (α_i) of real numbers such that $\sum_{i=1}^{\infty} \alpha_i f_i$ exists a.e. or in L_p for all sequences of independent identically distributed symmetric random variables with pth moment. Moreover, we also treat the case $\sup|\alpha_i f_i| < \infty$ a.e.

It is known that a sequence (α_i) belongs to $l_p\,(0 < p < \infty)$ if and only if $\sup|\alpha_i f_i| < \infty$ a.e. for all sequences (f_i) of independent symmetric 3-valued random variables, with $\sup E|f_i|^p < \infty$ [1]. We give a similar characterization of the spaces l_p, respectively $l_{p\infty}\,(1 \le p < \infty)$ by sequences (f_i) of independent identically distributed random variables. More precisely we describe the following spaces of sequences of real numbers:

$$A_p^0 = \{(\alpha_i) : \textstyle\sum_{i=1}^{\infty} \alpha_i f_i \quad \text{ex. a.e. for all } (f_i) \in \Phi_p\}$$

$$A_p^p = \{(\alpha_i) : \textstyle\sum_{i=1}^{\infty} \alpha_i f_i \quad \text{ex. in } L_p \text{ for all } (f_i) \in \Phi_p\}$$

$$A_p^\infty = \{(\alpha_i) : \sup|\alpha_i f_i| < \infty \text{ a.e. for all } (f_i) \in \Phi_p\}$$

where Φ_p is the set of sequences (f_i) of independent symmetric identically distributed random variables on some probability space $(\Omega, \mathscr{A}, P)$ having pth moment. It turns out that

$$A_p^0 = A_p^\infty = l_{p\infty}$$

$$A_p^p = l_p \qquad \text{for } 1 \le p < 2$$

and

$$A_p^0 = A_p^p = l_2$$

$$A_p^\infty = l_{p\infty} \qquad \text{for } 2 \le p < \infty.$$

1. Notation and preliminary results. By l_p we denote the space of sequences (α_i) with

$$\textstyle\sum_{i=1}^{\infty} |\alpha_i|^p < \infty$$

and $l_{p\infty}$ is the space of sequences (α_i) such that

$$\sup|\alpha_i^*|\, i^{1/p} < \infty$$

where (α_i^*) is the monotone rearrangement of (α_i) (i.e., $|\alpha_1^*| \ge |\alpha_2^*| \ge \cdots$). The space $L_p\,(0 < p < \infty)$ consists of all random variables f with pth moment, that is

$$\mathbf{E}|f|^p < \infty.$$

We denote by L_p^0 the subspace of L_p, which consists of all random variables f with $\mathbf{E}\, f = 0$ $(p \ge 1)$. L_0 is the space of all measurable functions. We define for a random variable f the so-called tail function $\tilde{F}_f$ by

$$\tilde{F}_f(x) = P\{f \ge x\}, \qquad x \in R_1.$$

Received March 20, 1979; revised March, 1980.
AMS 1970 *subject classifications.* Primary, 60G50; secondary, 60E05, 46E30.
Key words and phrases. Identically distributed random variables, convergence of weighted sums a.e., existence of sums in L_p, sequences of real numbers.

We now give some elementary lemmas.

LEMMA 1. *Let $1 \le p < q < 2$. Then*

$$l_p \subseteq A_p^p \subsetneq A_p^0 \subseteq A_p^\infty \subseteq l_q.$$

In the case $2 \le p < \infty$

$$A_p^p = A_p^0 = l_2.$$

PROOF. The inclusions

$$A_p^p \subseteq A_p^0 \subseteq A_p^\infty \qquad \text{for} \quad 1 \le p < \infty$$

are known (cf. [5]). Moreover if $1 \le p \le 2$ by [7] it follows that

$$\mathbf{E} \,|\sum_{i=n}^m \alpha_i f_i|^p \le \sum_{i=n}^m |\alpha_i|^p \,\mathbf{E}\,|f_i|^p \le c \sum_{i=n}^m |\alpha_i|^p.$$

