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PROPAGATION OF SINGULARITIES IN THE BROWNIAN SHEET

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We study certain singularities of the Brownian sheet which have the property that they propagate parallel to the coordinate axis. This is used to give an intuitive explanation of the fact that the Brownian sheet does not satisfy Lévy's sharp Markov property.

Let $\mathbb{R}^2_+$ be the positive quadrant of the plane provided with the usual partial order: $(u, v) < (s, t)$ if $u \le s$ and $v \le t$. We will use z and ζ for points in $\mathbb{R}^2_+$, and reserve s, t, u and v for real numbers. If $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$, $(z_1, z_2]$ will denote the rectangle $(s_1, s_2] \times (t_1, t_2]$. Let W be white noise on $\mathbb{R}^2_+$, that is, a random additive set function defined on the Borel sets of $\mathbb{R}^2_+$ such that $W\{A\}$ is a $N(0, |A|)$ random variable, and if $A \cap B = \phi$, $W\{A\}$ and $W\{B\}$ are independent. We define the Brownian sheet to be the process $\{W_z, z \in \mathbb{R}^2_+\}$, where $W_z = W\{(0, z]\}$. The Brownian sheet has a version which is continuous, and we will always take that version.

The coordinate axes have a great effect on the Brownian sheet, not only near the axes themselves, where $W_z = 0$, but in the interior of $\mathbb{R}^2_+$ as well. For instance, any unusual event has a tendency to propagate parallel to the axes, and the more unusual the event, the stronger the tendency. This leads to a phenomenon we call the propagation of singularities, in which certain singular events propagate with probability one.

The particular behavior we study here is specific to the Brownian sheet. It would be interesting to know if related processes, such as Lévy's multiparameter Brownian motion, have analogous properties.

1. Some preliminaries. Suppose the Brownian sheet is defined on a complete probability space $(\Omega, \mathscr{F}, P)$. We will assume in what follows that all σ-fields under discussion are augmented by the addition of all null-sets of $\mathscr{F}$. Let $\mathscr{F}_z = \sigma\{w_\zeta, \zeta < z\}$, $\mathscr{F}^1_{st} = \mathscr{F}_{s\infty} = \bigvee_v \mathscr{F}_{sv}$ and $\mathscr{F}^2_{st} = \mathscr{F}_{\infty t} = \bigvee_u \mathscr{F}_{ut}$, and define $\mathscr{F}^+_z = \mathscr{F}^1_z \vee \mathscr{F}^2_z$. Since $\mathscr{F}^1_{st}$ doesn't depend on t, we will often write $\mathscr{F}^1_s$ when there is no danger of confusion, and similarly, we may write $\mathscr{F}^2_t$ in place of $\mathscr{F}^2_{st}$.

DEFINITION. A random variable Z with values in $\mathbb{R}^2_+$ is a *weak stopping point* if for each $z \in \mathbb{R}^2_+$,

$$\{Z < z\} \in \mathscr{F}^+_z.$$

If Z is a weak stopping point, define

$$\mathscr{F}^+_Z = \{A \in \mathscr{F}: A \cap \{Z < z\} \in \mathscr{F}^+_z, \quad \text{all} \quad z \in \mathbb{R}^2_+\}.$$

One can easily verify that $\mathscr{F}^+_Z$ is a σ-field, that Z is $\mathscr{F}^+_Z$-measurable, and that if $Z_1 < Z_2$ are weak stopping points, then $\mathscr{F}^+_{Z_1} \subset \mathscr{F}^+_{Z_2}$. Finally, we define

$$W^Z_z = W\{(Z, Z + z]\}, \qquad z \in \mathbb{R}^2_+,$$

where the mass of the (random) rectangle $(Z, Z + z]$ is computed from W_z by the usual formula:

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$$W\{((u, v), (s, t)]\} = W_{st} - W_{ut} - W_{sv} + W_{uv}. \tag{1.1}$$

The following theorem gives one analogue (among many) of the familiar Brownian strong Markov property.

THEOREM 1. *Let Z be a weak stopping point. Then the process $\{W_z^Z, z \in \mathbb{R}_+^2\}$ is a Brownian sheet, independent of $\mathscr{F}_Z^+$.*

PROOF. The proof of this type of result is completely standard. One first approximates Z from above by countable-valued stopping points Z_n, proves the theorem for the Z_n, and then goes to the limit.

Write $Z = (T_1, T_2)$, let $T_i^n = j2^{-n}$ if $(j-1)2^{-n} \le T_i < j2^{-n}$, $j = 1, 2, \cdots, i = 1, 2$, and put $Z_n = (T_1^n, T_2^n)$. Let $\{r_k\}$ be an enumeration of the lattice points $(i2^{-n}, j2^{-n})$ and note that $\{Z_n = r_k\} \in \mathscr{F}_{r_k}^+$ for all k. By (1.1) and the continuity of the Brownian sheet,

$$W_z^Z = W\{(Z, Z + z]\} = \lim W\{(Z_n, Z_n + z]\} = \lim W_z^{Z_n}.$$

Let $A \in \mathscr{F}_Z^+ \subset \mathscr{F}_{Z_n}^+$ and let $B_1, \cdots, B_m$ be Borel sets in $\mathbb{R}$. If $z_1, \cdots, z_m \in \mathbb{R}_+^2$, then

$$P\{W\{(Z_n, Z_n + z_j]\} \in B_j, j = 1, \cdots, m; A\}$$
$$= \textstyle\sum_k P\{W\{(r_k, r_k + z_j]\} \in B_j, j = 1, \cdots, m; A \cap \{Z_n = r_k\}\}.$$

But $A \cap \{Z_n = r_k\} \in \mathscr{F}_{r_k}^+$, so by the independence properties of white noise, this is

$$= \textstyle\sum_k P\{W\{(r_k, r_k + z_j]\} \in B_j, j = 1, \cdots, m\} P\{A \cap \{Z_n = r_k\}\}$$
$$= P\{W_{z_j} \in B_j, j = 1, \cdots, m\} P\{A\}.$$

Thus W^{Z_n} is a Brownian sheet independent of $\mathscr{F}_Z^+$, hence so is its limit, W^Z.

