The Annals of Probability
1983, Vol 11, No. 4, 1048–1050

A MULTIDIMENSIONAL CLT FOR MAXIMA OF NORMED SUMS

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It is shown that if $S_{k,j} = \sum_{i=1}^{k} X_{ij}$, $1 \le j \le d$, $k \ge 1$ where $(X_{i1}, \cdots, X_{id})$, $i \ge 1$ are i.i.d. random vectors with positive mean vector $(\mu_1, \cdots, \mu_d)$ and finite covariance matrix Σ, then for any choice of α_j in $[0, 1)$, $1 \le j \le d$ the random vector whose jth component is $n^{\alpha_j - 1/2}(\max_{1\le k\le n} S_{k,j}/k^{\alpha_j} - \mu_j n^{1-\alpha_j})$ converges in law to a multinormal distribution with mean vector zero and covariance matrix Σ, thereby extending a result of Teicher when $d = 1$.

1. Introduction. In [7], it was proved that if $S_n = \sum_{i=1}^{n} X_i$, $n \ge 1$ where $\{X_i, i \ge 1\}$ are i.i.d. random variables with mean $\mu > 0$ and finite variance σ^2, then for any α in $[0, 1)$

$$n^{\alpha-1/2}\left(\max_{1\le k\le n} \frac{S_k}{k^{\alpha}} - \mu n^{1-\alpha}\right) \to_{\mathscr{D}} N_{0,\sigma^2} \tag{1}$$

where N_{0,σ^2} denotes a normal random variable with mean zero and variance σ^2 and $\mathscr{D}$ signifies convergence in distribution. Note that (1) remains true trivially when $\sigma = 0$ if the right side is interpreted in customary fashion as zero.

Here, it will be shown that (1) is susceptible of the following multivariate generalization. To say that a vector is positive will signify that all of its components are positive.

THEOREM 1. *If* $S_k = (S_{k1}, \cdots, S_{kd}) = \sum_{i=1}^{k} X_i$, $k \ge 1$ *where* $X_i = (X_{i1}, \cdots, X_{id})$, $i \ge 1$ *are* i.i.d. *random vectors with positive mean vector* $\mu = (\mu_1, \cdots, \mu_d)$ *and finite covariance matrix* Σ, *then for any constant vector* $\alpha = (\alpha_1, \cdots, \alpha_d)$ *whose components lie in* $[0, 1)$

$$\left(n^{\alpha_1-1/2}\left(\max_{1\le k\le n} \frac{S_{k1}}{k^{\alpha_1}} - \mu_1 n^{1-\alpha_1}\right), \cdots, n^{\alpha_d-1/2}\left(\max_{1\le k\le n} \frac{S_{kd}}{k^{\alpha_d}} - \mu_d n^{1-\alpha_d}\right)\right) \to_{\mathscr{D}} N_{0,\Sigma}^{d} \tag{2}$$

where $N_{0,\Sigma}^{d}$ *signifies a d-dimensional normal random vector with mean vector zero and covariance matrix* Σ.

Note that the only moment constraints are that the means be positive and the variances finite. If exactly r of the variances vanish then (2) is effectively a statement about a vector of dimension $d - r$. Even in the case of primary interest $r = 0$, the covariance matrix need not be positive definite.

2. Mainstream. In the course of establishing the theorem, a multivariate analogue of a central limit theorem of Siegmund [5] will be proved and this, in turn, necessitates a multivariate generalization of a result of Anscombe [2]. Define the stopping rules

$$T_{c,j}(\alpha) = \inf\{n \ge 1 : S_{nj} > cn^{\alpha}\}, \quad c > 0, 0 \le \alpha < 1. \tag{3}$$

Then, as is well known [4] when $\mu_j > 0$, setting $T_{c,j} = T_{c,j}(\alpha_j)$ where $0 \le \alpha_j < 1$, $1 \le j \le d$.

$$T_{c,j} \Big/ \left(\frac{c}{\mu_j}\right)^{1/1-\alpha_j} \to_{\text{a.c.}} 1 \quad \text{as} \quad c \to \infty, 1 \le j \le d. \tag{4}$$

Received June 1982; revised November 1982.

AMS 1980 *subject classifications.* Primary 60F05; secondary 60K05.

Key words and phrases. Multivariate CLT, maxima of normed sums, stopping rules.

[1] Research supported by National Science Foundation under grant MCS-8005481.

