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SOME MORE RESULTS ON INCREMENTS OF THE WIENER PROCESS

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Let $W(T)$ for $0 \leq T < \infty$ be a standard Weiner process and suppose that c_k and b_k are fixed sequences of real numbers satisfying $0 \leq c_k < b_k < \infty$. Let $K(\omega)$ be the set of limit points (as $T \to \infty$) of

$$\frac{W(b_k;\omega) - W(c_k;\omega)}{\{2(b_k - c_k)[\log(b_k/(b_k - c_k)) + \log\log b_k]\}^{1/2}}$$

where ω is a point in the probability space on which $W(T)$ is defined. We give necessary conditions on b_k and c_k to have $K(\omega) = [-1, 1]$ a.s. We also give some related results and discuss sharpness.

1. Introduction Let $W(T)$ for $0 \leq T < \infty$ be a standard Wiener process. Csörgő and Révész, in [1], studied the behavior of increments of the form $W(T) - W(T - a_T)$. In particular, they obtained conditions under which

(1.1) $$\limsup_{T\to\infty} \max_{0\leq t\leq T-a_T, 0\leq s\leq a_T} \frac{W(t+s) - W(t)}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}} \leq 1 \quad \text{a.s.}$$

and they proved ((5) of Theorem 2 of [1]) that if

(1.2a) $0 < a_T \leq T$,

(1.2b) a_T is nondecreasing, and

(1.2c) a_T/T is nonincreasing,

then

(1.3) $$\limsup_{T\to\infty} \frac{W(T + a_T) - W(T)}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}} \geq 1 \quad \text{a.s.}$$

We have been interested in variants of (1.1) and (1.3) motivated by a particular problem involving partial sums of independent random variables. In [2] we presented our theorems giving results similar to (1.1). The main purpose of this paper is to present results giving conclusions similar to (1.3).

Our results are stated and discussed in Section 2. They are proved in Section 3. We give only a few references at the end of this paper. A more extensive list is in [2].

2. Our results. Throughout this paper $\#(A)$ will denote the number of elements in the set A,

(2.1) $$\log x = \log(\max\{x, 1\}) \quad \text{and} \quad \log\log x = \log\log(\max\{x, e\})$$

so that $\log x \geq 0$ and $\log\log x \geq 0$ for all x, and C will denote various positive constants whose exact numerical values do not matter so that, for example, $1 + C = C$ might appear in this notation. We will write $A + B$ for $A \cup B$ and $\sum A_i$ for $\cup A_i$ when the sets involved in the union are pairwise disjoint (if we wish to emphasize the disjointness), and we will let

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μ denote Lebesgue measure restricted to the Borel measurable sets. We will write

$$\textstyle\sum c_k \lesssim \sum d_k \text{ (or } \eqsim \text{ or } \gtrsim)$$

if $c_k \le d_k$ for all sufficiently large k—so $\sum d_k$ diverges if $\sum c_k$ diverges.

Let ω be any point in the probability space on which $W(T)$ is defined. We use the following notation:

(2.2a) $L(\omega)$ is the set of limit points (as $T \to \infty$) of

$$\frac{W(T;\omega) - W(T - a_T;\omega)}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}}$$

and, if $\varepsilon > 0$ and $0 \le c_k < b_k < \infty$ for all k,

(2.2b) $K(\omega)$ is the set of limit points (as $k \to \infty$) of

$$\frac{W(b_k;\omega) - W(c_k;\omega)}{\{2(b_k - c_k)[\log(b_k/(b_k - c_k)) + \log\log b_k]\}^{1/2}};$$

(2.2c) $c_k^\varepsilon = \max\{c_k, \varepsilon b_k\}$;

(2.2d) $S_\varepsilon = \bigcup_{k=1}^{\infty} (c_k^\varepsilon, b_k]$, and $S_\varepsilon' = S_\varepsilon \cap [e, \infty)$.