The singularities we deal with in this article are associated with the law of the iterated logarithm, so we will introduce some convenient notation. Let

$$L(s; t; \omega) = \lim\sup_{h\downarrow 0} \frac{|W_{s+h,t} - W_{st}|}{\sqrt{2h \log\log 1/h}} \tag{1.2}$$

and

$$M(s; t, t'; \omega) = \lim\sup_{h\downarrow 0} \frac{|W_{s+h,t'} - W_{st'} - (W_{s+h,t} - W_{st})|}{\sqrt{2h \log\log 1/h}}. \tag{1.3}$$

We will usually suppress the variable ω. Notice that $L(s; t)$ is just the quantity of the law of the iterated logarithm for the Brownian motion $s \to W_{st}$ (where t is fixed), and M is the same thing for the "increment" $s \to (W_{st'} - W_{st})$. Both L and M are measurable in all variables, as can be seen, for instance, by writing:

$$L(s; t) = \lim_{n\to\infty}\left[\sup\left\{\frac{|W_{s+h,t} - W_{s,t}|}{\sqrt{2h \log\log 1/h}} : 0 < h \le \frac{1}{n}, h \in \mathbb{Q}\right\}\right]. \tag{1.4}$$

Let us record an elementary fact about the lim sup. If f and g are real-valued functions, and if either $\lim\sup|f|$ or $\lim\sup|g|$ is finite, then

$$\lim\sup|f| - \lim\sup|g| \le \lim\sup|f + g| \le \lim\sup|f| + \lim\sup|g|. \tag{1.5}$$

This leads to some relations between L and M. First note that

$$L(s; t) = M(s; 0, t). \tag{1.6}$$

Let $0 \le a \le b \le c$. Then $W_{sb} = W_{sa} + (W_{sb} - W_{sa})$ so that it follows from (1.5) that if either $L(s; a)$ or $M(s; a, b)$ is finite

$$L(s; a) - M(s; a, b) \le L(s, b) \le L(s; a) + M(s; a, b). \tag{1.7}$$

Similarly, if either $M(s; a, b)$ or $M(s; b, c)$ is finite,

$$M(s; a, b) - M(s; b, c) \le M(s; a, c) \le M(s; a, b) + M(s; b, c). \tag{1.8}$$

2. The propagation of singularities. Part (a) of the following theorem is due to G. Zimmerman [4], although the proof given here is new. Part (b) contains (a) as a special case, courtesy of (1.6), but is also an easy corollary of it, so we thought it worthwhile to separate the two.

THEOREM 2(a). *For each fixed $s \ge 0$, we have with probability one that $L(s; t) = \sqrt{t}$ simultaneously for all $t \ge 0$.* (b) *For each fixed $s \ge 0$, we have with probability one that $M(s; t, t') = \sqrt{t' - t}$ simultaneously for all $0 \le t \le t'$.*

PROOF. For each fixed t, the statement that $L(s; t) = \sqrt{t}$ a.s. is just the familiar law of the iterated logarithm. By Fubini, it holds for a.e. t with probability one. The whole problem is to show that it holds simultaneously for all t.

It follows from (1.4) that $\{L(s; t), t \ge 0\}$ is well-measurable with respect to the fields $(\mathscr{F}_t^2)_{t\ge 0}$, which are known to be continuous. Thus by Meyer's section theorem [1], if $P\{\exists t: L(s; t) \ne \sqrt{t}\} > 0$, then there exists a finite $(\mathscr{F}_t^2)$-stopping time T such that $P\{L(s; T) \ne \sqrt{T}\} > 0$. Let $B_{st} = W_{s,T+t} - W_{s,T}$. Note that $Z := (0, T)$ is a weak stopping point, so that by Theorem 1, $B = W^Z$ is again a Brownian sheet. Thus if $\delta > 0$,

$$M(s; T, T+\delta) = \limsup_{h\downarrow 0} \frac{|B_{s+h,\delta} - B_{s,\delta}|}{\sqrt{2h \log\log 1/h}} = \sqrt{\delta} \qquad \text{a.s.} \tag{2.1}$$

But $W_{s,T+\delta} = W_{sT} + B_{s\delta}$, so by (1.7) and (2.1)

$$L(s; T) - \sqrt{\delta} \le L(s; T+\delta) \le L(s; T) + \sqrt{\delta} \qquad \text{a.s.}$$

It follows that $L(s, T+\delta) \ne \sqrt{T+\delta}$ for a.e. small enough δ with positive probability, i.e. $L_t \ne \sqrt{t}$ for a t-set of positive Lebesgue measure, a contradiction. This proves (a). To prove (b), notice that for any $t_0 \ge 0$,

$$\hat{W}_{st} := W_{s,t_0+t} - W_{s,t_0}$$

is a Brownian sheet. If we apply part (a) to $\hat{W}$ and let $\hat{L}(s; t)$ be defined as in (1.4) with W replaced by $\hat{W}$, we see that $\hat{L}(s; t) = \sqrt{t}$ for all t. But $\hat{L}(s; t) \equiv M(s; t_0, t_0 + t)$.