[2] This paper was delivered at a statistical research conference dedicated to the memory of Jack Kiefer and Jacob Wolfowitz, held at Cornell University, July, 1983, sponsored by the ARO, NSF, ONR, IEEE and Cornell University.

THEOREM 2. *Under the hypothesis of Theorem* 1 *if* $\min_{1\le j\le d} c_j \to \infty$ *and*

$$(c_j/\mu_j)^{1/1-\alpha_j} \sim (c_k/\mu_k)^{1/1-\alpha_k}, \quad 1 \le j < k \le d \tag{5}$$

then the random vector whose jth *component is*

$$\frac{\mu_j(1-\alpha_j)[T_{c_j,j} - (c_j/\mu_j)^{1/1-\alpha_j}]}{(c_j/\mu_j)^{1/2(1-\alpha_j)}}$$

converges in distribution to $N^d_{0,\Sigma}$.

As alluded to, the proof ultimately rests upon a multivariate version of a central limit theorem for a sum of a random number of random variables.

THEOREM 3. *For each* $j = 1, 2, \cdots, d$ *let* $0 < b_{c,j} \uparrow \infty$ *as* $0 < c \uparrow \infty$ *and let* $T_{c,j}$ *be positive integer valued random variables such that as* $c_j \to \infty$, $1 \le j \le d$

$$T_{c_j,j}/b_{c_j,j} \to_P 1, \quad 1 \le j \le d \tag{6}$$

where

$$b_{c_j,j}/b_{c_k,k} \to 1, \quad 1 \le j \le k \le d. \tag{7}$$

Then, if $Y_i = (Y_{i1}, \cdots, Y_{id})$, $i \ge 1$ *are* i.i.d. *random vectors with mean vector zero and covariance matrix* Σ,

$$\left(T_{c_1,1}^{-1/2} \sum_{i=1}^{T_{c_1,1}} Y_{i1}, \cdots, T_{c_d,d}^{-1/2} \sum_{i=1}^{T_{c_d,d}} Y_{id}\right) \to_{\mathscr{D}} N^d_{0,\Sigma}. \tag{8}$$

PROOF. It suffices to consider the case that all variances are positive. If $k_{c_j,j}$ = greatest integer $\le b_{c_j,j}$, then $k_{c_j,j} \to \infty$ as $c_j \to \infty$, $1 \le j \le d$. In view of (7), $m \equiv k_{c_1,1} \sim k_{c_j,j}$, $1 \le j \le d$ as $\min_{1\le j\le d} c_j \to \infty$ and so (6) ensures that as $\min_j c_j \to \infty$

$$T_{c_j,j}/m \to_P 1, \quad 1 \le j \le d. \tag{9}$$

Now, setting $V(m, j) = \sum_{i=1}^m Y_{ij}$

$$\frac{V(T_{c_j,j}, j)}{T_{c_j,j}^{1/2}} = \left(\frac{m}{T_{c_j,j}}\right)^{1/2}\left[\frac{V(m,j)}{m^{1/2}} + \frac{V(T_{c_j,j}, j) - V(m,j)}{m^{1/2}}\right]. \tag{10}$$

For $j = 1, 2, \cdots, d$, via Kolmogorov's inequality and (9), the second term within brackets converges in probability to zero as $\min_j c_j \to \infty$. Moreover, the vector whose jth component is the first term within brackets converges in distribution to $N^d_{0,\Sigma}$. It follows that the same is true for the vector whose jth component is the left side of (10). □

A condition such as (7) is needed if the covariance matrix of the limit distribution is to remain unchanged [6].

PROOF OF THEOREM 2. To reduce the level of subscripts, denote $S_{n,j}$ by $S(n, j)$. According to (4) and Theorem 3, the random vector whose jth component is

$$\left\{T_{c_j,j}\left(\frac{\mu_j}{c_j}\right)^{1/1-\alpha_j}\right\}^{1/2}\left[\frac{S(T_{c_j,j}, j) - (\mu_j T_{c_j,j})}{T_{c_j,j}^{1/2}}\right] = \frac{S(T_{c_j,j}, j) - c_j T_{c_j,j}^{\alpha_j}}{(c_j/\mu_j)^{1/2(1-\alpha_j)}} + \frac{c_j T_{c_j,j}^{\alpha_j} - \mu_j T_{c_j,j}}{(c_j/\mu_j)^{1/2(1-\alpha_j)}}$$

conve.·ges in distribution to $N^d_{0,\Sigma}$. However, for $1 \le j \le d$ as $c_j \to \infty$

