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CHARACTERIZING ALL DIFFUSIONS WITH THE $2M - X$ PROPERTY

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If $(X_t)_{t\geq 0}$ is a Brownian motion on the real line, started at zero, if $M_t \equiv \max\{X_s;\, s \leq t\}$ and if $Y_t \equiv 2M_t - X_t$ for $t \geq 0$, then $(Y_t)_{t\geq 0}$ is a homogeneous strong Markov process equal in law to the radial part of Brownian motion in three dimensions. This result was discovered by Pitman, and recently Rogers and Pitman have found other one-dimensional diffusions X for which $2M - X$ is again a diffusion. This paper gives a complete characterisation of all such diffusions X.

1. Introduction. Let Ω be the set of continuous functions from $[0, \infty)$ to $\mathbb{R}$ and for each $t \geq 0$ define the mappings X_t, M_t and Y_t from Ω to $\mathbb{R}$, and Z_t from Ω to $[0, \infty)^2$ by

$$X_t(\omega) \equiv \omega(t),$$

$$M_t(\omega) \equiv \max\{X_s(\omega);\quad 0 \leq s \leq t\},$$

$$Y_t(\omega) \equiv 2M_t(\omega) - X_t(\omega),$$

$$Z_t(\omega) = (M_t(\omega) - X_t(\omega),\, M_t(\omega)).$$

For each $t \geq 0$, define the σ-fields $\mathscr{F}_t$ and $\mathscr{G}_t$ of subsets of Ω by $\mathscr{F}_t \equiv \sigma(\{X_s;\, 0 \leq s \leq t\})$ and $\mathscr{G}_t \equiv \sigma(\{Y_s;\, 0 \leq s \leq t\})$. It is clear that $\mathscr{G}_t \subseteq \mathscr{F}_t$. Defining $\mathscr{F} \equiv \sigma(\{X_s;\, s \geq 0\})$, we let $\mathscr{C}$ be the collection of probability measures P on $(\Omega, \mathscr{F})$ such that

(i) $P(X_0 = 0) = 1$;

(ii) under P, $(X_t, t \geq 0)$ is a regular conservative diffusion on an interval $I \subseteq \mathbb{R}$, started at 0;

(iii) under P, $(Y_t, t \geq 0)$ is a time homogeneous strong Markov process with respect to $\{\mathscr{G}_t\}$.

A diffusion X started at 0 is said to have the $2M - X$ property if its law is an element of $\mathscr{C}$.

This situation was first considered by Pitman (1975), who showed that Wiener measure is in $\mathscr{C}$ (and, in that case, Y is a BES(3) process), and more recently, Rogers and Pitman (1981) have found other measures in $\mathscr{C}$, notably the law of drifting Brownian motion (whether the drift be upward or downward) and a Brownian motion with mixed drift. These results were of interest because they led immediately to the path decompositions of the fundamental paper of Williams (1974), and it was natural to ask whether there were any other elements of $\mathscr{C}$ which might also yield striking decompositions of the paths of one-dimensional diffusions. The purpose of this paper is to prove that there are essentially no more elements of $\mathscr{C}$ than those which appeared in Rogers and Pitman (1981). After setting up notation and proving some simple preliminary results, we prove two results of central importance (Lemma 3 and Theorem 1) from which we calculate the possible scale functions and speed measures of elements of $\mathscr{C}$. Theorem 1 tells us that, whenever $(2M_t - X_t)_{t\geq 0}$ is Markov, the sufficient condition of Theorem 2 of Rogers and Pitman (1981) is operating. The main result, Theorem 2, gives a complete description of $\mathscr{C}$.

2. Notation and preliminary results. Statement (ii) in the definition of $\mathscr{C}$ should be understood in the sense of Freedman (1971); there exists a family $\{P^x;\, x \in I\}$ of probability measures on $(\Omega, \mathscr{F})$ such that (i) $P = P^0$ (ii) for each $x \in I$, $P^x(X(0) = x$,

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$X_t \in I \ \forall\, t \geq 0) = 1$ (iii) for each $y \in I$, $x \in I^0$, $P^x(X_t = y$ for some $t > 0) > 0$ (iv) for each $\{\mathscr{F}_{t+}\}$-optional time T, and each $x \in I$, the conditional P^x-distribution of $(X_{T+t})_{t\geq 0}$ given $\mathscr{F}_{T+}$ is P^{X_T} on $\{T < \infty\}$. We further require that for some $x \in I^0$, $P^0(X_t = x$ for some $t > 0)$ > 0, to exclude the trivial diffusion which remains at 0 for all time. It is well known (see, for example Freedman (1971)) that a regular conservative diffusion is determined by its scale function s and its speed measure m; this characterisation is essential to the proof of the main result, since we shall identify elements of $\mathscr{C}$ through scale and speed.

For any $\{\mathscr{F}_t\}$-adapted process $(W_t,\ t \geq 0)$, with continuous paths, define the $\{\mathscr{F}_t\}$-optional times

$$\tau_x(W) \equiv \inf\{t;\ W_t = x\} \qquad x \in \mathbb{R}$$

$$\tau_{ab}(W) \equiv \inf\{t;\ W_t \notin (a, b)\} \qquad a < b \in \mathbb{R},$$

where $\inf \varnothing \equiv \infty$. Where convenient, $\tau_x(W)$ (respectively $\tau_{ab}(W)$) will be shortened to τ_x (respectively τ_{ab}).

For each $\alpha, \beta > 0$, define the map $S_{\alpha,\beta}: \Omega \to \Omega$ by

$$(S_{\alpha,\beta}\omega)(t) = \alpha\omega(\beta t).$$

Clearly, if $P \in \mathscr{C}$, then $P \circ S_{\alpha,\beta}^{-1} \in \mathscr{C}$.

