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NOTE ON A THEOREM OF WOODROOFE'S

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This note proves a stronger result than Woodroofe's Theorem (1977) on uniform integrability which is a key tool to obtain second order approximations in certain sequential point and interval estimation problems. The conditions for the stronger conclusion are also weaker than the conditions used by Woodroofe (1977) for his theorem.

1. Introduction and statement of the theorem. Woodroofe (1977) has obtained the second order approximations for the mean of the stopping rules for sequential point and interval estimation. He has also obtained second order approximations for the risk and the coverage probabilities respectively for the above cases by a very delicate computation. One of the key tools in his argument is the following uniform integrability result.

WOODROOFE'S THEOREM: *Let $\{X_n, n \ge 1\}$ be a sequence of positive independent and identically distributed random variables with mean $E(X_1) = \mu$ and variance $\tau^2 = E(X_1^2) - \mu^2$ which are both finite and positive. For each $c > 0$, let*

$$t_c = \inf\{n \ge m : S_n < cn^\alpha L(n)\}, \tag{1}$$

where $S_n = X_1 + \cdots + X_n$, $\alpha > 1$, $m \ge 1$ and for some constant L_0.

$$L(n) = 1 + L_0 n^{-1} + o(n^{-1}), \qquad \text{as } n \to \infty. \tag{2}$$

And c is often allowed to approach zero. Assume that $E(X_1^r) < \infty$ where $r \ge 2$. If

$$P[X_1 \le x] \le Bx^a, \qquad \textit{for all } x, \qquad \textit{for some } a > 0, B > 0, \tag{3}$$

$$ma > \frac{s}{2(\alpha - 1)}, \tag{4}$$

$$\textit{where} \qquad 0 < s < \min\left(r, \frac{r}{2}(2\alpha - 1)\right), \tag{5}$$

then

$$\left\{ \left| c^{\frac{1}{2(\alpha-1)}} \left(t_c - \left(\frac{\mu}{c}\right)^{\frac{1}{(\alpha-1)}} \right) \right|^s \right\} \textit{ is uniformly integrable (in c).} \tag{6}$$

A closely related problem in extended renewal theory is treated in Chow, Hsiung and Lai (1979). By adapting their techniques to the Woodroofe's theorem, we have been able to weaken conditions (3), (4) and (5) to that

$$P[S_m \le x] = o(x^{\frac{r}{2(\alpha-1)}}), \qquad \text{as } x \to o, \tag{7}$$

and strengthen the conclusion to that

$$\left\{ \left| \left(c^{\frac{1}{2(\alpha-1)}} \left(t_c - \left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} \right) \right) \right|^r \right\} \text{ is uniformly integrable.} \tag{8}$$

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In fact, that (7) is a necessary condition for (8) can easily be seen as follows: for any positive $\epsilon < 1$,

$$\int_{t_c<\epsilon\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}}} \left| c^{\frac{1}{2(\alpha-1)}}\left(t_c - \left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}}\right)\right|^r dP \geq K_1 \int_0^{\left(\frac{\mu}{c}\right)^{\frac{1}{2(\alpha-1)}}} x^{r-1}P[t_c = m]\, dx \tag{9}$$

$$\geq K_2 c^{-\frac{r}{2(\alpha-1)}} P[S_m < cm^\alpha L(m)],$$

for some constants K_1 and K_2 and for small c.

2. Proof of the theorem. The proof will be broken into the following steps.

STEP 1. As $c \to o$, for any constant K,

$$P[\inf_{m\leq j\leq -K\log c} S_j/j^\alpha L(j) \leq c] = o\left(c^{\frac{r}{2(\alpha-1)}}\right). \tag{10}$$

PROOF. If c is small

$$P[\inf_{m\leq j\leq -K\log c} S_j/j^\alpha L(j) \leq c]$$

$$\leq P[S_m < cm^\alpha L(m)] + \textstyle\sum_{j=m+1}^{2m} P[S_j < cj^\alpha L(j)] + \sum_{2m<j\leq -K\log c} P[S_j < cj^\alpha L(j)]$$

$$\leq o\left(c^{\frac{r}{2(\alpha-1)}}\right) + o\left(c^{\frac{r}{2(\alpha-1)}}\right) + \Sigma P[S_m < cK_1(K\log c)^\alpha] P[S_{2m} - S_m < cK_1(K\log c)^\alpha]$$

$$\leq o\left(c^{\frac{r}{2(\alpha-1)}}\right) - K\log c\, o\left((c(\log c)^\alpha)^{\frac{r}{2(\alpha-1)}}\right) o\left((c(\log c)^\alpha)^{\frac{r}{2(\alpha-1)}}\right)$$

$$= o\left(c^{\frac{r}{2(\alpha-1)}}\right),$$

where K_1 is some constant.

