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# LAPLACE FUNCTIONAL APPROACH TO POINT PROCESSES OCCURRING IN A TRAFFIC MODEL

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This paper deals with a wide class of point processes which are subsumed under the name of $z$-processes. These processes are generalizations, in the sense that the initial distribution of the vehicles are not necessarily stationary Poisson, of point processes occurring in a traffic model of Rényi (1964). Using the Laplace functional, we derive the distributions of various $z$-processes when the initial process is stationary Poisson and prove a weak convergence result to the doubly stochastic Poisson process when the initial process is not necessarily Poisson distributed.

**1. Introduction.** Rényi (1964) proposed the following simple model for traffic flow on an infinite straight road without any intersections. Vehicles appear at an arbitrary fixed point on the road, usually taken as the point zero on the real line, at the sequence of random instants $\{\tau_i\}$, $i \geq 1$, distributed as a stationary Poisson process. The $i$th vehicle then proceeds to travel along the road at the constant velocity $v_i$, the assumption here being that the random variables $v_i$ are independent and identically distributed and are also independent of the sequence $\{\tau_i\}$. The trajectories of the above model in the time-road diagram are straight lines, and the points of intersection between the lines give the times and positions where overtakings occur. Such overtakings, known as "free overtakings," can only be done "without delay" in the sense that there can be no change in velocity as one vehicle approaches another prior to overtaking. If $\{\tau_i^+\}$ are the instants when a fixed vehicle overtakes other vehicles with lower velocities and $\{\tau_i^-\}$ the instants when faster vehicles overtake the fixed vehicle, then provided we consider overtakings by vehicles entering the road at any time $t > 0$, Rényi showed in the same paper that $\{\tau_i^+\}$ and $\{\tau_i^-\}$ are distributed as two independent Poisson processes.

Brown (1969a, b), using an invariance property of the Poisson process, derived other processes from the traffic model of Rényi besides $\{\tau_i^+\}$ and $\{\tau_i^-\}$. Amongst these are the positions of the vehicles at any time $t$ and the instants at which vehicles pass a given point on the highway. Solomon and Wang (1972), using a geometrical approach based on the notion of a Poisson field of random lines, also derived the distributions of the above processes.

On close inspection, it is clear that most of the derived processes mentioned above are in fact point processes on the real line where the position of the $i$th point is given by

$$z_i = a(v_i)\alpha_i + b(v_i, t), \qquad t > 0.$$

Here $a(v) \neq 0$ and $b(v, t)$ are some measurable functions, $\{\alpha_i\}$ is a realization of an initial point process, and the random variables $v_i$ are assumed to be independent and identically distributed. Point processes generated in this manner will be called *z-processes* in the sequel.

The objective of this paper is to apply the Laplace functional method in considering $z$-processes. We are motivated here by Kallenberg (1975), who has shown how the Laplace functional can be used to provide a compact formulation of many important properties of random measures and point processes, as well as a useful tool in proving weak convergence results.

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In Section 2, we present some basic definitions and preliminary results. The reader is referred to Jagers (1974), Kallenberg (1975) and Kerstan *et al* (1974) for details. In Section 3, the Laplace functional of the general $z$-processes is derived, and we used it to obtain the distribution of some $z$-processes. When the initial process is neither Poisson nor mixed Poisson distributed, the resultant $z$-process will in general not have a Poisson distribution, so in Section 4 we give a necessary and sufficient condition under which a sequence of $z$-processes converges in distribution to a subordinated Poisson process (also called a Cox or doubly stochastic Poisson process). When $a(v) = 1$, our results also subsume some classical results on random translations (for example, Doob (1953), Goldman (1967) and Thedeen (1964)).

**2. Preliminaries.** Denote by $\mathscr{B}$ and $\mathscr{B}^b$ the Borel subsets and the ring of bounded Borel subsets of the real line $R$ respectively. Let $M$ be the set of measures on $\mathscr{B}$ that are finite on $\mathscr{B}^b$ and $m$ the smallest $\sigma$-algebra of subsets of $M$ such that the mappings $\mu \to \mu(A)$, $A \in \mathscr{B}^b$, are measurable. Let $N$ be the subset of $M$ consisting of the non-negative, integer-valued measures of the form $\mu = \sum_{i=1}^{k} \delta_{\alpha_i}$ ($\delta_\alpha(A) = 1_A(\alpha)$, where $1_A$ is the indicator function of the set $A$), $k \in \bar{Z}_+$, and $\{\alpha_i\}$ is a sequence of points in $R$ without any limit points.