For each $\xi > 0$, we define the measure $\Lambda(\xi, \cdot)$ on Borel subsets of $[0, \infty)^2$, concentrated on $\{(u, v);\ u + v = \xi\}$, by

$$\Lambda(\xi, A) \equiv P(Z_{\tau_\xi(Y)} \in A,\ \tau_\xi(Y) < \infty) \tag{1}$$

and for $\xi = 0$, $\Lambda(\xi, \cdot)$ is a unit mass on $(0, 0)$. We shall use the notation for $\xi, x \geq 0$

$$\begin{aligned} \Lambda_\xi(x) &\equiv \Lambda(\xi, [0, \infty) \times [0, x]) = P(M_{\tau_\xi(Y)} \leq x,\ \tau_\xi(Y) < \infty) \\ \bar{\Lambda}_\xi(x) &\equiv P(\tau_\xi(Y) < \infty) - \Lambda_\xi(x). \end{aligned} \tag{2}$$

We shall later see (Corollary 1) that $\tau_\xi(Y) < \infty$ for all $\xi > 0$ P-a.s. whenever $P \in \mathscr{C}$. This makes $\Lambda(\cdot, \cdot)$ into a Markov kernel from $[0, \infty)$ to $[0, \infty)^2$ because $\tau_x(Y) \uparrow \tau_\xi(Y)$ as $x \uparrow \xi$ P-a.s., and it follows, since Z is continuous, that if f: $[0, \infty)^2 \to \mathbb{R}$ is bounded continuous, then the function

$$t \to \Lambda f(t) \equiv \int \Lambda(t, dy) f(y)$$

is left continuous, so measurable. Standard monotone class arguments now extend the measurability to $\Lambda f(\cdot)$ for arbitrary bounded measurable f.

Notice that, if ϕ: $[0, \infty)^2 \to [0, \infty)$ by $\phi(u, v) = u + v$, and if Φ is the Markov kernel from $[0, \infty)^2$ to $[0, \infty)$ defined by $\Phi((u, v), A) = I_A(u + v)$, then the kernel Λ satisfies

$$\Lambda\Phi = I,$$

which is condition (5a) of Theorem 2 in Rogers and Pitman (1981).

Our analysis begins with the remark that, since $0 \in I$, $\gamma \equiv \sup\{x;\ x \in I\}$ is nonnegative. Let $\mathscr{C}_0$ be the set of probability measures on $(\Omega, \mathscr{F})$ under which $(X_t,\ t \geq 0)$ is a diffusion on some interval $I \subseteq (-\infty, 0]$, starting at 0. Thus if $P \in \mathscr{C}_0$, $M_t = 0$ for all t P-a.s. and $Y_t = -X_t$ for all t P-a.s., so $P \in \mathscr{C}$. $\mathscr{C}_0$ is a subset, albeit a dull one, of $\mathscr{C}$. Henceforth, we assume that $\gamma > 0$.

LEMMA 1. *If $P \in \mathscr{C}$ and $\gamma \equiv \sup\{x;\ x \in I\} > 0$, then* $\inf\{x;\ x \in I\} = -\infty$.

REMARK. If $\alpha \equiv \inf\{x;\ x \in I\} > -\infty$, then the value of M at the time $\tau_\xi(Y)$ when Y first reaches $\xi > 0$ must be at least ½ $(\xi + \alpha) \vee 0$, so whatever level $c < \xi$ the Y process subsequently dropped to, we would know that it could never go below ½ $(\xi + \alpha)$ — but it is possible for Y to reach level c without going up to ξ, and, if ξ, c are chosen suitably, it is possible for Y to drop below $\frac{1}{2}(\xi + \alpha)$ after $\tau_c(Y)$. This contradicts the strong Markov property of Y.

PROOF. Pick some $b \in (0, \gamma \wedge (-\alpha))$, and let $\xi \equiv 2b - \alpha$. Now choose $\epsilon \in (0, \frac{1}{4}b)$ so small that $c \equiv b + 2\epsilon \in (0, \gamma \wedge (-\alpha))$. Thus $\alpha < \alpha + 2\epsilon < 0 < \frac{1}{2}b < \frac{1}{2}b + \epsilon < b - \epsilon < b < c \equiv b + 2\epsilon < \gamma \wedge (-\alpha) < \xi$, and all of these real numbers, except perhaps the first and the last two, lie in I^0. Consider now the $\{\mathscr{G}_t\}$-optional times

$$T_0 \equiv \tau_\xi(Y),$$

$$T_1 \equiv \inf\{t > T_0;\ Y_t = c\},$$

$$T_2 \equiv \tau_c(Y),$$

and the $\mathscr{F}$-measurable events

$$A_1 \equiv \{\omega;\ \tau_{b+\epsilon}(X) < \tau_{\alpha+2\epsilon}(X) < \tau_c(X) < \infty\},$$

$$A_2 \equiv \{\omega;\ \tau_{b-\epsilon}(X) < \infty, X_t < 0 \quad \text{for some} \quad \tau_{\epsilon+b/2}(X) < t < \tau_{b-\epsilon}(X)\}.$$

By the regularity of X, and the fact that $\alpha + 2\epsilon$, $b + 2\epsilon = c$ are in I^0, it follows that $P(A_1)\ P(A_2) > 0$, and for every $\omega \in A_1$, the path of $Y.(\omega)$ rises to ξ, and subsequently drops back to c, so $P(T_1 < \infty) \geq P(A_1) > 0$.

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$$Y_{T_0} = \xi = 2M_{T_0} - X_{T_0} \leq 2M_{T_0} - \alpha \quad \text{on } \{T_0 < \infty\}$$

so $M_{T_0} \geq b$ on $\{T_0 < \infty\}$, and by the strong Markov property of Y

$$\begin{aligned} (3) \qquad & P(T_1 < \infty,\ Y_{T_1+t} = b - \epsilon \quad \text{for some} \quad t > 0) = 0 \\ &= P(T_1 < \infty)\cdot P(Y_{T_2+t} = b - \epsilon \quad \text{for some} \quad t > 0 \mid T_2 < \infty) \end{aligned}$$

since $Y_{T_1} = c = Y_{T_2}$ wherever T_1 and T_2 are finite.

On the other hand, for each $\omega \in A_2$, the path of $Y.(\omega)$ rises to at least level c, and then drops down to $b - \epsilon$, so, again by the strong Markov property of Y,

$$\begin{aligned} 0 < P(A_2) &\leq P(T_2 < \infty,\ Y_{T_2+t} = b - \epsilon \quad \text{for some} \quad t > 0) \\ &= P(T_2 < \infty)\cdot P(Y_{T_2+t} = b - \epsilon \quad \text{for some} \quad t > 0 \mid T_2 < \infty) \end{aligned}$$

which contradicts (3). Hence $\alpha = -\infty$.