STEP 2. Suppose that $\alpha > 1$, $c > o$ and $t \geq o$

$$\text{(i)} \qquad ct^\alpha - t \geq (\alpha - 1)\left(t - c^{-\frac{1}{\alpha-1}}\right). \tag{11}$$

(ii) For $\lambda \in (0, \alpha - 1)$, there is a $\theta \in (o, 1)$ such that if $t \geq (1-\theta)c^{-\frac{1}{\alpha-1}}$, then

$$ct^\alpha - t \leq -\lambda\left(c^{-\frac{1}{\alpha-1}} - t\right). \tag{12}$$

These facts can easily be proved by using an elementary calculus argument.

STEP 3.

$$\left\{\left| c^{\frac{1}{2(\alpha-1)}} (\mu t_c - ct_c^\alpha) \right|^r\right\} \text{ is uniformly integrable.} \tag{13}$$

The argument in Chow, et al., (1979) can be modified to prove (13). Hence the details of this proof are omitted.

STEP 4. The main result:

$$\left\{ \left| c^{\frac{1}{2(\alpha-1)}} \left(t_c - \left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} \right) \right|^r \right\} \text{ is uniformly integrable.} \tag{14}$$

PROOF. By (11) (i) in Step 2,

$$\left(\frac{c}{\mu}\right) t_c^\alpha - t_c \geq (\alpha - 1)\left(t_c - \left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} \right), \tag{15}$$

hence by (13) in Step 3,

$$\left\{ \left(c^{\frac{1}{2(\alpha-1)}} \left(t_c - \left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} \right)^+ \right)^r \right\} \text{ is uniformly integrable.} \tag{16}$$

Next consider $t_c \geq (1-\theta)\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}}$ for some $\theta \in (o, 1)$. By (11) (ii) in Step 2, choose λ and θ suitably, then

$$\left(\frac{\mu}{c}\right) t_c^\alpha - t_c \leq -\lambda\left(\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} - t_c \right). \tag{17}$$

Hence uniform integrability follows from (13) in Step 3 for the part where $t_c \geq (1-\theta)\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}}$. It remains to consider $t_c \leq (1-\theta)\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}}$.

Let $n = \left[(1-\theta)\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} \right]$ and M be some constant. Then

$$\begin{aligned} P\left[t_c \leq (1-\theta)\left(\frac{\mu}{c}\right)^{\frac{1}{\alpha-1}} \right] &= P[\inf_{m\leq j\leq n} S_j/j^\alpha L(j) < c] \\ &\leq P[\inf_{m\leq j\leq M\log n} S_j/j^\alpha L(j) < c] \\ &\quad + P[\inf_{M\log n\leq j\leq n} S_j/j^\alpha L(j) < c]. \end{aligned} \tag{18}$$

And

$$\begin{aligned} &P[\inf_{M\log n\leq j\leq n} S_j/j^\alpha L(j) < c] \\ &\qquad = P[S_j < cj^\alpha L(j), \text{ for some } j \text{ between } M\log n \text{ and } n] \\ &\qquad \leq P[S_j < (1-\theta)^{\alpha-1}\mu j, \text{ for some } j \text{ between } M\log n \text{ and } n], \end{aligned} \tag{19}$$

for a possibly smaller θ. For each j, define

$$Y_j = X_j I_{[X_j \leq Q]}, \tag{20}$$

where Q is a positive constant chosen so that

$$EY_j \geq (1-\theta)^{\alpha-1}\mu + \epsilon, \qquad \text{for some positive } \epsilon. \tag{21}$$

Since X_j's and hence Y_j's are positive, (19) is

$$\leq P[Y_1 + \cdots + Y_j < (1-\theta)^{\alpha-1}\mu j, \text{ for some } j \text{ between } M\log n \text{ and } n]$$

$$\leq \textstyle\sum_{M\log n\leq j\leq n} P[\sum_{i=1}^j Y_i < (1-\theta)^{\alpha-1}\mu j]$$

$$\le \sum_{M \log n \le j \le n} P[\sum_{i=1}^{j} (Y_i - EY_i) < -\epsilon j]$$

$$= \sum_{M \log n \le j \le n} O(e^{-\beta j})$$

$$= nO(e^{-\beta M \log n}) = O(n^{1-\beta M}),$$

by the Kolmogorov exponential bound (see, e.g. Loève 1963, page 254) for some positive β, or by some large deviation result. By (10) in Step 1, the first term on the right side of (18) is

$$P[\inf_{m \le j \le M \log n} S_j / j^{\alpha} L(j) < c] = o\left(c^{\frac{r}{2(\alpha-1)}}\right), \tag{22}$$

and we choose M large enough so that (19) is $o\left(c^{\frac{r}{2(\alpha-1)}}\right)$. These complete the proof. The proof can be generalized to cover the case, where the X_n's are merely independent, under certain uniform integrability conditions on the X_n's. This has been done in Yu (1978), by a further refinement.

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