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ON THE SUPREMUM OF A CERTAIN GAUSSIAN PROCESS

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Let $W(t)$, $0 \le t \le 1$, be the Wiener process tied down at $t = 0$, $t = 1$; $W(0) = W(1) = 0$. We find the distribution of $\sup_{0\le t\le 1} W(t) - \int_0^1 W(t)\,dt$ in terms of the zeros of the Airy function and the positive stable density of exponent ⅔. This corresponds to the distribution of the supremum of a certain stationary, mean zero, periodic Gaussian process. It is also the limiting distribution of an optimal test statistic for the isotropy of a set of directions, proposed by G. S. Watson.

1. Statement of theorem. Let $W(t)$, $0 \le t \le 1$, be the "tied down" Wiener process, $W(0) = W(1) = 0$. In this paper we find the distribution of the random variable

$$G = \sup_{0\le t\le 1} W(t) - \int_0^1 W(t)\,dt. \tag{1.1}$$

THEOREM. *Let* $0 < \alpha_1 < \alpha_2 < \cdots$ *be the zeros of the function*

$$J_{1/3}(\alpha) + J_{-1/3}(\alpha) \tag{1.2}$$

where J_ν *is the standard Bessel function, and let* $\psi(x)$ *denote the density of the positive stable distribution of exponent* ⅔; *i.e.*,

$$\int_0^\infty \exp(-\lambda x)\psi(x)\,dx = \exp(-\lambda^{2/3}), \quad \lambda \ge 0. \tag{1.3}$$

Then

$$P(G < x) = \frac{4\sqrt{\pi}}{3} \sum_{n=1}^\infty \frac{1}{\alpha_n}\, \psi\left(\frac{\sqrt{8}x}{3\alpha_n}\right), \quad x \ge 0. \tag{1.4}$$

This is a reasonably explicit formula, but it seems difficult to get even qualitative information from it. The writer is preparing a numerical tabulation.

The problem of finding the distribution of G, given by (1.1), was proposed by G. S. Watson (1976); it is the limiting distribution of an optimal test statistic for the uniformity of the distribution on a circle.

2. A stationary Gaussian process. With $W(t)$ as above define

$$Y(t) = W(t) - \int_0^1 W(\tau)\,d\tau, \quad 0 \le t \le 1. \tag{2.1}$$

Then $Y(t)$ is Gaussian, mean 0, and a straightforward calculation yields for its covariance

$$E(Y(t_1)Y(t_2)) = r(t_1 - t_2)$$

where

$$r(t) = \frac{1}{2}\left(|t| - \frac{1}{2}\right)^2 - \frac{1}{24}, \quad |t| \le 1. \tag{2.2}$$

Consequently if $Y^*(t)$ is the process $Y(t)$ of (2.1) extended periodically over $-\infty < t < \infty$,

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it is seen that $Y^*(t)$ is the stationary, mean zero, Gaussian process with the covariance (2.2) likewise extended periodically. Plainly $G = \sup Y^*(t)$, and the theorem adds another member to the small list of non-Markovian stationary Gaussian processes for which the distribution of the supremum is known. According to Shepp and Slepian (1976), there were two such cases known in 1976, to which these authors added a third.

3. Proof of the Theorem. Let $X(t)$, $t \geq 0$ be the usual Wiener process, with $X(0) = 0$. We will have occasion to use $X(t)$ "tied down" at 0 and t with respective values a and b. That is, we consider the process

$$\bar{X}(\tau) = X(\tau) + a - \frac{\tau}{t}(X(t) - b + a), \quad 0 \leq \tau \leq t.$$

Since we are only going to evaluate expectations of functionals of $\bar{X}(\tau)$ over $0 \leq \tau \leq t$, we simplify the notation by using $X(\tau)$ and conditioning these expectations by $X(0) = a$, $X(t) = b$. The absence of a condition at $t = 0$ implies $X(0) = 0$.

In the definitions of the following functions, the variables x, y, z, λ, t are all non-negative.

$M(t) = \sup_{0 \leq \tau \leq t} X(\tau)$, $m(t) = \inf_{0 \leq \tau \leq t} X(\tau)$, I_A = indicator function of the event A

$$g(z, y, t) = E\left(\exp\left(-\lambda \int_0^t (z - X(\tau))\, d\tau\right) I_{(M(t) < z)} \,|\, X(t) = y\right) \tag{3.1}$$

$$h(x, y, t) = E\left(\exp\left(-\lambda \int_0^t X(\tau)\, d\tau\right) I_{(m(t) > 0)} \,|\, X(0) = x, X(t) = y\right). \tag{3.2}$$

By noting that $-X(t)$ has the same distribution as $X(t)$, and considering the process $z - X(\tau)$ in (3.1), it is seen that

$$g(z, y, t) = h(z, z - y, t). \tag{3.3}$$

In (3.1), we integrate on z over $0 \leq z < \infty$; by Fubini's theorem we can integrate under the expectation sign. Since

$$\int_0^\infty \exp(-\lambda z t) I_{(M(t) < z)}\, dz = \frac{1}{\lambda t} \exp(-\lambda t M(t))$$

we get

$$E\left(\exp\left(-\lambda\left[tM(t) - \int_0^t X(\tau)\, d\tau\right]\right) \middle|\, X(t) = y\right) = \lambda t \int_0^\infty g(z, y, t)\, dz. \tag{3.4}$$

Hence if we can determine the function g in (3.1) and evaluate the integral on the right hand side of (3.4), we can find, by setting $y = 0$ and $t = 1$, the Laplace transform of G given by (1.1), i.e., $E(\exp(-\lambda G))$.