It follows by Fubini that for a.e. ω,

$$M(s; t, t'; \omega) = \sqrt{t' - t} \tag{2.2}$$

for all rational t and all real $t' \ge t$. Let us fix such an ω, and show that (2.2) holds for all real t. If t is irrational, let $t_n < t$ be rational. Then $M(s, t_n, t') = \sqrt{t' - t_n} < \infty$, so by (1.8),

$$|M(s; t_n, t') - M(s; t, t')| \le M(s; t_n, t) = \sqrt{t - t_n}.$$

Now we can let $t_n \uparrow t$ to see that $M(s; t, t') = \sqrt{t' - t}$, and we are done.

It is the converse to this theorem which gives us the announced propagation of singularities.

THEOREM 3. *Let $t_0 > 0$ and let $S \ge 0$ be an $\mathscr{F}_{t_0}^2$-measurable random variable. Then for* a.e. ω

(a) *if* $t \ge t_0$, $L(S(\omega); t, \omega) = \infty$ iff $L(S(\omega); t_0, \omega) = \infty$

(b) *if* $t_0 \le t < t'$, $M(S(\omega); t, t', \omega) = \sqrt{t' - t}$.

PROOF. Let $Z = (S, t_0)$. Z is a weak stopping point, for if $z = (s, t)$, then if $t < t_0$, $\{Z \prec z\} = \phi \in \mathscr{F}_z^+$, while if $t \ge t_0$, $\{Z \prec z\} = \{S \le s\} \in \mathscr{F}_{t_0}^2 \subset \mathscr{F}_z^+$. By Theorem 1, $\{W_z^Z\}$ is a Brownian sheet. Applying Theorem 2, we see that for a.e. ω

$$\limsup_{h\downarrow 0} \frac{|W_{ht}^Z|}{\sqrt{2h \log\log 1/h}} = \sqrt{t}, \qquad \forall t > 0.$$

Set $t' = t_0 + t$. Then

$$W_{S+h,t'} - W_{S,t'} = W_{S+h,t_0} - W_{S,t_0} + W^Z_{h,t}.$$

Thus for all $t' > t_0$, (1.7) implies:

$$L(S; t_0) - \sqrt{t} \leq L(S; t') \leq L(S; t_0) + \sqrt{t},$$

which proves (a). (b) follows upon noticing that, if M^Z is defined as in (1.3) with W replaced by W^Z, then if $t_0 \leq t < t'$,

$$M(S, t, t') = M^Z(0, t - t_0, t' - t_0) = \sqrt{t' - t}. \qquad \square$$

A point s for which $L(s; t, \omega) = \infty$ is called a *singular point* for $W_{st}(\omega)$. More generally, if B is any Brownian motion, we say s is *singular for* $B(\omega)$ if

$$\limsup_{h \downarrow 0} \frac{|B_{s+h}(\omega) - B_s(\omega)|}{\sqrt{2h \log \log 1/h}} = \infty.$$

REMARK 1: There are many singular points for W_{st}, and random singular times do exist. Indeed, Orey and Taylor [3] showed that the set of singular points for any Brownian motion is dense, uncountable, and even of Hausdorff dimension one. Thus, if B is a Brownian motion, let

$$\mathscr{S} = \{(s, \omega) : s \text{ is singular for } B(\omega)\}.$$

Then $\mathscr{S}$ is $\mathscr{B}_+ \times \sigma\{B_s, s \geq 0\}$-measurable, where $\mathscr{B}_+$ is the Borel field of R_+; see e.g. (4). It follows by [1, page 18] that there exists a r.v. S for which $(S(\omega), \omega) \in \mathscr{S}$ for a.e. ω, i.e. S is a.s. singular for B. If we take B_s to be W_{st_0}, S will be $\mathscr{F}^2_{t_0}$-measurable, and $L(S(\omega); t_0, \omega)$ will be infinite for a.e. ω.

REMARK 2: We can think of S as a way of choosing a singularity at random. What we have shown is that if S is a singular point of W_{st_0}, it is singular for W_{st} for all $t \geq 0$, i.e. the singularity propagates vertically, parallel to the t-axis. If we think of the sample function of W as a sheet, then we can visualize this propagating singularity as a wrinkle running parallel to the t-axis.

By symmetry, there are wrinkles running horizontally, parallel to the s-axis. To rephrase Theorem 3: the wrinkles in the Brownian sheet are parallel to the edges of the bed.

REMARK 3: A similar argument shows that singularities of the type

$$\limsup_{h \downarrow 0} \frac{|W_{s+h,t} - W_{s,t}|}{\sqrt{2h \log \log 1/h}} > (1 + \varepsilon)\sqrt{t}$$

propagate vertically for short distances.

REMARK 4: A word of warning: the exact measurability conditions on S are not mere technicalities. They govern the type of singularity we choose. This will become clear in Theorems 4 and 5 and the associated remarks.

3. Genesis. Let us turn to the question: "How do these propagating singularities start?" We can get the beginnings of an answer to this by a simple time-reversal argument.