Denote by $K$ and $K_c$ the set of non-negative, bounded and measurable functions on $R$ with bounded supports, and the subset of $K$ consisting of the continuous functions respectively. In the sequel, we define $\mu(f)$, $\mu \in M$ and $f \in K$, as the integral $\int f\, d\mu$. Let the topology on $M$ be the *vague topology*, that is, the coarsest topology making the mappings $\mu \to \mu(f)$, $f \in K_c$, continuous. Under the vague topology, $M$ is a Polish space, and the Borel subsets generated by the open sets are identical to $m$. Also, $N$ is a vaguely closed subset of $M$.

A measurable mapping $\xi$ from a probability space $(\Omega, \mathscr{A}, P)$ into $(M, m)$ is a *random measure*. If the range of $\xi$ is contained in $N$ a.s., it is a *stochastic point process* (s.p.p.). The *Laplace functional* (L.f.) of a s.p.p. $\xi$ is the mapping from $K$ into $R_+ (R_+ = [0, \infty))$ defined by

$$\mathscr{L}_\xi(f) = E(e^{-\xi(f)}), \qquad f \in K. \tag{1}$$

The L.f. of a s.p.p. $\xi$ determines the distribution of $\xi$ uniquely. For $\mu \in M$, the Poisson s.p.p. $\xi$ with mean measure $\mu$ is defined as the s.p.p. with L.f.

$$\mathscr{L}_\xi(f) = \exp(-\mu(1 - e^{-f})), \qquad f \in K. \tag{2}$$

If the measure $\mu$ in (2) is a random measure, we obtain by mixing a *subordinated Poisson process $\xi$ directed by $\mu$* with L.f.

$$\mathscr{L}_\xi(f) = \mathscr{L}_\mu(1 - e^{-f}), \qquad f \in K. \tag{3}$$

Using the same notation as Kallenberg (1975), we denote a subordinated Poisson process directed by $\mu$ as SP($\mu$).

We note that the $k$th moment ($k = 1, 2, \cdots$) of a s.p.p. $\xi$ on a set $A \in \mathscr{B}$, $E(\xi(A)^k)$, is obtained by applying the formulae:

$$E(\xi(A)^k) = (-1)^k \left\{ \frac{d^k}{ds^k} \mathscr{L}_\xi(s1_A) \right\}_{s=0} \tag{4}$$

where $s$ is a non-negative real variable. If the *first moment measure* $\nu_\xi$ of a s.p.p. $\xi$ is absolutely continuous with respect to the Lebesgue measure $\lambda$, the corresponding Radon-Nikodym derivative $d\nu_\xi/d\lambda = i_\xi$ will be referred to as the *intensity* of $\xi$.

Finally, a sequence of random measures $\xi_n$ *converges in distribution* to a random measure $\xi$, written $\xi_n \to_d \xi$, if the corresponding distribution of $\xi_n$ *converges weakly* to the distribution of $\xi$ (Billingsley (1968)). Let $\partial A$, $A \in \mathscr{B}$, denotes the boundary of a set $A$, and for any random measure $\xi$, define $\mathscr{B}_\xi^b = \{B \in \mathscr{B}^b : \xi(\partial B) = 0 \text{ a.s.}\}$. Then $\mathscr{B}_\xi^b$ is a ring (see Lemma 4.3 in Kallenberg (1975)) and contains the semi-ring $\mathscr{I}_\xi$ of half open intervals. The equivalence of the following statements is well known: (i) $\xi_n \to_d \xi$, (ii) $\mathscr{L}_{\xi_n}(f) \to \mathscr{L}_\xi(f)$, $f \in K_c$, and (iii) $(\xi_n(B_1), \cdots, \xi_n(B_k)) \to_d (\xi(B_1), \cdots, \xi(B_k))$ for each finite sequence $B_1, \cdots, B_k$ in $\mathscr{B}^b_\xi$. In fact, it suffices that (iii) holds for pairwise disjoint sets in $\mathscr{I}_\xi$. We note that the above convergence criterion for random measures also hold for sequence of s.p.p.