We turn to the function h defined by (3.2), taking its Laplace transform with the transition density of the Wiener process. Namely, define

$$r(x, y) = \int_0^\infty e^{-st} h(x, y, t) \exp\left(-\frac{(y - x)^2}{2t}\right) \frac{dt}{\sqrt{2\pi t}}. \tag{3.5}$$

As a function of the variable y, r is the Green's function of the differential equation

$$-\frac{1}{2}\frac{d^2 r}{dy^2} + \lambda y r + s r = 0, \quad y \neq x,$$

i.e., r satisfies the differential equation except at $y = x$, r vanishes at $y = 0$, $r \in L_2(0, \infty)$, r

is continuous on $0 \le y < \infty$, but dr/dy has a jump discontinuity of magnitude 2 at $y = x$. Such a solution exists and is unique. For these points and a general account of the distribution of functionals of the form $\int_0^t V(X(\tau))\, d\tau$ see M. Kac (1949, 1951), Darling and Siegert (1956).

The Sturm-Liouville system

$$(3.6) \qquad -\frac{1}{2}\frac{d^2\varphi}{dy^2} + \lambda y\varphi = -s\varphi, \qquad 0 \le y < \infty; \qquad \varphi(0) = 0, \qquad \varphi \in L_2(0, \infty)$$

is known to have a discrete spectrum $\{s_1, s_2, \cdots\}$ with

$$(3.7) \qquad s_n = -\frac{\lambda^{2/3}}{2^{1/3}}\sigma_n$$

where σ_n is the nth zero of the Airy function

$$(3.8) \qquad J_{1/3}\left(\frac{2}{3}\sigma^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\sigma^{3/2}\right)$$

$0 < \sigma_1 < \sigma_2 < \cdots$ in which J_ν is the standard Bessel function. See E. Titchmarsh (1946), Section 4.12, where this example is worked out in detail. If $\varphi_1, \varphi_2, \cdots$ are the corresponding normalized eigenfunctions of (3.6) (given explicitly in Titchmarsh, 1946), the Green's function is given by

$$(3.9) \qquad r(x, y) = \sum_{n=1}^\infty \frac{\varphi_n(x)\varphi_n(y)}{s - s_n}.$$

This series converges absolutely and, as a function of either variable, uniformly; Titchmarsh (1946), Section 2.13 et seq., Kac (1951). Since $(s - s_n)^{-1}$ is the Laplace transform of $\exp(ts_n)$ we obtain from (3.5) and (3.9)

$$h(x, y, t) = \sum_{n=1}^\infty e^{ts_n}\varphi_n(x)\varphi_n(y)\sqrt{2\pi t}\exp\left(\frac{(y - x)^2}{2t}\right).$$

If we set $y = 0$, $x = z$ and use (3.3), (3.7) we get

$$g(z, 0, t) = h(z, z, t) = \sqrt{2\pi t}\sum_{n=1}^\infty \exp\left(-\frac{t\lambda^{2/3}\sigma_n}{2^{1/3}}\right)\varphi_n^2(z),$$

and if we integrate on $0 \le z < \infty$, recalling that the φ_n are normalized eigenfunctions, we obtain

$$\int_0^\infty g(z, 0, t)\, dz = \sqrt{2\pi t}\sum_{n=1}^\infty \exp\left(-\frac{t\lambda^{2/3}\sigma_n}{2^{1/3}}\right).$$

Setting here $t = 1$ and using (3.4), (1.1)

$$(3.10) \qquad \begin{aligned} &E\left(\exp\left(-\lambda\left[M(1) - \int_0^1 X(\tau)\, d\tau\right]\right) \Big| X(1) = 0\right) \\ &\quad = E(\exp(-\lambda G)) = \lambda\sqrt{2\pi}\sum_{n=1}^\infty \exp\left(-\frac{\lambda^{2/3}\sigma_n}{2^{1/3}}\right). \end{aligned}$$

To conclude the proof of (1.4) we note that $\lambda^{-1}E(\exp(-\lambda G))$ is the Laplace transform of the function $P(G < x)$ while $\exp(-\lambda^{2/3}\sigma_n/2^{1/3})$ is the Laplace transform of the density of $\sigma_n^{3/2}T/\sqrt{2}$, where T has the density $\psi(x)$ determined by (1.3). From (3.8) we have $\alpha_n = (2/3)\,\sigma_n^{3/2}$, where α_n are the zeros of (1.2). Thus (3.10) yields

$$\int_0^\infty e^{-\lambda x}P(G < x)\, dx = \sqrt{2\pi}\sum_1^\infty E\left(\exp\left(-\frac{\lambda 3\alpha_n T}{\sqrt{8}}\right)\right)$$

and finally

$$\int_0^\infty e^{-\lambda x} P(G < x)\, dx = \sqrt{2\pi} \sum_1^\infty \int_0^\infty \exp\left(\frac{-\lambda 3\alpha_n x}{\sqrt{8}}\right) \psi(x)\, dx$$

$$= \frac{4\sqrt{\pi}}{3} \int_0^\infty e^{-\lambda x} \sum_1^\infty \frac{1}{\alpha_n} \psi\left(\frac{\sqrt{8}\, x}{3\alpha_n}\right) dx \quad .$$

so that we obtain (1.4).

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