Thus $l_p \subseteq A_p^p$. Let $1 \le p < q < 2$. Taking for the f_i's q-stable random variables we obtain from [4]

$$A_p^\infty \subseteq l_q.$$

In the case $2 \le p < \infty$ we get $l_2 \subseteq A_p^p$ from the inequality

$$(\mathbf{E}\,|\sum_{i=n}^m \alpha_i f_i|^p)^{1/p} \le K_p \max((\sum_{i=n}^m |\alpha_i|^p \,\mathbf{E}\,|f_i|^p)^{1/p}, \sum_{i=n}^m |\alpha_i|^2 \,\mathbf{E}\,|f_i|^2)^{1/2})$$

(Theorem 3 [7]). Taking for (f_i) a Bernoulli sequence Khinchin's inequalities assert

$$A_p^0 \subseteq l_2.$$

Thus the lemma is proved.

LEMMA 2. *If the sequence (α_i) belongs to A_p^0, A_p^p, or A_p^∞, then for any permutation π of the natural numbers $(\alpha_{\pi(i)})$ also belongs to A_p^0, A_p^p or A_p^∞, respectively.*

PROOF. We only give the proof for the space A_p^0. The other cases are analogous. Let $(\alpha_i) \in A_p^0$ and $(f_i) \in \Phi_p$. Then for any natural numbers k, l the inequality

$$P\{|\sum_{i=k}^l \alpha_{\pi(i)} f_i| > \vartheta\} \le 2\, P\{|\sum_{i=k_\pi}^{l_\pi} \alpha_i f_i| > \vartheta\}$$

is valid, where

$$k_\pi = \inf_{k \le i \le l} \quad \pi(i) \qquad \text{and} \qquad l_\pi = \sup_{k \le i \le l} \quad \pi(i)$$

(Theorem 2.3 [5]). Thus by our assumption $(\sum_{i=1}^n \alpha_{\pi(i)} f_i)$ is a fundamental sequence in L_0. The required result follows from the completeness of L_0 and Theorem 2.4 of [5].

Because the α_i's tend to zero (this follows by taking Gaussian random variables) we will henceforth assume that $(|\alpha_i|)$ is monotonely decreasing.

LEMMA 3. *Let f be a nonnegative random variable. Then $\mathbf{E}\, f^p < \infty$ if and only if $\sum_{i=0}^\infty \tilde{F}_f(i^{1/p}) < \infty$. Moreover*

$$\mathbf{E}\, f^p \le \sum_{i=0}^\infty \tilde{F}_f(i^{1/p}) \le \mathbf{E}\, f^p + 1.$$

This lemma can be proved by standard methods.

2. Characterization of the spaces A_p^0 and A_p^∞. To characterize the spaces A_p^0 and A_p^∞ we need

LEMMA 4. *Let* (g_i) *be a sequence of independent symmetric random variables and let* (ζ_i) *be a bounded sequence of real numbers. Then the existence of* $\sum_{i=1}^{\infty} g_i$ a.e. *implies the existence of* $\sum_{i=1}^{\infty} \zeta_i g_i$ a.e.

For the proof see [6]. We are now prepared to prove

THEOREM 1. *If* $1 \le p < 2$ *then*

$$l_{p\infty} \subseteq A_p^0.$$

PROOF. It is only necessary to prove the theorem in the special case $\alpha_i = i^{-1/p}$, $i = 1, 2, \cdots$. The general case follows from Lemma 4 by setting

$$g_i = i^{-1/p} f_i \qquad \text{and } \zeta_i = \alpha_i i^{1/p}.$$

The special case can be proved by the Kolmogoroff's 3-series test (cf. [3]) and is a known result of Marcinkiewicz and Zygmund (see [2], page 115).
Before we can give the characterization of the space A_p^{∞} we need the following lemma.