Define $\tilde{W}_{st} = tW_{s1/t}$. Then $\tilde{W}$ is again a Brownian sheet, for it is a mean-zero Gaussian process with covariance function

$$E\{\tilde{W}_{st}\tilde{W}_{uv}\} = (s \wedge u)(t \wedge v).$$

If $\tilde{L}(s; t)$ is defined as in (1.2) with W replaced by $\tilde{W}$, then

$$\tilde{L}(s; t) = tL(s; 1/t). \tag{3.1}$$

Furthermore, if $t < t' \leq t_0$,

(3.2) $$W_{st'} - W_{st} = t'(\tilde{W}_{s,1/t'} - \tilde{W}_{s,1/t}) + (t' - t)\tilde{W}_{s,1/t}.$$

Apply (1.5): if $\tilde{M}(s; 1/t', 1/t)$ is finite, then

(3.3) $$(t' - t)\tilde{L}(s; 1/t) - t'\tilde{M}(s; 1/t', 1/t) \le M(s; t, t') \le (t' - t)\tilde{L}(s; 1/t) + t'\tilde{M}(s; 1/t', 1/t).$$

THEOREM 4. *Let $t_0 > 0$ and let $S \ge 0$ be measurable with respect to $\sigma\{W_{st}, s \ge 0, t \ge t_0\}$. Then for* a.e. ω,

(a) *if* $t \le t_0$, $L(S(\omega); t, \omega) = \infty$ iff $L(S(\omega); t_0, \omega) = \infty$;

(b) *if* $t < t' \le t_0$, $M(S(\omega); t, t', \omega) = \infty$ iff $L(S(\omega); t_0, \omega) = \infty$.

PROOF. This reduces to Theorem 3 by time reversal. If we define $\tilde{W}$ as above, with $\tilde{\mathscr{F}}_t^2$ defined in the obvious way, then S is $\tilde{\mathscr{F}}^2_{1/t_0}$-measurable, so (a) follows by applying Theorem 3(a) to $\tilde{W}$ instead of W, and using (3.1). By Theorem 3(b), $\tilde{M}(S; 1/t', 1/t) = \sqrt{1/t - 1/t'}$, which is finite, so (3.3) holds, and part (b) follows immediatetly from (3.1). □

Theorem 3 only applies to S which are measurable with respect to $\sigma\{W_{st}, t \le t_0\}$, and it doesn't tell us if the singularity propagates into the region $t < t_0$, while Theorem 4 applies to S which are $\sigma\{W_{st}, t \ge t_0\}$-measurable, and it doesn't tell us anything about propagation into the region $t > t_0$. Indeed, there is little to be said about this at this level of generality, as one can see from Theorem 7 and its proof. However, if we take the random variable S to be measurable with respect to $\sigma\{W_{st_0}, s \ge 0\}$, then both theorems apply, and we can get a rather complete description of the singularity and its propagation.

THEOREM 5. *Let $S \ge 0$ be a $\sigma\{W_{st_0}, s \ge 0\}$-measurable random variable. Then for* a.e. ω

(a) *for any* $t > 0$, $L(S(\omega); t, \omega) = \infty$ iff $L(S(\omega); t_0, \omega) = \infty$;

(b) $M(S(\omega); t, t', \omega) = \sqrt{t' - t}$ *for all* $t_0 \le t < t'$;

(c) *if* $t < t_0$, $M(S(\omega); t, t', \omega) = \infty$ iff $L(S(\omega); t_0, \omega) = \infty$.

PROOF. Part (a) follows from parts (a) of Theorem 3 (for $t \ge t_0$) and 4 (for $t < t_0$). Part (b) follows from Theorem 3(b), and part (c) follows from Theorem 4(b) if $t < t' \le t_0$. If $t < t_0 < t'$, then by (1.8),

$$M(S; t, t_0) - M(S; t_0, t') \le M(S; t, t') \le M(S; t, t_0) + M(S; t_0, t').$$

By part (b), $M(S; t_0, t') = \sqrt{t' - t_0}$, and we are done, since $M(S; t, t_0)$ is infinite iff $L(S; t_0)$ is, by the case just proved.

REMARK. If we choose S in Theorem 5 to be a singularity, then, whatever the value of t_0, the singularity starts at the origin and propagates to infinity. Still, the exact type of singularity it is depends on t_0. Indeed, we can recover t_0 by looking closely at the singularity we have chosen: $t_0 = \inf\{t : M(S; t, t + 1) < \infty\}$.

There are some rather surprising properties of the sets of singularities embodied in the two foregoing theorems. We can illustrate this by considering the singular points of some independent Brownian motions. Let $B^1, \cdots, B^n$ be independent standard Brownian motions, and let $B = B^1 + \cdots + B^n$. For a given j, let $\mathscr{S}_j$ be the set of points singular for B^j, and let $\mathscr{S}$ be the set of points singular for B.

COROLLARY 6. *Let S_j and S be positive random variables measurable with respect to $\sigma\{B_s^j, s \ge 0\}$ and $\sigma\{B_s, s \ge 0\}$ respectively. Then with probability one*:

(a) $S_j \in \mathscr{S}$ iff $S_j \in \mathscr{S}_j$;

(b) $S_j \notin \mathscr{S}_k$ if $k \neq j$;

(c) $S \in \mathscr{S}$ iff $S \in \cup_j \mathscr{S}_j$ iff $S \in \cap_j \mathscr{S}_j$.