**3. Laplace Functional of $z$-processes and some applications.** For each $t > 0$, let $T_t$ be the measurable mapping from $R \times R_+$ into $R$ defined by $T_t(\alpha, v) = a(v)\alpha + b(v, t)$ where $a(v) \neq 0$ and $b(v, t)$ are some measurable functions. Let $\xi_t$ be the $z$-process generated from an initial s.p.p. $\eta$, i.e. a typical realization of $\xi_t$ consists of a sequence of points in $R$ with the $i$th point occupying the position $z_i = T_t(\alpha_i, v_i)$ where $\sum_{i=1}^k \delta_{\alpha_i}$ is a realization of $\eta$. Here $\{v_i\}$ is a sequence of non-negative, independent and identically distributed random variables with distribution $G_t$, independent of $\eta$. For each fixed $t$, we shall assume that the functions $a(v)$, $b(v, t)$ are continuous a.s. and are such that

$$\nu_\eta \times G_t(T_t^{-1}(B)) < \infty, \qquad B \in \mathscr{B}^b. \tag{5}$$

It is routine to check, using the following theorem and (4), that (5) is nothing more than the usual assumption that a $z$-process $\xi_t$ has an a.s. finite number of points in any bounded set.

THEOREM 1. *Let $\xi_t$ be the $z$-process generated from a* s.p.p. $\eta$ *and define*

$$g_t^f(\alpha) = \int_0^\infty \exp(-f \circ T_t(\alpha, v))\, dG_t(v), \qquad f \in K, \tag{6}$$

*where $f \circ T_t$ denotes the composition between $f$ and $T_t$. Then*

$$\mathscr{L}_{\xi_t}(f) = \mathscr{L}_\eta(-\ln g_t^f). \tag{7}$$

PROOF. For any fixed non-random $\eta = \sum_{i=1}^k \delta_{\alpha_i}$,

$$\mathscr{L}_{\xi_t}(f) = \textstyle\prod_{i=1}^k E(\exp(-f \circ T_t(\alpha, v))) = \exp(\eta(\ln g_t^f))$$

where the expectation is taken with respect to $G_t$. Therefore, by mixing

$$\mathscr{L}_{\xi_t}(f) = E(\exp(\eta(\ln g_t^f))) = \mathscr{L}_\eta(-\ln g_t^f). \qquad \square$$

From (2), if $\eta$ is Poisson with mean measure $\mu$, the resulting $z$-process has L.f.

$$\mathscr{L}_{\xi_t}(f) = \exp\left(-\int_{R\times R_+} (1 - e^{-f\circ T_t(\alpha, v)})\, d(\mu \times G_t)\right) \tag{8}$$

and more generally from (3), if $\eta$ is SP($\mu$),

$$\mathscr{L}_{\xi_t}(f) = \mathscr{L}_\mu(1 - g_t^f). \tag{9}$$

COROLLARY 1. *Let $\eta$ be a stationary Poisson process with intensity $\beta > 0$ and consider only outcomes of $\eta$ on some fixed interval $(\gamma_1, \gamma_2)$. Set*

$$T_t(\gamma_{1^+}, v) = \lim_{\alpha\downarrow\gamma_1} T_t(\alpha, v), \qquad T_t(\gamma_{2^-}, v) = \lim_{\alpha\uparrow\gamma_2} T_t(\alpha, v),$$
$$h_t(v) = \max\{T_t(\gamma_{1^+}, v), T_t(\gamma_{2^-}, v)\}, \qquad d_t(v) = \min\{T_t(\gamma_{1^+}, v), T_t(\gamma_{2^-}, v)\},$$

*and $h_t = \sup_v h_t(v)$, $d_t = \inf_v d_t(v)$. Then the $z$-process generated from $\eta$ is Poisson with intensity*

$$i_t(z) = \beta \int_{A_t^z} dG_t(v)/|\,a(v)\,| \tag{10}$$

*with outcomes in the interval* $(d_t, h_t)$. *Here*

(11) $$A_t = \{(v, z): d_t(v) < z < h_t(v), \quad 0 < v < \infty\}$$

*and* $A_t^z = \{v: v \in R_+, (v, z) \in A_t\}$.