LEMMA 5. *Let* (α_i) *be a sequence of* A_p^{∞}. *Then*

$$\lim \alpha_i f_i = 0 \text{ a.e.} \qquad \textit{for all} \quad (f_i) \in \Phi_p.$$

PROOF. Fix $(\alpha_i) \in A_p^{\infty}$.
(1) First we prove that there exists a constant c with

$$\limsup |\alpha_i f_i| \le c(\mathbf{E}|f_1|^p)^{1/p} \text{ a.e.}$$

for all $(f_i) \in \Phi_p$($\limsup |\alpha_i f_i|$ is a constant a.e.). On the space L_p^0 we define a *quasinorm* by

$$\zeta(f) = \max((\mathbf{E}|f|^p)^{1/p}, \zeta_0(\sup|\alpha_i f_i|)),$$

where (f_i) is a sequence of independent random variables, identical distributed as f, and ζ_0 is the usual quasinorm on L_0 ([8], page 38). By standard methods one can prove that L_p^0 with this quasinorm is an F-space (see [8], page 52). Then the quasinorm ζ is equivalent to the usual norm on L_p (closed graph theorem). Thus we get that

$$\lim_{n\to\infty} \mathbf{E}|f^n|^p = 0$$

implies

$$\lim_{n\to\infty} \zeta_0(\sup_i|\alpha_i f_i^n|) = 0.$$

This yields

$$\limsup|\alpha_i f_i| \le c(\mathbf{E}|f_1|^p)^{1/p} \text{ a.e.}$$

for all $(f_i) \in \Phi_p$ with some constant $c \ge 0$.
(2) In the second step of the proof we show

$$\lim \alpha_i f_i = 0 \text{ a.e. for all } (f_i) \in \Phi_p.$$

Assume there is a sequence $(g_i) \in \Phi_p$ with $\mathbf{E}|g_1|^p \le 1$, which does not satisfy the condition $\lim \alpha_i g_i = 0$ a.e. Then there exists a constant t_0 with $0 < t_0 < c$, such that $\limsup|\alpha_i g_i| > t_0$, where c is the same constant as above.
We define a new sequence $(f_i) \in \Phi_p$ by

$$P\{f_1 \ge t\} = (t_0/c)^p P\{g_1 \ge (t_0/c)t\}, \qquad t \in (0, \infty)$$

and

$$P\{f_1 = 0\} = 1 - P\{f_1 > 0\}.$$

Obviously $\mathbf{E}|f_1|^p = \mathbf{E}|g_1|^p \leq 1$. For each natural number n we obtain

$$\begin{aligned}\textstyle\sum_{i=1}^n P\{|\alpha_i g_i| > t_0\} &= \textstyle\sum_{i=1}^n (c/t_0)^p P\{|\alpha_i f_i| > c\} \\ &\leq (c/t_0)^p \textstyle\sum_{i=1}^\infty P\{|\alpha_i f_i| > c\} < \infty.\end{aligned}$$

This contradicts our assumption $\limsup |\alpha_i g_i| > t_0$ thus proving Lemma 5.

THEOREM 2. *Let $1 \leq p < \infty$. Then*

$$l_{p\infty} = A_p^\infty.$$

PROOF. By the Borel-Cantelli lemma the supremum is finite if and only if the sum

$$\textstyle\sum_{i=1}^\infty P\{|\alpha_i f_i| > t\}$$

exists for some $t > 0$ (cf. [1]). Then Lemma 3 yields the existence of the supremum for a sequence (α_i) of $l_{p\infty}$. Thus

$$l_{p\infty} \subseteq A_p^\infty$$

holds.