PROOF. Note that $(B_s^1, \cdots, B_s^n, B_s)$ has the same distribution as $(W_{s1}, W_{s2} - W_{s1}, \cdots, W_{sn} - W_{sn-1}, W_{sn})$, so we may as well suppose that $B_s^j = W_{sj} - W_{sj-1}, j = 1, \cdots, n$, and $B_s = W_{sn}$.

(a) Suppose first that $j = 1$. Note that $S_1 \in \mathscr{S}$ iff $\infty = L(S_1; n) = M(S_1; 0, n)$, and the result follows from Theorem 5b with $t_0 = t = 1$ and $t' = n$. Since the initial ordering of the $B^1, \cdots, B^n$ was arbitrary, (a) must also hold for $j = 2, \cdots, n$.

(b) $S_1 \in \mathscr{S}_k$ iff $M(S_1; k - 1, k) = \infty$. But by Theorem 5b (with $t_0 = 1$), this never happens for $k \geq 2$, i.e. (b) holds for $j = 1$. By symmetry, it holds for all j.

(c) Note that $S \in \mathscr{S}$ iff $L(S; n) = \infty$, and $S \in \mathscr{S}_j$ iff $M(S; j - 1, j) = \infty$. But $L(S; n)$ and $M(S; j - 1, j)$ are finite or infinite together by Theorem 5c (with $t_0 = n$), and we are done.

REMARKS. The sets $\mathscr{S}$ and $\mathscr{S}_j$ above have Lebesgue measure zero. The $\mathscr{S}_j$ are independent, and even though they have Hausdorff dimension one, one might suspect that $\mathscr{S}_i \cap \mathscr{S}_j$ was negligibly small relative to $\mathscr{S}_i \cup \mathscr{S}_j$. (In fact, we once conjectured that it was empty.) Parts (a) and (b) of Corollary 6 would seem to support this, but part (c) indicates the opposite: in some sense, $\cap_j \mathscr{S}_j$ is almost as large as $\cup_j \mathscr{S}_j$. If there is any moral to this, it is that one must be careful in arguing naively about the size of various subsets of the singularities.

Just to reduce naiveté to absurdity, let us look at the Brownian sheet from the same viewpoint. Let $\mathscr{S}'$ be the set of singularities which propagate vertically from zero to infinity, and, for each t, $0 < t \leq \infty$ (note that $t = \infty$ is included) let $\mathscr{S}_t$ be the subset of $\mathscr{S}'$ consisting of points s for which $\inf\{\tau : M(s; \tau, \tau + 1) < \infty\} = t$. The $\mathscr{S}'_t$ are disjoint, and $\mathscr{S}' = \cup_{t \leq \infty} \mathscr{S}'_t$. If we choose a singularity S which is $\sigma\{W_{st}, s \geq 0\}$-measurable, then by Theorem 5, $s \in \mathscr{S}'_t$ a.s. This is true for each t, so that if we really believed that S was a singularity chosen "at random" and if we were to use it to measure the size of various subsets of $\mathscr{S}'$, we would be forced to conclude that $\mathscr{S}'$ is an uncountable disjoint union of subsets, each of which is as large as the whole!

The singularities we have studied so far all propagate across the whole plane, and it is natural to ask if this is always the case. The answer is "no", as the following theorem shows.

THEOREM 7. *Let $0 \leq t_0 < t_1$ be finite or infinite reals. Then with probability one there exist singularities which start at t_0 and propagate exactly to t_1.*

PROOF. Let $t_2 > t_0$, let $B_s = W_{st_2} - W_{st_0}$, and let $S \geq 0$ be a $\sigma\{B_s, s \geq 0\}$-measurable random variable such that S is a.s. singular for B. Define $\hat{W}_{st} = W_{s,t_0+t} - W_{st_0}$. Then $\hat{W}$ is a Brownian sheet, S is $\sigma\{\hat{W}_{s,t_2-t_0} : s \geq 0\}$-measurable, and S is singular for $\hat{W}_{s,t_2-t_0}$, i.e. $\hat{L}(S; t_2 - t_0) = \infty$. By Theorem 4, $\hat{L}(S; t) = \infty$ for all t. In terms of the original process,

$$\infty = \hat{L}(S; t) = M(S; t_0, t_0 + t) \quad \text{for all} \quad t.$$

Now B_s is independent of $\{W_{st}, s \geq 0, 0 \leq t \leq t_0\}$ so by an easy modification of Theorem 2, $L(S; t) = \sqrt{t}$ a.s. for all $t \leq t_0$. But for $t > t_0$, by (1.7) $L(S; t) \geq M(S; t_0; t) - L(S; t_0) = \infty - \sqrt{t_0} = \infty$. Thus

$$L(S; t) = \begin{cases} \sqrt{t} & \text{if} \quad t \leq t_0 \\ \infty & \text{if} \quad t > t_0, \end{cases}$$

so the singularity propagates from t_0 to infinity.

In order to construct a singularity ending at t_1, just reverse time and consider $\tilde{W}_{st} = tW_{s1/t}$. As $\tilde{W}$ is a Brownian sheet, it must have a singularity S which propagates from $1/t_1$ to infinity, i.e.

$$\tilde{L}(S; 1/t) = \begin{cases} \sqrt{\frac{1}{t}} & \text{if } \frac{1}{t} \le \frac{1}{t_1} \\ \infty & \text{if } \frac{1}{t} > \frac{1}{t_1}. \end{cases}$$

By (3.1)

$$L(S; t) = \begin{cases} \sqrt{t} & \text{if } t \ge t_1 \\ \infty & \text{if } t < t_1, \end{cases}$$

which is exactly what we wanted.