Thus the interval I must be one of $(-\infty, \gamma)$ or $(-\infty, \gamma]$ for some $0 < \gamma < \infty$, or else the whole of $\mathbb{R}$.

PROPOSITION 1. *If $P \in \mathscr{C}$, and $\gamma > 0$, then for each $b > 0$, $P(\tau_b(Y) < \infty) > 0$, and if $a > 0$, $P(Y_{\tau_b+t} = a$ for some $t \geq 0 \mid \tau_b = \tau_b(Y) < \infty) > 0$.*

PROOF. Let $\beta \equiv \frac{1}{2}(\gamma \wedge a)$, $\alpha \equiv -(b \vee a)$, both in I^0. Regularity of X implies $P(A) > 0$, where A is the event $\{\tau_{\alpha\beta}(X) = \tau_\alpha(X),\ \tau_\beta(X) < \infty\}$, and for each $\omega \in A$ the path of $Y.(\omega)$ visits b, and later a.

LEMMA 2. *Suppose $P \in \mathscr{C}$ and $\gamma > 0$, and let $(\Omega', (\mathscr{F}'_t, t \geq 0), \mathscr{F}')$ be a copy of $(\Omega, (\mathscr{F}_t, t \geq 0), \mathscr{F})$. Then there exists a family $\{Q^x; x > 0\}$ of probability measures on $(\Omega', \mathscr{F}')$ such that*

(i) *$\{Q^x; x > 0\}$ is a regular diffusion on $(0, \infty)$;*
(ii) *The P-distribution of $(Y_{\tau_x+t}, t \geq 0)$ conditional on $\{\tau_x(Y) < \infty\}$ is the same as the Q^x-distribution of $(X'_t, t \geq 0)$, for each $x > 0$.*

PROOF. Fix $x > 0$, let $\Omega_x \equiv \{\omega \in \Omega; \tau_x(Y) < \infty\} \in \mathscr{F}$ and define $\phi: \Omega_x \to \Omega'$ by $\phi(\omega)(t) = Y_{\tau_x+t}(\omega)$ for $t \geq 0$, $\omega \in \Omega_x$. Then $\phi^{-1}(\mathscr{F}'_t) \subseteq \mathscr{G}_{\tau_x+t}$ for each $t \geq 0$, ϕ is measurable, and we define Q^x on $(\Omega', \mathscr{F}')$ by $Q^x(A) = P(\phi^{-1}(A) \mid \tau_x(Y) < \infty)$. By the above proposition, $P(\tau_x(Y) < \infty) > 0$, so this is well-defined, and condition (ii) is satisfied, by construction. To check the first condition, the regularity of the diffusion follows from Proposition 1 also, and only the strong Markov property need be checked. But if T' is a $\{\mathscr{F}'_{t+}\}$-optional time, $\tau_x(Y) + T' \circ \phi$ is a $\{\mathscr{G}'_{t+}\}$-optional time, and the strong Markov property of $(X'_t, t \geq 0)$ follows from the strong Markov property of $(Y_t, t \geq 0)$.

COROLLARY 1. *For each* $x > 0$, $\tau_x(Y) < \infty$ P-a-s.

PROOF. Since $\{Q^x; x > 0\}$ is a regular diffusion on $(0, \infty)$, it follows for each $n \in N$, $Q^x(\tau_{1/n,n}(X') < \infty) = 1$ (see Freedman (1971) page 112) so, by Lemma 2, using the Markov property of $(Y_t, t \geq 0)$ at the time 1, and the fact that $M_1 > 0$ P-a.s. we learn that

$$P(\text{for large enough } n, \exists\, t_n \geq 1 \text{ s.t. either } Y_{t_n} = n \text{ or } Y_{t_n} = n^{-1}) = 1.$$

But $\inf\{Y_t; t \geq 1\} \geq M_1 > 0$ P-a.s., so there can be only finitely many $n \in N$ such that $Y_{t_n} = n^{-1}$ for some $t_n \geq 1$, P-a.s.

COROLLARY 2. *There exists a continuous strictly increasing function* ρ: $[0, \infty) \to [0, \infty)$ *with* $\rho(0) = 0$, $\lim_{t\to\infty}\rho(t) = \infty$ *and, for* $0 < a < b$,

$$Q^b(\tau_a(X') < \infty) = \rho(a)/\rho(b).$$

PROOF. Let σ be the scale function of the diffusion $\{Q^x; x > 0\}$. For any $0 < a < b$, $Q^a(\tau_b(X') < \infty) = 1$ by Corollary 1, so $\lim_{\epsilon\downarrow 0} \sigma(\epsilon) = -\infty$ and, since it is easy to see that there exists $0 < a < b$ s.t. $Q^b(\tau_a(X') < \infty) < 1$, it must be that $\lim_{t\to\infty} \sigma(t) < \infty$, so without loss of generality $\lim_{t\to\infty} \sigma(t) = 0$. Now let $\rho(t) \equiv -\sigma(t)^{-1}$ for $t > 0$, $\rho(0) = 0$.

3. Two key results. As explained earlier, the following results play a central role, particularly Theorem 1. Proposition 2 is a technical result, a springboard to the all-important Theorem 1.

PROPOSITION 2. *Suppose* $B > 0$, *and* r *is a continuous strictly decreasing function from* $(-\infty, B]$ *to* $[0, \infty)$. *If* G *is a right-continuous function of bounded variation on* $[0, B]$ *such that* $G(0) = 0$ *and*

$$\int_{(0,\,a]} G(du)\, r(2u - a) = 0 \qquad 0 \leq a < B,$$

then $G(u) = 0$ *for* $0 \leq u < B$.