$$0 \le \frac{S(T_{c_j,j}, j) - c_j T_{c_j,j}^{\alpha_j}}{(c_j/\mu_j)^{1/2(1-\alpha_j)}} \le \frac{X_{T_{c_j,j},j}}{T_{c_j,j}^{1/2}}\left[T_{c_j,j}\left(\frac{\mu_j}{c_j}\right)^{1/1-\alpha_j}\right]^{1/2} \to_{\text{a.c.}} 0$$

in view of $\sum_{n=1}^\infty P\{X_{nj}^2 > n\varepsilon\} < \infty$, $\varepsilon > 0$. Hence recalling (4), the vector whose jth component is

$$\frac{c_jT^{\alpha_j}_{c_j,j} - T_{c_j,j}\mu_j}{(c_j/\mu_j)^{1/2(1-\alpha_j)}} = \frac{\mu_jT_{c_j,j}\left\{\left[T^{-1}_{c_j,j}\left(\frac{c_j}{\mu_j}\right)^{1/1-\alpha_j}\right]^{1-\alpha_j} - 1\right\}}{(c_j/\mu_j)^{1/2(1-\alpha_j)}}$$

$$= \frac{-\mu_j(1-\alpha_j)[T_{c_j,j} - (c_j/\mu_j)^{1/1-\alpha_j}](1+o(1))}{(c_j/\mu_j)^{1/2(1-\alpha_j)}}$$

converges in distribution to $N^d_{0,\Sigma}$ as $\min_j c_j \to \infty$ which is tantamount to the conclusion of Theorem 2.

We are now in a position to proceed with the

PROOF OF THEOREM 1. For $x_j \neq 0$, $1 \leq j \leq d$, define

$$c_j = c_j(n) = n^{1-\alpha_j}\mu_j + x_j n^{1/2-\alpha_j}. \tag{11}$$

Then, setting $q_j = (c_j/\mu_j)^{1/2(1-\alpha_j)}/\mu_j(1-\alpha_j)$,

$$(c_j/\mu_j)^{1/1-\alpha_j} - n \sim q_j x_j, \quad 1 \leq j \leq d \tag{12}$$

as $n \to \infty$ and, in particular, (5) holds as $n \to \infty$. Consequently, Theorem 2 is applicable and so

$$P\left\{\cap^d_{j=1}\left[\max_{1\leq k\leq n}\frac{S_{k,j}}{k^{\alpha_j}} - \mu_j n^{1-\alpha_j} \leq x_j n^{1/2-\alpha_j}\right]\right\} = P\left\{\cap^d_{j=1}\left[\max_{1\leq k\leq n}\frac{S_{k,j}}{k^{\alpha_j}} \leq c_j\right]\right\}$$

$$= P\{\cap^d_{j=1}[T_{c_j,j} > n]\} = P\left\{\cap^d_{j=1}\left[\frac{T_{c_j,j} - (c_j/\mu_j)^{1/1-\alpha_j}}{q_j} > \frac{n - (c_j/\mu_j)^{1/1-\alpha_j}}{q_j}\right]\right\}$$

$$\to \Phi_d(x_1, \cdots, x_d; \Sigma)$$

as $n \to \infty$ where $\Phi_d(x_1, \cdots, x_d; \Sigma)$ denotes the normal distribution with mean vector zero and covariance matrix Σ. □

REFERENCES

AHMAD, I. A. (1981). A note on rates of convergences in the multidimensional CLT for maximum partial sums. *Ann. Inst. Statist. Math.* **33** 375–383.

ANSCOMBE, F. (1952). Large sample theory of sequential estimation. *Proc. Cambr. Philos. Soc.* **48** 600–607.

CHOW, Y. S. and HSIUNG, C. A. (1976). Limiting behavior of max $S_j j^{-\alpha}$ and the first passage times in a random walk with positive drift. *Bull. Inst. Math., Academia Sinica* **4** 35–44.

CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory: Independence, Interchangeability, Martingales.* Springer, Berlin.

SIEGMUND, D. (1968). On the asymptotic normality of one sided stopping rules. *Ann. Math. Statist.* **39** 1493–1497.

TEICHER, H. (1965). On random sums of random vectors. *Ann. Math. Statist.* **36** 1450–1458.

TEICHER, H. (1973). A classical limit theorem without invariance or reflection. *Ann. Probab.* **1** 702–704.

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