This takes care of the case where $t_0 = 0$ and/or $t_1 = \infty$. To handle the case $0 < t_0 < t_1 < \infty$, we need a different transformation of the Brownian sheet. To simplify notation, suppose $t_0 = 1$ and $t_1 = 2$. Put

$$\bar{W}_{st} = \begin{cases} W_{s,t+1} - W_{s1} & \text{if } 0 \le t \le 1 \\ \bar{W}_{s1} + W_{s,t-1} & \text{if } 1 \le t \le 2 \\ W_{st} & \text{if } t > 2 \end{cases}$$

so that

$$W_{st} = \begin{cases} \bar{W}_{s,t+1} - \bar{W}_{s1} & \text{if } 0 \le t \le 1 \\ W_{s1} + \bar{W}_{s,t-1} & \text{if } 1 \le t \le 2 \\ \bar{W}_{st} & \text{if } t > 2. \end{cases} \tag{3.4}$$

What we have done is to interchange the strips $\{(s, t): 0 < t \le 1\}$ and $\{(s, t): 1 < t \le 2\}$. It is not hard to verify that $\bar{W}$ is again a Brownian sheet. Thus $\bar{W}$ must have a singularity S which propagates from zero to one, i.e.

$$\bar{L}(S; t) = \begin{cases} \infty & \text{if } t < 1 \\ \sqrt{t} & \text{if } t \ge 1. \end{cases} \tag{3.5}$$

But from (3.4), (3.5) and (1.5), if $0 \le t \le 1$ then

$$L(S; t) \le \bar{L}(S; t + 1) + \bar{L}(S; 1) = \sqrt{t + 1} + 1 < \infty.$$

If $t \ge 2$, then $L(S; t) = \bar{L}(S; t) = \sqrt{t} < \infty$, while if $1 < t < 2$, $L(S; t) \ge \bar{L}(S; t - 1) - L(S; 1) = \infty - 1 - \sqrt{2} = \infty$, so that the singularity at S propagates exactly from one to two. □

4. A Refinement. We can refine the foregoing results by looking at higher-order singularities. Let φ be a continuous increasing function on $\mathbb{R}^+$ with $\varphi(0) = 0$, such that $\lim_{h\downarrow 0}\varphi(h)(2h \log \log 1/h)^{-1/2} = \infty$.

The quantities corresponding to L and M respectively are

$$K(s; t) = \limsup_{h\downarrow 0} \frac{|W_{s+h,t} - W_{st}|}{\varphi(h)}$$

and

$$N(s; t, t') = \limsup_{h\downarrow 0} \frac{|W_{s+h,t'} - W_{s+h,t} - W_{st'} + W_{st}|}{\varphi(h)}.$$

Notice that $K(s; t)$ and $N(s; t, t')$ vanish if $L(s; t)$ and $M(s; t, t')$ respectively are finite. The most interesting choice of φ is $\varphi(h) = \sqrt{2h \log 1/h}$. In this case $K(s; t) \le \sqrt{t}$ a.s. for all s by Lévy's modulus of continuity, while Orey and Taylor [3] have shown that if $o \le a \le 1$, the set of s for which $K(s; t) \ge a\sqrt{t}$ has Hausdorff dimension $1 - a^2$. Thus K and N can take on non-trivial values and it is possible to choose, say, a random variable S such that $0 < K(S; t) \le \sqrt{t}$ a.s.

The theorems of Sections two and three give us some information about K and N, but we can greatly refine this with a small amount of additional work. We have the following two results.

THEOREM 8. Let $t_0 > 0$ and let $S \geq 0$ be $\mathscr{F}^2_{t_0}$-measurable. Then for a.e. ω,

$$\text{(a)} \quad K(S(\omega); t; \omega) = K(S(\omega); t_0; \omega) \quad \text{for all} \quad t \geq t_0;$$

$$\text{(b)} \quad N(S(\omega); t, t'; \omega) = 0 \quad \text{for all} \quad t' > t \geq t_0.$$

THEOREM 9. *Let $t_0 > 0$ and let $S \geq 0$ be $\sigma\{W_{st_0}, s \geq o\}$ – measurable. Then for* a.e. *ω, we have for all $t' > t \geq 0$:*

$$\text{(a)} \quad t_0 K(S(\omega); t; \omega) = (t \wedge t_0) K(S(\omega); t_0; \omega);$$

$$\text{(b)} \quad t_0 N(S(\omega); t, t'; \omega) = (t' \wedge t_0 - t \wedge t_0) K(S(\omega); t_0; \omega).$$

These theorems are closely connected, and we will prove them together.

PROOF. Theorem 8(b) is immediate from Theorems 1 and 2(a). Then part (a) follows because

$$K(S; t_0) - N(S; t_0, t) \leq K(S; t) \leq K(S; t_0) + N(S; t_0, t),$$

which is the analogue of (1.7).