PROOF. For $0 \leq a < B$, write $r_a(u) \equiv r(2u - a)$ $(u \leq a)$. We are told that for $a \in [0, B)$,

$$0 = \int_{(0,a]} G(du) r_a(u) = G(a)r(a) - \int_0^a r_a(du) G(u)$$

$$= G(a)r(a) - \int_{-a}^{a} r(dy) G\left(\frac{a+y}{2}\right)$$

so

$$\int_{-a}^{a} -r(dy) G\left(\frac{a+y}{2}\right) = -G(a)r(a) \qquad 0 \leq a < B.$$

Suppose $\sup\{|G(x)|; x \in [0, B)\} = \delta > 0$. Then for $0 \le \epsilon < \delta$, we define

$$a_\epsilon \equiv \inf\{t; |G(t)| > \epsilon\}.$$

It follows that $a_\epsilon < B$ and for $\epsilon > 0$

$$\begin{aligned} 0 < r(a_\epsilon)\cdot\epsilon \le |r(a_\epsilon)G(a_\epsilon)| &= \left|\int_{-a_\epsilon}^{a_\epsilon} - r(dy)G\left[\frac{a_\epsilon + y}{2}\right]\right| \\ &\le \int_{2a_0 - a_\epsilon}^{a_\epsilon} - r(dy)\cdot\epsilon \\ &= \epsilon(r(2a_0 - a_\epsilon) - r(a_\epsilon)). \end{aligned}$$

Hence $2r(a_\epsilon) \le r(2a_0 - a_\epsilon)$. Now let $\epsilon \downarrow 0$; $a_\epsilon \downarrow a_0$ and by continuity of r we get $0 < 2r(a_0) \le r(a_0)$ which is impossible. Thus $G(u) = 0$ for $0 \le u < B$.

The next result is stated in slightly greater generality than we need.

Lemma 3. *Suppose I is an interval of $\mathbb{R}$ with* $\sup\{x; x \in I\} \equiv \gamma > 0$, $\inf\{x; x \in I\} = -\infty$, *and suppose $\{P^x; x \in I\}$ is any regular diffusion on I, with scale function s. Then for each $b > 0$, taking $P = P^0$, $\bar{\Lambda}_b$ has the explicit form*

$$\begin{aligned} \bar{\Lambda}_b(u) &= \exp\left[-\int_0^u s(dy)\{s(y) - s(2y - b)\}^{-1}\right] \\ &\quad - P(\tau_b(Y) = \infty) \quad \text{for} \quad 0 \le u < b \wedge \gamma \\ &= 0 \qquad\qquad\qquad\quad\;\; \text{for} \quad u \ge b \wedge \gamma. \end{aligned} \tag{4}$$

Proof. We begin by noting that, since $\gamma > 0$ and the diffusion is regular, $\Lambda_b(0) = 0$.

It is plain from the definitions of Λ_b and γ that $\bar{\Lambda}_b(u) = 0$ if $u \ge b \wedge \gamma$, so we have only got to account for Λ_b on $[0, b \wedge \gamma)$. Fix $b > 0$, and let $t \in (0, b \wedge \gamma)$. By the strong Markov property of Z at the P-a.s. finite $\{\mathscr{F}_t\}$-optional time $\tau_b(Y) \wedge \tau_{-b,t}(X)$ we see that

$$\begin{aligned} P(\tau_{-b,t}(X) = \tau_{-b}(X)) &\equiv (s(t) - s(0))/(s(t) - s(b)) \\ &= \int_{(0,t)} \Lambda_b(dy)P^{2y-b}(\tau_{-b,t}(X) = \tau_{-b}(X)) \\ &= \int_{(0,t)} \Lambda_b(dy)(s(t) - s(2y - b))/(s(t) - s(b)) \end{aligned}$$

whence

$$s(t) - s(0) = s(t)\,\Lambda_b(t-) - \int_{(0,t)} \Lambda_b(dy)\,s(2y - b). \tag{5}$$

Now this is true for all $0 \le t < b \wedge \gamma$, and the left side of (5) is a continuous function of t in this range, and the jump of the right side at $t \in (0, b \wedge \gamma)$ is $\{s(t) - s(2t - b)\}\Delta\Lambda_b(t) = 0$. Strict monotonicity of s implies that $\Delta\Lambda_b(t) = 0$ so Λ_b is continuous on $(0, b \wedge \gamma)$. Now differentiate each side of (5) to learn that

$$s(dt)(1 - \Lambda_b(t)) = (s(t) - s(2t - b))\,\Lambda_b(dt). \tag{6}$$

The unique bounded solution to (6) satisfying $\Lambda_b(0) = 0$ is given by

$$1 - \Lambda_b(t) = \exp\left[-\int_0^t s(du)\{s(u) - s(2u - b)\}^{-1}\right] \qquad 0 \le t < b \wedge \gamma$$

from which (4) follows.

REMARKS. If $b < \gamma$, $\tau_b(Y) \le \tau_{-b,b}(X) < \infty$ P-a.s., simplifying (4). Perhaps a little surprisingly, it is possible for Λ_b to be discontinuous at b if $b < \gamma$, though it must of course be continuous elsewhere, by (4) and continuity of s. As an example of such behaviour, take $\gamma = \infty$, fix $b > 0$ and let the diffusion on $\mathbb{R}$ have scale function

$$s(x) = -e^{-(b-x)^{-1}}(x < b),\ s(x) = e^{-(x-b)^{-1}}(x > b),\ s(b) = 0.$$

Henceforth, we shall suppose that $P \in \mathscr{C}$, $\gamma > 0$, and that s is the scale function of the regular diffusion $(X_t, t \ge 0)$. We have to consider separately two cases:

(I) $s(-\infty) > -\infty$. In this case, we suppose $s(-\infty) = 0$, $s(0) = 1$, so that for $a < b < \gamma$,

$$P^a(\tau_b(X) < \infty) = s(a)/s(b).$$

(II) $s(-\infty) = -\infty$. In this case, $P^a(\tau_b(X) < \infty) = 1$ for all $a < b < \gamma$.

THEOREM 1. *Suppose $P \in \mathscr{C}$ and $\gamma > 0$. Let T be a $\{\mathscr{G}_t\}$-optional time. Then on $\{T < \infty\}$, $\Lambda(Y_T, \cdot)$ is a regular conditional P-distribution for Z_T given $\mathscr{G}_T$.*

PROOF. Define the process $(V_t, t \ge 0)$ by $V_t \equiv \inf\{Y_s; s \ge t\}$ and use the notation $E(U; \Lambda)$ for $E(UI_\Lambda)$ for any $U \in L^1$ and $\mathscr{F}$-measurable Λ. We consider firstly case I, where $s(-\infty) = 0$, $s(0) = 1$.