PROOF. From (8), we have

$$\mathscr{L}_{\xi_t}(f) = \exp\left(-\beta \int_{\gamma_1}^{\gamma_2} \int_0^\infty (1 - e^{-f\circ T_t(\alpha, v)})\, d\lambda(\alpha)\, dG_t(v)\right), \qquad f \in K.$$

By the translation invariance of $\lambda$, monotonicity of $T_t(\alpha, v)$ in $\alpha$ for fixed $v$, and Fubini's theorem

$$\mathscr{L}_{\xi_t}(f) = \exp\left(-\beta \int_0^\infty \int_{d_t(v)}^{h_t(v)} (1 - e^{-f(z)})\, d\lambda(z - b(v, t))/a(v)\right) dG_t(v)$$

(12) $$= \exp\left(-\beta \int_0^\infty \int_{d_t(v)}^{h_t(v)} (1 - e^{-f(z)})\, d\lambda(z)\, dG_t(v)/| a(v) |\right)$$

$$= \exp\left(-\beta \int_{d_t}^{h_t} \int_{A_t^z} (1 - e^{-f(z)})\, dG_t(v)\, d\lambda(z)/| a(v) |\right).$$

The proof is completed on comparing (12) with (2). □

Note that when $a(v) = 1$ and $(\gamma_1, \gamma_2) = R$, $i_t(z) = \beta$ by (10). This invariance property of the stationary Poisson process under translation was first shown by Doob (1953), page 404. As remarked by Goldman (1967) and easily proved using the above method, this invariance property remains valid for mixtures of stationary Poisson processes. Corollary 1 also provides the starting point to deriving many of the results for the simple traffic model of Rényi obtained by other methods in Rényi (1964), Brown (1969b) and Solomon and Wang (1972). We shall demonstrate this by three examples, and in each of these, the initial process is assumed to be stationary Poisson with intensity $\beta > 0$ and the distribution $G_t \equiv G$ is independent of time.

EXAMPLE A. Vehicles start off at a fixed point on the road at times $(\tau_i)$ in $[0, t)$. We are interested in the distribution of the vehicles at time $t$. The location of the $i$th vehicle at time $t$ is given by

$$z_i = v_i(t - \tau_i) = -v_i\tau_i + v_i t.$$

Also, $[\gamma_1, \gamma_2) = [0, t)$, $d_t(v) = 0$, $h_t(v) = vt$ and

$$A_t = \{(v, z): 0 < z < vt, 0 < v < \infty\}.$$

From (10), the generated $z$-process is Poisson with intensity

(13) $$i_t(z) = \beta \int_{z/t}^\infty dG(v)/v, \qquad 0 < z < \infty.$$

Integrating (13) over any interval $[x, y)$ with respect to $\lambda$, one concludes that the $z$-process in $[x, y)$ is Poisson with mean

$$\nu_t[x, y) = \beta \int_{x/t}^{y/t} (t - x/v)\, dG(v) + \beta \int_{y/t}^\infty (y - x)/v\, dG(v).$$

EXAMPLE B. Vehicles start off at $t = 0$ at different locations $(\alpha_i)$ in an interval $[a, b)$. We are interested in the distribution of these vehicles in a fixed stretch of road $[x, y)$, $x \geq a$, at time $t$. Here

$$z_i = \alpha_i + v_i t$$

and so $d_t(v) = a + vt$, $h_t(v) = b + vt$, $d_t = a$ and $h_t = \infty$. Therefore

$$A_t = \{(v, z): a + vt < z < b + vt, 0 < v < \infty\}$$

and

$$A_t^z = \begin{cases} \varnothing, & 0 < z < a \\ \{v: 0 < v < (z-a)/t\}, & a \leq z \leq b, \\ [v: (z-b)/t < v < (z-a)/t\}, & z > b, \end{cases} \tag{14}$$

where $\varnothing$ is the empty set. Applying (10) to (14) and then integrating the resulting intensity with respect to $\lambda$ over $[x, y)$, it is routine to check that the generated $z$-process is Poisson with mean

$$\nu_t[x, y) = \begin{cases} \int_0^{(x-a)/t} (y-x)\, dG(v) + \int_{(x-a)/t}^{(y-a)/t} (y-a-vt)\, dG(v), & [x, y) \subset [a, b), \\ \int_0^{c_2} (b + vt - x)\, dG(v) + e\int_{c_1}^{c_2} dG(v) + \int_{c_2}^{(y-a)/t} (y-a-vt)\, dG(v), & x < b, \quad y > b, \\ \int_{(x-b)/t}^{(y-b)/t} (b+vt-x)\, dG(v) + \int_{(y-b)/t}^{(x-a)/t} (y-x)\, dG(v) + \int_{(x-a)/t}^{(y-a)/t} (y-a-vt)\, dG(v), & b < x < y. \end{cases}$$

Here $c_1 = \min\{(x-a)/t, (y-b)/t\}$, $c_2 = \max\{(x-a)/t, (y-b)/t\}$ and $e$ equals $b - a$ when $c_1 = (x-a)/t$ and $y - x$ otherwise.