We now show that the converse inclusion is true. We assume that the sequence (α_i) does not belong to $l_{p\infty}$. Then there exists a sequence of natural numbers (i_k) having the following properties:

(1) $4^p\, i_k \leq i_{k+1}$

(2) $|\alpha_{i_k}| \geq k/(i_k^{1/p})$ $\qquad k = 1, 2, \ldots.$

If

$$\beta_k = i_k^{1/p}/k \qquad k = 1, 2, \cdots$$

then condition 1 ensures $\beta_k \to \infty$. Using (i_k) we construct a tail function $\tilde{F}$.

$$\begin{aligned}\tilde{F}(t) &= \frac{1}{2k(i_k - i_{k-1})} && \text{for } \beta_{k-1} < t \leq \beta_k && k = 2, 3, \cdots \\ &= \frac{1}{2} && \text{for } -\beta_1 < t \leq \beta_1 \\ &= 1 - \frac{1}{2k(i_k - i_{k-1})} && \text{for } -\beta_k < t \leq -\beta_{k-1} && k = 2, 3, \cdots .\end{aligned}$$

A random variable f with this tail function is symmetrical and possesses a pth moment because of (we define $\beta_0 = i_0 = 0$)

$$\begin{aligned}\textstyle\sum_{i=1}^\infty \tilde{F}(i^{1/p}) &= \textstyle\sum_{k=1}^\infty \sum_{\beta_{k-1}^p < i \leq \beta_k^p} \tilde{F}(i^{1/p}) \\ &\leq 2 \textstyle\sum_{k=1}^\infty \tilde{F}(\beta_k)(\beta_k^p - \beta_{k-1}^p) \\ &\leq 2\, \beta_1^p\, \tilde{F}(\beta_1) + \textstyle\sum_{k=2}^\infty \frac{1}{k(i_k - i_{k-1})}\left(\frac{i_k}{k^p} - \frac{i_{k-1}}{(k-1)^p}\right) \\ &\leq \beta_1^p + \textstyle\sum_{k=2}^\infty \frac{1}{k^{p+1}} < \infty.\end{aligned}$$

By the monotonicity of the sequence $(|\alpha_i|)$ and the second condition

$$\tilde{F}(|\alpha_i|^{-1}) \geq \tilde{F}(i_k^{1/p}/k) \qquad \text{for} \quad i_{k-1} < i \leq i_k$$

holds. Hence it follows

$$\begin{aligned}\textstyle\sum_{i=1}^\infty \tilde{F}(|\alpha_i|^{-1}) &\geq \textstyle\sum_{k=1}^n \tilde{F}(i_k^{1/p}/k)(i_k - i_{k-1}) \\ &\geq \textstyle\sum_{k=1}^n 1/2k\end{aligned}$$

for each natural number n. By the Lemma 5 the required result follows.

By Lemma 1, Theorems 1 and 2 the spaces A_p^0 and A_p^∞ are completely characterized.

THEOREM 3. *The following relations are valid:*

$$A_p^0 = A_p^\infty = l_{p\infty} \qquad 1 \le p < 2$$
$$A_p^0 = l_2 \qquad 2 \le p < \infty$$
$$A_p^\infty = l_{p\infty}.$$

3. Characterization of the spaces A_p^p. Before we can give the characterizations of the spaces A_p^p we need

LEMMA 6. *Let (α_i) be a sequence of A_p^p. Then there exists a constant c such that for each sequence $(f_i) \in \Phi_p$ the inequality*

$$\mathbf{E}\,|\textstyle\sum_{i=1}^\infty \alpha_i f_i|^p \le c\,\mathbf{E}\,|f_1|^p$$

holds.

PROOF. Fix $(\alpha_i) \in A_p^p$. By symmetrization one can prove that $\sum_{i=1}^\infty \alpha_i f_i$ exists in L_p for all sequences (f_i) of independent identical distributed random variables with pth moment and $\mathbf{E}\, f_1 = 0$. Thus we define on L_p^0 a norm by

$$\|f\| = (\mathbf{E}\,|\textstyle\sum_{i=1}^\infty \alpha_i f_i|^p)^{1/p}.$$

Then L_p^0 with this norm is complete and thus the norm is equivalent to the usual norm on L_p (closed graph theorem). This proves our lemma.