Let U be a bounded $\mathscr{G}_T$-measurable random variable, and let $b > a > 0$. Then

$$E(U;\ V_T \le a,\ T < \infty,\ Y_T > b) = E(UQ^{Y_T}(V_0 \le a);\ T < \infty,\ Y_T > b), \tag{7}$$

by the strong Markov property of $(Y_t, t \ge 0)$. But by the strong Markov property of $(Z_t, t \ge 0)$,

$$\begin{aligned} E(U;\ V_T \le a,\ T < \infty,\ Y_T > b) &= E\left(UP(V_T \le a \mid \mathscr{F}_T);\ T < \infty,\ Y_T > b\right) \\ &= E\left(UI_{\{M_T \le a\}} \frac{s(2M_T - Y_T)}{s(2M_T - a)};\ T < \infty,\ Y_T > b\right) \qquad (8) \\ &= E\left(UE\left(I_{\{M_T \le a\}} \frac{s(2M_T - Y_T)}{s(2M_T - a)} \mid \mathscr{G}_T\right);\ T < \infty,\ Y_T > b\right). \end{aligned}$$

Comparing (7) and (8), since $U \in \mathscr{G}_T$ was arbitrary, we conclude that, on $\{T < \infty,\ Y_T > b\}$,

$$E\left(I_{\{M_T \le a\}} \frac{s(2M_T - Y_T)}{s(2M_T - a)} \mid \mathscr{G}_T\right) = Q^{Y_T}(V_0 \le a) \quad \text{P-a.s.} \tag{9}$$

If we let $(P|\mathscr{G}_T)$: $\Omega \times \mathscr{B}([0, \infty)) \to [0, 1]$ be a regular conditional P-distribution for $M_T I_{\{T<\infty\}}$ given $\mathscr{G}_T$, then from (9) and the definition of $\{Q^y; y > 0\}$

$$\int_{(0, a\wedge\gamma]} (P|\mathscr{G}_T)(du) \frac{s(2u - Y_T)}{s(2u - a)} = \int_{(0, a\wedge\gamma]} \Lambda_{Y_T}(du) \frac{s(2u - Y_T)}{s(2u - a)} \tag{10}$$

P-a.s. on $\{T < \infty,\ Y_T > b\}$.

Thus there exists a P-null subset $N \in \mathscr{G}_T$ where (10) fails for some rational $a < b$, and right continuity in a of each side of (10) implies that, for $\omega \notin N$, (10) holds for all $0 \le a < b$. By Proposition 2, taking $r \equiv s^{-1}$, $(P|\mathscr{G}_T)((0, x]) = \Lambda_{Y_T}(x)$ for all $\omega \notin N$, and $0 \le x < b \wedge \gamma$, and hence we conclude that for all ω not in some P-null set N',

$$(P|\mathscr{G}_T)((0, x]) = \Lambda_{Y_T}(x) \qquad \text{for all} \quad 0 \le x < Y_T \wedge \gamma \quad \text{on} \quad \{T < \infty\}. \tag{11}$$

This proves the result in case I; in case II, $V_t = M_t\ \forall t \ge 0$ P-a.s., so the argument is even easier. The details are left to the reader.

COROLLARY. *Z_T is conditionally independent of $\mathscr{G}_T$ given Y_T, with $\Lambda(Y_T, \cdot)$ providing a regular conditional distribution for Z_T given $\mathscr{G}_T$.*

4. The scale function. Here we exploit Lemma 3 and Theorem 1 to determine possible scale functions. As before, we suppose $\gamma > 0$, $P \in \mathscr{G}$, s is the scale function of X under P, and we consider the two cases separately.

Case I; $s(-\infty) = 0$, $s(0) = 1$. We define the positive continuous strictly decreasing function $r: I \to (0, \infty)$ by $r(t) \equiv s(t)^{-1}$. The analysis begins from the crucial, but obvious, equality valid for $0 < a \le b$,

$$\rho(a)/\rho(b) = \int_{(0, a \wedge \gamma]} \Lambda_b(du) s(2u - b)/s(2u - a) \tag{12}$$

which holds since each side is equal to $P(Y_{\tau b + t} = a$ for some $t > 0)$. Notice that we can deduce from (12) and (4) that, if $\gamma < \infty$, then $s(\gamma-) = \infty$, and so $\gamma \notin I$; by fixing $b > \gamma$ in (12), and letting $a \uparrow \gamma$, monotone convergence and continuity of ρ imply that Λ_b is continuous at γ, so, since $s(y) - s(2y - b)$ is bounded away from 0 on $[0, \gamma]$, (4) and continuity of Λ_b imply $s(\gamma-) = \infty$. Now we apply Theorem 1 using the $\{\mathscr{G}_T\}$-optional time $T \equiv \inf\{t > \tau_b(Y); Y_t = a\}$. For $x \le a \wedge \gamma$,

$$\begin{aligned} P(T < \infty, M_T \le x) &= P(T < \infty) \cdot P(M_T \le x \mid T < \infty) \\ &= P(T < \infty) \Lambda_a(x) \\ &= \rho(a) \rho(b)^{-1} \Lambda_a(x) \\ &= \int_0^x \Lambda_b(du) s(2u - b)/s(2u - a), \end{aligned} \tag{13}$$

the last line coming from considering the value of M at $\tau_b(Y)$. Now using the differential equation (6) satisfied by Λ_a, Λ_b, we learn that

$$\frac{s(du)\bar{\Lambda}_b(u)}{s(u) - s(2u - b)} \cdot \frac{s(2u - b)}{s(2u - a)} = \frac{s(du)\bar{\Lambda}_a(u)}{s(u) - s(2u - a)} \cdot \frac{\rho(a)}{\rho(b)}, \quad 0 \le u < a \wedge \gamma,$$

and since s is strictly increasing, we deduce that

$$\frac{\bar{\Lambda}_b(u)\rho(b)}{r(2u - b) - r(u)} = \frac{\bar{\Lambda}_a(u)\rho(a)}{r(2u - a) - r(u)}. \qquad 0 \le u < a \wedge \gamma. \tag{14}$$

Put another way, for any $0 < B < \gamma$ and for each $0 \le u \le B$,

$$\frac{\bar{\Lambda}_b(u)\rho(b)}{r(2u - b) - r(u)} \quad \text{is the same for all} \quad b < B. \tag{15}$$