Moving to Theorem 9, in case $t_0 \leq t < t'$, part (b) follows from Theorem 8(b). If $t < t' \leq t_0$, we apply a time reversal argument: let $\hat{W}_{st} = tW_{s,1/t}$. Then

$$W_{st'} - W_{st} = (t' - t)\hat{W}_{s,1/t} + t'(\hat{W}_{s,1/t'} - \hat{W}_{s,1/t}).$$

It follows from (1.5) that, if $\hat{K}$ and $\hat{N}$ are defined in the obvious way,

$$(t' - t)\hat{K}(S; 1/t) - t'\hat{N}(S; 1/t', 1/t) \leq N(S; t, t')$$
$$\leq (t' - t)\hat{K}(S; 1/t) + t'\hat{N}(S; 1/t', 1/t).$$

But $1/t' \geq 1/t_0$ so by Theorem 8 applied to $\hat{W}$, $\hat{N}(S; 1/t', 1/t) = 0$, while $\hat{K}(S; 1/t) = \hat{K}(S; 1/t_0) = 1/t_0 K(S; t_0)$, which gives part (b) in this case. The remaining case occurs when $t < t_0 < t'$, when the analogue of (1.8) gives

$$N(S; t, t_0) - N(S; t_0, t') \leq N(S; t, t') \leq N(S; t, t_0) + N(S; t_0, t').$$

We have just seen that $N(S; t_0, t') = 0$ so that $N(S; t, t') = N(S; t, t_0) = (1 - t/t_0)K(S; t_0)$. This proves (b), and (a) follows because $K(s; t) \equiv N(s; 0, t)$.

REMARKS. The singularities build up with surprising regularity. For instance, take $t_0 = 1$: if $S \geq 0$ is $\sigma\{W_{s1}, s \geq 0\}$ –measurable, and if $K(S; 1) > 0$, then not only is each "increment" $(W_{st'} - W_{st})$ below $t = 1$ singular, but its degree of singularity is known once $K(S; 1)$ is known. Indeed, for all $t < t' \leq 1$, $N(S; t, t') = (t' - t)K(S; 1)$.

5. Some remarks on Lévy's Markov property. If $A \subset \mathbb{R}^2_+$, let $\mathscr{G}_A = \sigma\{W_z, z \in A\}$, and $\mathscr{G}^+_A = \cap\, \mathscr{G}_G$, where the intersection is over all open G which contain A. $\mathscr{G}^+_A$ is called the *germ-field* and $\mathscr{G}_A$ the *sharp field.* We say that W satisfies *Lévy's Markov property* if, for every domain D with a piecewise-smooth boundary, $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$ are conditionally independent given $\mathscr{G}^+_{\partial D}$. If they are conditionally independent given $\mathscr{G}_{\partial D}$, we say it satisfies *Lévy's sharp Markov property.* In one parameter, Lévy's sharp Markov property corresponds to the ordinary Markov property, while the other is often called a higher-order Markov property, so that the sharp Markov property might be said to be the rule rather than the exception. The situation is reversed in several parameters: non-trivial processes satisfying Lévy's sharp Markov property are rare, and to find examples one is forced to turn to generalized processes [2]. Thus, in general, $\mathscr{G}^+_{\partial D}$ is strictly bigger than

$\mathscr{G}_{\partial D}$. The two fields are often equal in one parameter—this is just Blumenthal's zero-one law. Evidently the zero-one law fails in several parameters. In order to see why one might expect this, we will consider the Brownian sheet.

The Brownian sheet satisfies Lévy's Markov property, and it even satisfies the sharp Markov property for domains which are finite unions of rectangles with sides parallel to the axes. But these are all. The sharp Markov property does not hold for other domains. For example, consider the domain $D = \{(s, t): s \geq t\}$, whose boundary is the diagonal $s = t$. D is unbounded, but this is irrelevant to our discussion. Write $W_{ss} = W_s^1 + W_s^2$, where $W_s^1 = W\{(u, v): v \leq u \leq s\}$ and $W_s^2 = W\{(u, v): u \leq v \leq s\}$. Note that W^1 and W^2 are $\mathscr{G}_D$ measurable. Indeed, one can approximate $\{(u, v): v \leq u \leq s\}$ by a union T_n of tall thin rectangles of the form $[k/n, k + 1/n] \times [0, k/n]$. If V_n is a $1/n$-neighborhood of ∂D, $W(T_n) \in \mathscr{G}_{V_n}$, and $W(T_n) \to W_s^1$, so $W_s^1 \in \cap_n \mathscr{G}_{V_n} = \mathscr{G}_{\partial D}^+$. The same argument shows that W^1 is measurable with respect to $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$, so that the sharp Markov property can only hold if W_s^1 is $\mathscr{G}_D$-measurable. But it is not. All the processes under discussion are Gaussian, so that the fact that $E\{(W_s^1 - W_{ss}/2) W_{tt}\} = 0$ for all t implies that $E\{W_s^1 \mid \mathscr{G}_{\partial D}\} = W_{ss}/2 \neq W_s^1$.

This proves that $\mathscr{G}_{\partial D}^+$ is strictly larger than $\mathscr{G}_{\partial D}$, but it gives little insight into why. We would like a more intuitive proof, based on some local behavior—some type of convergence, perhaps—which is clearly measurable with respect to the germ field, but not with respect to the sharp field. The propagation of singularities provides us with just this.