THEOREM 4. *The following relations are true:*

$$A_p^p = l_p \qquad \textit{for} \quad 1 \le p < 2$$
$$A_p^p = l_2 \qquad \textit{for} \quad 2 \le p < \infty.$$

PROOF. The case $2 \le p < \infty$ was proved in Lemma 1, also

$$A_p^p \supseteq l_p \qquad \text{for} \quad 1 \le p < 2.$$

We now show the converse inclusion. For an arbitrary fixed natural number n we regard the sequence $(f_i^n) \in \Phi_p$ which is distributed with

$$P\{f_i^n = 0\} = 2^{-1/n}$$

and
$$P\{f_i^n = -\,b_n^{1/p}\} = P\{f_i^n = -b_n^{1/p}\} = \frac{1}{2}\,(1 - 2^{-1/n})$$

where
$$b_n = 2^{1/n}(2^{1/n} - 1)^{-1}.$$

Obviously $\mathbf{E}\,|f_i^n|^p = 1$.

By Lemma 6 there is a constant c with

$$c \ge 2\,\mathbf{E}\,|\textstyle\sum_{i=1}^\infty \alpha_i f_i^n|^p + 1 \quad \text{for any } n.$$

Corollary 3.2 of [5] implies

$$c \ge \mathbf{E}(\sup_i |\alpha_i f_i^n|)^p + 1$$
$$\ge \mathbf{E}(\sup_{i\le n} |\alpha_i f_i^n|)^p + 1.$$

Applying Lemma 3 we get

$$c \ge \textstyle\sum_{j=0}^\infty P\{\sup_{i\le n} |\alpha_i f_i^n| > j^{1/p}\}$$
$$\ge \textstyle\sum_{j=0}^\infty (P\{|\alpha_1 f_1^n| > j^{1/p}\} + \sum_{i=2}^n P\{|\alpha_i f_i^n| > j^{1/p}, \sup_{k\le i-1} |\alpha_k f_k^n| \le j^{1/p}\}).$$

Since the random variables f_i^n are independent we obtain

$$c \geq |\alpha_1|^p + \textstyle\sum_{j=0}^{\infty} \sum_{i=2}^{n} P\{|\alpha_i f_i^n| > j^{1/p}\} \prod_{k=1}^{i-1} P\{|\alpha_k f_k^n| \leq j^{1/p}\}$$

$$\geq |\alpha_1|^p + \frac{1}{2} \textstyle\sum_{i=2}^{n} \sum_{j=0}^{\infty} P\{|\alpha_i f_i^n| > j^{1/p}\}$$

$$\geq |\alpha_1|^p + \frac{1}{2} \textstyle\sum_{i=2}^{n} \mathbf{E}|\alpha_i f_i^n|^p$$

$$\geq \frac{1}{2} \textstyle\sum_{i=1}^{n} |\alpha_i|^p.$$

This proves Theorem 4.

Acknowledgment. I'd like to thank W. Linde for many discussions and remarks, which simplified the original version of this paper. I would like to express my gratitude to the referee for his helpful comments.

REFERENCES

[1] CHOBANJAN, S. A., LINDE, W. and TARIELADZE, V. I. (1978). A probabilistic characterization of absolutely summing operators. *Mat. Zametki.* To appear.

[2] CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory.* Springer, Berlin, Heidelberg, New York.

[3] DUDLEY, R. M. (1969). Random linear functionals. *Trans. Amer. Math. Soc.* **136** 1–24.

[4] GARLING, D. J. H. (1977). Sums of Banach space valued random variables. Lecture notes, Dept. of Math., Ohio State Univ., Columbus, Ohio.

[5] HOFFMANN-JØRGENSEN, J. (1975). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

[6] JAIN, N. C. and MARCUS, M. B. (1975). Integrability of infinite sums of independent vector valued random variables. *Trans. Amer. Math. Soc.* **212** 1–36.

[7] ROSENTHAL, H. P. (1970). On the subspaces of $L_p(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.* **8** 273–303.

[8] YOSIDA, K. (1965). *Functional Analysis.* Springer, Berlin, Göttingen, Heidelberg.

FRIEDRICH-SCHILLER-UNIVERSITAET
SEKTION MATHEMATIK
69 JENA DDR
GERMAN DEMOCRATIC REPUBLIC