By taking $u = 0$ in (15), we see that there exists a nonzero constant c such that

$$\rho(b) = c\{r(-b) - 1\} \qquad \text{for all} \quad b \ge 0.$$

Returning to (4), monotone convergence implies that for each $0 \le u < \gamma$

$$\lim_{b \to \infty} \bar{\Lambda}_b(u) = r(u)/r(0) = r(u) \in (0, 1].$$

This implies, returning to (15), that for each $0 \le u < \gamma$,

$$\lim_{b \to \infty} \frac{r(-b) - 1}{r(2u - b) - r(u)} = \lim_{b \to \infty} \frac{r(-b)}{r(2u - b)} \equiv h(u), \quad \text{say},$$

exists, is finite and ≥ 1, and h is an increasing function of $u \in [0, \gamma)$. Moreover, if u_1, u_2 and $u_1 + u_2$ are all in $[0, \gamma)$, we see that

$$h(u_1)h(u_2) = h(u_1 + u_2)$$

so there exists $\mu \geq 0$ such that $h(u) = e^{4\mu u}$ $(0 \leq u < \gamma)$. Thus for any $0 \leq u < b \wedge \gamma$,

(16) $$\bar{\Lambda}_b(u)(r(-b) - 1) = e^{4\mu u} r(u)(r(2u - b) - r(u)).$$

Taking logs, using (4) and the notation $r_b(u) \equiv r(2u - b)$, we obtain after rearrangement

$$4\mu u = \int_0^u (2r(dv) - r_b(dv))(r_b(v) - r(v))^{-1} \qquad 0 \leq u < b \wedge \gamma.$$

Hence

(17) $$2r(u) - r(2u - b) = \int_0^u 4\mu(r_b(v) - r(v))dv + 2 - r(-b),$$

valid for $0 \leq u < b \wedge \gamma$. r is a continuous strictly decreasing function on $(-\infty, \gamma)$ and as such is differentiable a.e. If $0 \leq u < \gamma$, pick some $b_0 \in (u, \gamma)$ such that r is differentiable at $2u - b_0$. Then, from (17), r is differentiable at u. Similarly, we conclude that r is differentiable in $(-\infty, \gamma)$, so, differentiating (17) with respect to u gives

$$2r'(u) - 2r'(2u - b) = 4\mu(r(2u - b) - r(u)) \qquad 0 \leq u < b \wedge \gamma$$

and by setting $c \equiv b - u$, we have that

(18) $$r'(u) - r'(u - c) = -2\mu(r(u) - r(u - c)) \qquad 0 \leq u < \gamma,\ c > 0.$$

Accordingly, since $r(0) = 1$,

(19) $$r(u) - r(u - c) = (1 - r(-c))e^{-2\mu u} \qquad 0 \leq u < \gamma,\ c > 0$$

so, dividing either side by c and letting $c \downarrow 0$, we conclude that

$$r'(u) = r'(0)e^{-2\mu u} \qquad 0 \leq u < \gamma,$$

and this implies that

(20) $$\begin{aligned} r(u) &= A_\mu + B_\mu e^{-2\mu u} \qquad (0 \leq u < \gamma) \quad \text{if} \quad \mu > 0, \\ r(u) &= A_0 + B_0 u \qquad (0 \leq u < \gamma) \quad \text{if} \quad \mu = 0. \end{aligned}$$

By using (18), we can extend r to the whole of $(-\infty, \gamma)$ and the requirements that $r(0) = 1$ and $r(\gamma-) = 0$ when $\gamma < \infty$ determine the constants A_μ and B_μ. We can conclude that there are essentially only three possibilities in case I.

(Ia) $\gamma < \infty$, $\mu > 0$, and for $x \in (-\infty, \gamma)$

$$r(x) \equiv s(x)^{-1} = (e^{2\mu(\gamma - x)} - 1)(e^{2\mu\gamma} - 1)^{-1}.$$

In this case, for $0 \leq x < b \wedge \gamma$, $\Lambda_b(x) = r(u - b)r(\gamma - u)/r(\gamma - b)$;

(Ib) $\gamma < \infty$, $\mu = 0$, and for $x \in (-\infty, \gamma)$

$$r(x) = \gamma^{-1}(\gamma - x).$$

In this case, for $0 \leq x < b \wedge \gamma$, $\Lambda_b(x) = x(\gamma + b - x)/\gamma b$;

(Ic) $\gamma = \infty$, there exists, some $0 < q \leq 1$, with $p \equiv 1 - q$, such that

$$r(x) = p + qe^{-2\mu x}.$$

In this case, for $0 \leq x < b$, $\Lambda_b(x) = r(x)(e^{2\mu x} - 1)r(b)^{-1}(e^{2\mu b} - 1)^{-1}$.

Case II. As with Case I, we begin with the equation analogous to (12), which is even simpler in this case:

(21) $$\rho(a)/\rho(b) = \int_{(0, a \wedge \gamma]} \Lambda_b\,(du) \qquad 0 < a \leq b.$$

Strict monotonicity of ρ implies immediately that $\gamma = \infty$ is the only possibility in this case, and that $\Lambda_b(a) = \rho(a)/\rho(b)$ for $0 < a \le b$. From (6),

$$\rho(da) = \Lambda_b(da) \cdot \rho(b) = s(da) \frac{\rho(b) - \rho(a)}{s(a) - s(2a - b)} \qquad 0 < a < b$$

whence we conclude that, for each $a > 0$,

(22) $\qquad (\rho(b) - \rho(a))/(s(a) - s(2a - b)) \qquad$ is the same for every $b > a$.