The reason is simple, at least at the intuitive level. We know that the plane is criss-crossed by singularities propagating horizontally and vertically like wrinkles. When the diagonal crosses one of these winkles, the process on the diagonal will have a singularity. If one looks only at the diagonal, it is impossible to tell (symmetry!) if the singularity is propagating horizontally or vertically. But if we look in any neighborhood of the diagonal, we can see which way it is travelling. In other words, the direction of propagation is measurable with respect to the germ-field, but not with respect to the sharp field.

We will have to be careful when we make this statement rigorous, for it has some subtle complications, but first, let us remark that it indicates why one might expect the germ-field to be strictly larger than the sharp field. This is because for the two to be equal, some sort of zero-one law would have to hold simultaneously at each of the uncountably many points of ∂D, and, in general, this is simply too much to ask. In the example above, the probability is zero that a singularity will pass through a given point of ∂D, but it is sure that some singularities will cross the line ∂D.

Here is a way of making precise the claim that the direction of propagation is measurable with respect to $\mathscr{G}_{\partial D}^+$, but not $\mathscr{G}_{\partial D}$.

PROPOSITION 10. (a) *For any* $\mathscr{G}_{\partial D}$-measurable S *such that* $s \to W_{ss}$ *is* a.s. *singular at* S, *S propagates both vertically and horizontally.* (b) *There exists a* $\mathscr{G}_{\partial D}^+$*-measurable T such that* $s \to W_{ss}$ *is* a.s. *singular at T, and T propagates vertically but not horizontally.*

Before proving this, we need to generalize part of Corollary 6.

LEMMA 11. *Let* X_t *and* Y_t *be independent, continuous, mean-zero processes of independent increments, and put* $Z_t = X_t + Y_t$. *Suppose* $(d/dt)E\{X_t^2\} = a(t)$ *and* $(d/dt)E\{Y_t^2\} = b(t)$, *where a and b are strictly positive, continuously-differentiable functions. If S is* $\sigma\{Z\}$*-measurable (or, more generally,* $\sigma\{Z, U\}$*-measurable, where U is independent of* (X, Y)*), then for* a.e. ω:

(a) *If one of X, Y, Z is singular at S, so are the other two*;

(b) *if T is* $\sigma\{X, U\}$*-measurable, Y is non-singular at T.*

PROOF. Let

$$V_t = \int_0^t b(s)/a(s)\, dX_s - Y_t.$$

A direct calculation with stochastic integrals shows that $E\{V_t Z_s\} = 0$ for all s and t. Since Z and V are Gaussian and mean zero, this implies that they are independent processes. But now

$$Z_t + V_t = \int_0^t (1 + b(s)/a(s))\, dX_s.$$

The random variable S is independent of V so that it is a (randomized) stopping time for V, and we can apply the strong Markov property to see that

$$\limsup_{h \downarrow 0} \frac{|V_{S+h} - V_S|}{\sqrt{2h \log \log /h}} = \left(b(S) + \frac{b^2(S)}{a(S)} \right)^{1/2} < \infty.$$

It follows from (1.5) that $Z + V$ is singular at S iff Z is. But note that $\int (1 + b/a)\, dX$ is singular at S iff X is, since by Ito's formula on integration by parts

$$\int_0^t (1 + b/a)\, dX = (1 + b(t)/a(t))X_t - \int_0^t X_s \left(\frac{b(s)}{a(s)} \right)' ds,$$

and the last integral is never singular, while $(1 + b/a)X$ is singular at S iff X is. This shows that Z is singular at S iff X is, and the same is true of Y by symmetry, which proves (a). Part (b) follows by applying the strong Markov property to Y instead of V as above. □

PROOF OF PROPOSITION 10. We will only consider propagation from the diagonal back towards the axes. Let $\mathscr{H} = \sigma\{W_s^1, W_s^2, s \geq 0\}$, and note that $\mathscr{H} \subset \mathscr{G}_{\partial D}^+$. Fix $t > 0$ and let $X_s = W_{st} - W_{tt}$, $Z_s = W_s^1 - W_t^1$, and $Y_s = Z_s - X_s$ for $s \geq t$. Notice that the processes $\{X_s, s \geq t\}$ and $\{Y_s, s \geq t\}$ are independent, and are independent of $\{W_v^2, v \geq 0\}$ and of $\{W_u^1, 0 \leq u \leq t\}$. Suppose S is $\mathscr{H}$-measurable. By Lemma 11a, a.e. on the set $\{S > t\}$, X is singular at S iff Z is. It follows by Fubini that

(i) $L(S; t) = \infty$ for a.e. $t < S$ on $\{S$ is singular for $W^1\}$;

(ii) $L(S; t) < \infty$ for a.e. $t < S$ on $\{S$ is non-singular for $W^1\}$.

An argument similar to that of Theorem 2 shows that we can replace "a.e. t" by "every t" above.

Now suppose that S is $\mathscr{G}_{\partial D}$-measurable, and note that $\mathscr{G}_{\partial D} \subset \mathscr{H}$. If W_{ss} is singular at S, we can apply Lemma 11 with $X_s = W_s^1$, $Y_s = W_s^2$ and $Z = W_{ss}$ to see that S is a.s. singular for both W^1 and W^2, hence by (i) above, the singularity propagates both vertically and horizontally back to the axes. This proves (a). To prove (b), choose T to be $\sigma\{W_s^1, s > 0\}$ $-$measurable such that T is a.s. singular for W^1, which we can do. By Lemma 11b, T is not singular for W^2, so by (i), T propagates vertically, and by the statement symmetric to (ii), it does not propagate horizontally. □

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