EXAMPLE C. Suppose vehicle $B$ enters the road at time $\tau_0$ and position $x = 0$ and then proceeds to travel at a fixed velocity $v_0$. We shall consider the distribution of the perpendicular projections (onto the $t$ axis) of the points of intersections between the trajectories of the vehicles entering the road at the instants $(\alpha_i)$ distributed as a stationary Poisson process with fixed rate $\beta$ and that of $B$. The $z$-process so generated consists of the sequence of times $(\tau_i^+) \cup (\tau_i^-)$ described in Section 1. Here

$$z_i = \begin{cases} -v_i\alpha_i/(v_0 - v_i) + v_0\tau_0/(v_0 - v_i), & v_i < v_0, \quad \alpha_i < \tau_0, \\ v_i\alpha_i/(v_i - v_0) - v_0\tau_0/(v_i - v_0), & v_i > v_0, \quad \alpha_i > \tau_0, \\ \infty & \text{otherwise} \end{cases}$$

In this case,

$$A = \{(v, z): \tau_0 < z < v_0\tau_0/(v_0 - v), 0 < v < v_0\} \cup \{(v, z): \tau_0 < z < \infty, v_0 < v < \infty\}$$

and (10) implies that the $z$-process is Poisson with intensity

$$i(z) = \beta \int_{v_0 - v_0\tau_0/z}^{\infty} |(v - v_0)/v|\, dG(v). \tag{15}$$

In fact, $(\tau_i^+)$ is stationary Poisson with intensity $\beta \int_{v_0}^{\infty} (v - v_0)/v\, dG(v)$, and $(\tau_i^-)$ is stationary Poisson with intensity $\beta \int_{v_0-\tau_0 v_0/z}^{v_0} (v_0 - v)/v \, dG(v)$, and by the complete randomness property of the Poisson process, the two subprocesses are also independent.

On integrating (15) over $[x, y)$, $x \geq \tau_0$,

$$\nu[x, y) = \int_{v_0-v_0\tau_0/x}^{v_0-v_0\tau_0/y} (v_0(\tau_0 - x)/v + x) \, dG(v) + (y - x) \int_{v_0-v_0\tau_0/y}^{\infty} |(v - v_0)/v| \, dG(v),$$

and so the $z$-process in $[x, y)$ is Poisson with the above mean. □

**4. Weak convergence of $z$-processes.** Let $B_t$, $B \in \mathscr{B}$, denote the set $T_t^{-1}(B)$ and $B_t^x = \{v \in R_+ : (x, v) \in B_t\}$. For any $\eta \in N$, let us define the set function $\psi_t(\cdot)$ by

$$\psi_t(B) = \int G_t(B_t^x) \, d\eta(x), \qquad B \in \mathscr{B}. \tag{16}$$

It is easy to check, using the properties of inverse mappings and (5), that if $\eta$ is a s.p.p., then $\psi_t$ is a random measure.

In Chapter 8 of Kallenberg (1975), the author used an elegant method in proving the weak convergence of a sequence of compounded point processes to a Cox process. We use a similar method to prove our next result.

THEOREM 2. *For every $B \in \mathscr{B}^b$, let the distributions $G_t$ satisfy*

$$\lim_{t\to\infty} \sup_{x\in R} G_t(B_t^x) = 0 \tag{17}$$

*and for each $t > 0$, let $\xi_t$ be the z-process generated from some* s.p.p. $\eta_t$. *Then $\xi_t \to_d \xi$* iff *$\psi_t \to_d \psi$, and in that case $\xi =_d$* SP($\psi$) *where $=_d$ denotes equality in distribution.*