Letting $a \downarrow 0$ shows that for some positive β,

$$\rho(b)(s(0) - s(-b))^{-1} = \beta \qquad \forall\, b > 0.$$

Substituting into (22) gives for each $a > 0$, $b > a$, that

$$\frac{s(-a) - s(-b)}{s(a) - s(2a - b)} = \lambda_a, \qquad \text{a constant depending only on } a.$$

Writing $\xi \equiv b - a$, this says that for all $\xi > 0$,

(23) $$s(-\xi - a) = \lambda_a s(-\xi + a) + s(-a) - \lambda_a s(a).$$

Writing c for $s(-a) - \lambda_a\, s(a)$, considering ξ and a as fixed and writing $x_n \equiv s(-\xi - (2n - 1)a)$ for each $n = 0, 1, 2, \ldots$, it follows from (23) that

$$x_{n+1} = \lambda_a x_n + c.$$

Either $\lambda_a = 1$, when $x_n = x_0 + nc$, or $\lambda_a \ne 1$, when $x_n = c' + \lambda_a^n(x_0 - c')$, $c' \equiv c(1 - \lambda_a)^{-1}$. Holding ξ fixed and successively halving a, we deduce from the continuity of s that, at least on $(-\infty, -\xi]$, in the first case s is linear, and in the second case, $s(x) = A + Be^{-2\mu x}$ for some $A, B, \mu \in \mathbb{R}$. Since $a > 0$ was arbitrary, we deduce that this holds throughout $\mathbb{R}$, so in case II, there are but two possibilities:

(IIa) $s(x) = x$ $\qquad (\Lambda_b(x) = x/b$ for $0 \le x < b)$;

(IIb) $s(x) = -e^{-2\mu x}$ $\qquad$ for some $\mu > 0$ $(\Lambda_b(x) = (e^{2\mu x} - 1)(e^{2\mu b} - 1)^{-1}$ for $0 \le x < b)$.

REMARKS. IIa gives the scale function of Brownian motion, IIb the scale function of Brownian motion with drift $\mu > 0$ upward and Ic, with $q = 1$, gives the scale function of Brownian motion with drift $\mu > 0$ downward. If $0 < q < 1$, we get the Brownian motion with the mixture of drifts μ upward and μ downward: this process is discussed in detail in Rogers and Pitman (1981)—it has generator $(½)D^2 + \mu \tanh(\mu x + \theta)D$, where $pq^{-1} = e^{2\theta}$. Ia and Ib are also the scale functions of diffusions considered in Rogers and Pitman (1981); if $(R_t, t \ge 0)$ is the radial part of a three-dimensional Brownian motion with drift of magnitude $\mu \ge 0$, and if $T \equiv \tau_\gamma(R)$, where $\gamma > 0$, then the process $(\gamma - R_{T+t}, t \ge 0)$ is a diffusion on $(-\infty, \gamma)$ starting at zero. If $\mu > 0$, the scale function of this diffusion is given by Ia, and if $\mu = 0$, its scale function is given by Ib. It is proved in Rogers and Pitman (1981) that for each of these diffusions, $(Y_t, t \ge 0)$ is a strong Markov process. Now our task is to prove that there can be no diffusion with one of these scale functions, but a different speed measure, for which $(Y_t, t \ge 0)$ is a strong Markov process.

5. The speed measures. In this section, we use Theorem 1 to identify possible speed measures for elements of $\mathscr{C}$. The main results on speed measures appear in Freedman (1971), and we state here briefly the relevant properties, in a form best suited to the present application.

Let $\{P^x; x \in I\}$ be a regular diffusion with scale function s on interval I. For $a < b \in I^0$, define the Green's function G: $[a, b] \times [a, b] \to [0, \infty)$ by

(24) $$G_{ab}(x, y) = (s(b) - s(a))^{-1}(s(x) - s(a))(s(b) - s(y)) \quad \text{for} \quad a \le x \le y \le b$$
$$= (s(b) - s(a))^{-1}(s(y) - s(a))(s(b) - s(x)) \quad \text{for} \quad a \le y \le x \le b.$$

There exists a strictly increasing right continuous function $m: I^0 \to \mathbb{R}$ such that for any $a < x < b \in I^0$,

$$E^x(\tau_{ab}(X)) = \int_a^b m(dy) G_{ab}(x, y), \tag{25}$$

and

$$\begin{aligned} E^x(\tau_{ab}(X); \tau_b(X) < \tau_a(X)) &= \int_a^b m(dy) G_{ab}(x, y) P^y(\tau_b < \tau_a) \\ &= \int_a^b m(dy) G_{ab}(x, y)(s(y) - s(a))(s(b) - s(a))^{-1}. \end{aligned} \tag{26}$$

REMARK. If we define the functions h_1 and h_2 on I^0 by $h_1(x) = \int_a^x s(dy) m(y)$ and $h_2(x) = \int_a^x m(dy)(s(x) - s(y))(s(y) - s(a))$, the strong Markov property of $(X_t, t \geq 0)$ implies that (25) and (26) are equivalent (respectively) to the statements that $(h_1(X_t) - t, t \geq 0)$ and $(h_2(X_t) - (s(X_t) - s(a))t, t \geq 0)$ are local martingales. Martingales such as these have been considered by Arbib (1965).

Let us now combine (25) and (26) with Theorem 1 to give the result we seek. Firstly, we shall consider Case I, where $s(-\infty) = 0$, $s(0) = 1$. As before, Case II is a lot easier. The idea is to exploit the fact that M_T is conditionally independent of $\mathscr{G}_T$ given Y_T; this means, in particular, that looking at the time spent by $(Y_t, 0 \leq t \leq T)$ in various regions should tell us no more about the value of M_T than we would learn by looking just at Y_T. So let us fix $0 < a < x < b$, and define the $\{\mathscr{G}_t\}$-optional times T_1 and T_2 by

$$T_1 \equiv \inf\{t > \tau_x(Y); \quad Y_t \notin (a, b)\}$$
$$T_2 \equiv \inf\{t > \tau_x(Y); \quad Y_t = a\}.$$

By Theorem 1, if $0 < u < a \wedge \gamma$,

$$E(T_1 - \tau_x(Y); T_2 < \infty, M_{T_2} \leq u) = E(T_1 - \tau_x(Y); T_2 < \infty)\, \Lambda_a(u) \tag{27}$$

and, by the strong Markov property of $(Z_t, t \geq 0)$ at $\tau_x(Y)$, the left side of (27) is equal to

$$\begin{aligned} \int_0^u \Lambda_x(dv) \{ & E^{2v-x}(\tau_{2v-b,2v-a}; \tau_{2v-a} < \tau_{2v-b}) \\ & + E^{2v-x}(\tau_{2v-b,2v-a}; \tau_{2v-b} < \tau_{2v-a}) s(2v-b)/s(2v-a)\}. \end{aligned} \tag{28}$$

In each of cases Ia, Ib and Ic, the right side of (27) is a C^∞ function of $u \in (0, a \wedge \gamma)$ so, differentiating (27) and (28) with respect to u, and using the notation $c_1 \equiv E(T_1 - \tau_x(Y); T_2 < \infty)$, $\alpha \equiv 2u - a$, $\xi \equiv 2u - x$, $\beta \equiv 2u - b$, we have

$$\begin{aligned} c_1 \frac{\Lambda_a'(u)}{\Lambda_x'(u)} s(\alpha) &= s(\alpha) E^\xi(\tau_{\beta\alpha}; \tau_\alpha < \tau_\beta) + s(\beta) E^\xi(\tau_{\beta\alpha}; \tau_\beta < \tau_\alpha) \\ &= \int_\beta^\alpha m(dy) G_{\beta\alpha}(\xi, y) s(y). \end{aligned} \tag{29}$$

Now in each of cases Ia, Ib and Ic, there is some constant c_2 (depending on a, x and b, but not on u) such that

$$\Lambda_a'(u)/\Lambda_x'(u) = c_2\, s(\xi)/s(\alpha). \tag{30}$$

Putting (29) and (30) together gives that for some c independent of u,

$$cs(\xi)(s(\alpha)-s(\beta)) = (s(\alpha)-s(\xi))\int_\beta^\xi m(dy)(s(y)-s(\beta))s(y) \qquad (31)$$
$$+ (s(\xi)-s(\beta))\int_\xi^\alpha m(dy)(s(\alpha)-s(y))s(y).$$

From here, the three cases proceed their separate, but similar, ways. In each case we differentiate either side of (31) with respect to u, using the differential equations satisfied by s (in case Ia, take $r(x) = e^{2\mu(\gamma-x)} - 1$, when $s' = 2\mu s(s+1)$; in case Ib, take $r(x) = \gamma - x$, when $s' = s^2$; in case Ic, taking $s = 1 + \tanh(\mu x + \theta)$ where $pq^{-1} \equiv e^{2\theta}$, gives $s' = \mu s(2-s)$). Then we eliminate c from (31) and its derivative. As an example, in case Ib, differentiating (31) with respect to $u \in (0, a \wedge \gamma)$ then eliminating c yields

$$\int_\xi^\alpha s(y)^2 m(dy)s(\alpha)(s(\alpha)-s(\xi))^{-1} = \int_\beta^\xi s(y)^2 m(dy)s(\beta)(s(\beta)-s(\xi))^{-1}. \qquad (32)$$

Now if we regard u, x and b as fixed, the right side of (32) is independent of a, so the left side of (32) is the same for every $u < a < x$, or, put another way, there is some finite positive constant k_1 such that for all $\alpha \in (\xi, u)$,

$$\int_\xi^\alpha s(y)^2 m(dy) = k_1(s(\alpha)-s(\xi))s(\alpha)^{-1}. \qquad (33)$$

From (33), we deduce that m is differentiable, and differentiating and substituting for s gives the result that for some finite positive constant k, for all $\alpha \in (\xi, u)$,

$$m(d\alpha) = k(\gamma-\alpha)^2\, d\alpha.$$

The extension to $\alpha \in (-\infty, \gamma)$ is routine, and cases Ia and Ic are similarly proven; in each case, m is absolutely continuous with respect to Lebesgue measure, with density proportional to $\sinh^2\mu(\gamma - x)$ in case Ia, proportional to $\cosh^2(\mu x + \theta)$ in Ic if $q < 1$, and proportional to $e^{-2\mu x}$ in Ic if $q = 1$.

Turning now to Case II, we consider the same two stopping times T_1 and T_2, for which (27) is, of course, still valid, but the equation (28) following from the strong Markov property of $(Z_t, t \geq 0)$ collapses to

$$\int_0^u \Lambda_x(dv)E^{2v-x}(\tau_{2v-b,2v-a}), \qquad (34)$$

and the analogue of (29) becomes

$$\lambda = E^\xi(\tau_{\beta\alpha}) \equiv \int_\beta^\alpha m(dy)G_{\beta\alpha}(\xi, y),$$

where λ is constant depending on a, b and x but not u—in this case, the ratio $\Lambda_a'(u)/\Lambda_x'(u)$ does not depend on u. A few calculations now prove that the only possible measures in Case IIa are multiples of Lebesgue measure, and, in Case IIb, multiples of the measure with density $e^{2\mu x}$ with respect to Lebesgue measure.

REMARKS. For each of the five possible types of scale function discovered in Section 3, we have proved that there is, to within a multiplicative constant, only one speed measure corresponding to each scale function which might give rise to an element of $\mathscr{C}$. That these measures must be the speed measures of the corresponding diffusions discussed in Rogers and Pitman (1981) follows from the fact proven in that paper that these diffusions are in $\mathscr{C}$; that they are the speed measures is a comfort.

Finally, we state what we have proved.

THEOREM 2. *Using the notation of Rogers and Pitman* (1981), *the elements of* $\mathscr{C}$ *are precisely the following*:

all elements of $\mathscr{C}_0$, *the set of laws on* $(\Omega, \mathscr{F})$ *under which* $(X_t, t \geq 0)$ *is a nonpositive diffusion started at* 0;

all measures obtained by one of the deterministic transformations $S_{\alpha\beta}$ *of time and space from one of the following*:

(i) *the measure on* $(\Omega, \mathscr{F})$ *under which* $(X_t, t \geq 0)$ *is Brownian motion*;

(ii) *the measure on* $(\Omega, \mathscr{F})$ *under which* $(X_t - t, t \geq 0)$ *is Brownian motion*;

(iii) *the measure on* $(\Omega, \mathscr{F})$ *under which* $(X_t + t, t \geq 0)$ *is Brownian motion*;

(iv) *any convex combination of the measures* (ii) *and* (iii);

(v) *the measure on* $(\Omega, \mathscr{F})$ *under which* $(\gamma - X_t, t \geq 0)$ *is a* $\mathrm{BES}^{\gamma}(3)$ *process, where* γ *is any positive real*;

(vi) *the measure on* $(\Omega, \mathscr{F})$ *under which* $(\gamma - X_t, t \geq 0)$ *is a* $\mathrm{BES}^{\gamma}(3, 1)$ *process, where* γ *is any positive real.*

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