PROOF. Assume $\psi_t \to_d \psi$, we have to prove $\xi_t \to_d \xi = \mathrm{SP}(\psi)$. Let $h(\cdot) = \sum_{j=1}^m s_j 1_{I_j}(\cdot)$, $s_j > 0$ and the $I_j$'s are pairwise disjoint sets in $\mathscr{I}_\psi$, then by (3), (7) and as $\mathscr{B}_\xi^b = \mathscr{B}_\psi^b$, it suffices to show

$$E(\exp(\eta_t \ln g_t^h)) \to E(\exp(-\psi(1 - e^{-h}))).$$

By the continuity theorem for Laplace transforms, it is enough to show

$$\begin{aligned} -\eta_t \ln g_t^h &= -\int \ln(1 - \textstyle\sum_{j=1}^m G_t(I_{j,t}^x)(1 - e^{-s_j})) \, d\eta_t(x) \to_d \psi(1 - e^{-h}) \\ &= \textstyle\sum_{j=1}^m (1 - e^{-s_j})\psi(I_j), \end{aligned} \tag{18}$$

where $I_{j,t} = T_t^{-1}(I_j)$, $1 \leq j \leq m$. From Theorem 5.5 in Billingsley (1968), (18) will follow if it holds in the sense of ordinary convergence on assuming $\psi_t \to_v \psi$. Using the simple inequalities

$$x < -\ln(1 - x) < x(1 + 2\varepsilon)$$

valid for $0 < \varepsilon < ½$ and $x < \varepsilon$, it follows that

$$\textstyle\sum_{j=1}^m (1 - e^{-s_j})\psi_t(I_j) < -\eta_t \ln g_t^h < \sum_{j=1}^m (1 - e^{-s_j})\psi_t(I_j)(1 + 2\varepsilon) \tag{19}$$

provided $t$ is large enough so that $m \max_{1\leq j\leq m} \sup_{x\in R} G_t(I_{j,t}^x) < \varepsilon$. But since $\psi_t \to_v \psi$ implies $\psi_t(I) \to \psi(I)$, $I \in \mathscr{I}_\psi$, (see A7.2 in Kallenberg (1975)), the desired convergence follows from (19).

Conversely, suppose $\xi_t \to_d \xi$; we shall first prove that this will imply $(\psi_t)$ is relatively compact. Fix $B \in \mathscr{B}_\xi^b$, then $\xi_t(B) \to_d \xi(B)$ by assumption and for each $\varepsilon > 0$, there exists some $s$ close to zero such that

$$E(e^{-s\xi(B)}) > 1 - \varepsilon(1 - e^{-1}). \tag{20}$$

Using first the inequality $\ln(1 - x) \le -x$, $x \le 1$, and then Chebyshev's inequality,

$$\begin{aligned}\Pr\{\psi_t(B) \ge (1 - e^{-s})^{-1}\} &= \Pr\{-(1 - e^{-s})\psi_t(B) \le -1\} \\ &\le \Pr\left\{\int \ln(1 - G_t(B_t^x)(1 - e^{-s}))\, d\eta_t \le -1\right\} \\ &= \Pr\{1 - \exp(\eta_t \ln g_t^{s1_B}) \ge 1 - e^{-1}\} \\ &\le (1 - E(e^{-s\xi_t(B)}))/(1 - e^{-1}).\end{aligned}$$

Hence, $\limsup_{t\to\infty} \Pr\{\psi_t(B) \ge (1 - e^{-s})^{-1}\} \le (1 - E(e^{-s\xi B}))/(1 - e^{-1})$, and by (20),

$$\lim_{s\downarrow 0} \limsup_{t\to\infty} \Pr\{\psi_t(B) \ge (1 - e^{-s})^{-1}\} < \varepsilon.$$

But $B \in \mathscr{B}_\xi$ was arbitrary, it therefore follows that $(\xi_t)$ is relatively compact (see Lemma 4.5 in Kallenberg (1975)).

By the relative compactness of $(\psi_t)$, any sequence $N \subset T$ contains a further subsequence $N'$ such that $\psi_n \to_d \psi$, $n \in N'$, and the direct part of the theorem now imply $\xi = \mathrm{SP}(\psi)$. But as $\psi$ and $\xi$ implies each other uniquely (see Corollary 3.2 in Kallenberg (1975)), Theorem 2.3 in Billingsley (1968) implies $\psi_t \to_d \psi$ and the proof is complete. □

Finally, we remark that when $T_t(\alpha, v) = \alpha + b(v, t)$ and we define $F_t(B) = G_t\{v: b(v, t) \in B\}$, $B \in \mathscr{B}^b$, condition (17) specializes to

$$\lim_{t\to\infty} \sup_{x\in R} F_t(B - x) = 0$$

and for any s.p.p. $\eta$, the random measure defined by (16) is now the convolution between $\eta$ and $F_t$. In this case, Theorem 2 generalizes a result given by Goldman (1967) on the weak convergence of a sequence of randomly translated s.p.p. to the mixed Poisson process.

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