THEOREM 2.1. *Suppose* $0 \le c_k < b_k < \infty$ *for* $k = 1, 2, \cdots$ *and that there is some* ε *in* $(0, 1)$ *such that one of the following holds*:

(2.3a) $\liminf_{T\to\infty} T^{-1}\mu([0, T] \cap S_\varepsilon) > 0$,

(2.3b) $$\int_{S_\varepsilon'} \frac{dt}{t(\log t)} = +\infty, \text{ or}$$

(2.3c) $$\int_{S_\varepsilon'} \frac{dt}{t(\log t)^\gamma} = +\infty \text{ for all } \gamma < 1.$$

Then $P\{K(\omega) \supset [-1, 1]\} = 1$. *If, in addition,* $(b_k - c_k)(b_k)^\alpha \to \infty$ *as* $k \to \infty$ *for every* $\alpha > 0$, *then* $P\{K(\omega) = [-1, 1]\} = 1$.

THEOREM 2.2. *Suppose* a_T *is measurable and that* $0 < a_T \le T$ *for all* $T > 0$. *Then* $P\{L(\omega) \supset [-1, 1]\} = 1$. *If, in addition,* $a_T T^\alpha \to \infty$ *as* $T \to \infty$ *for all* $\alpha > 0$, *then* $P\{L(\omega) = [-1, 1]\} = 1$.

COROLLARY 2.1. *If* a_T *is measurable and* $0 < a_T \le T$ *for all* $T > 0$, *then*

(2.4) $$\limsup_{T\to\infty} \frac{W(T) - W(T - a_T)}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}} \ge 1 \quad \text{a.s.}$$

COROLLARY 2.2. *Suppose* c_T *is measurable,* $0 \le c_T < b_T$ *for all* T, b_T *is continuous, and* $b_T \to \infty$ *as* $T \to \infty$. *Then*

(2.5) $$\limsup_{T\to\infty} \frac{W(b_T) - W(c_T)}{\{2(b_T - c_T)[\log(b_T/(b_T - c_T)) + \log\log b_T]\}^{1/2}} \ge 1 \quad \text{a.s.}$$

Now let k_N and N be positive integers and let $K(\omega)$ be the set of limit points of the sequence $[W(N) - W(N - k_N)]/\{2k_N[\log(N/k_N) + \log\log N]\}^{1/2}$.

COROLLARY 2.3. *If* $1 \le k_N \le N$ *for all* N *then*

(2.6) $$\limsup_{N\to\infty} \frac{W(N) - W(N - k_N)}{\{2k_N[\log(N/k_N) + \log\log N]\}^{1/2}} = 1 \quad \text{a.s.}$$

and, in fact, $P\{K(\omega) = [-1, 1]\} = 1$.

For notational convenience we define

(2.7) $$d(t, a) = \{2a[\log(t/a) + \log\log t]\}^{1/2}.$$

In [2] we used the denominator $\delta(T, a) = \{2a[\log(T/a) + \log\log a]\}^{1/2}$ instead of the denominator $d(T, a)$ which is used in this paper and by Csörgő and Révész in [1]. (I.e., in that paper we used the term log log a instead of log log T.) Now $\delta(T, a)/d(T, a) \to 1$ as $T \to \infty$ uniformly for $0 < a \le T$ (for fixed γ in (0, 1) consider separately the cases $a \le T^\gamma$ and $T^\gamma \le a$) so that, since our results in this paper are asymptotic, we get the same results with either denominator.

If we think of $W(T + a_T) - W(T)$ as being a "lead increment" (in the sense that the interval $(T, T + a_T]$ leads T) and $W(T) - W(T - a_T)$ as being a "lag increment", then when $t = T - a_T$ in the various Csörgő and Révész theorems from [1] they get the weighted lag increment $[W(T) - W(T - a_T)]/d(T, a_T)$. This would have been the "natural" thing for them to look at in their Theorem 2; in fact, if one looks at their proof starting with "Step 2" on page 735, they actually deal with lag increments instead of with lead increments. Their results are stated in terms of lead increments, apparently because they want to compare their results with those of Lai in [3] and [4]. Note that, when dealing with lag increments, it is possible to obtain a lower bound of one on $\limsup_{T\to\infty}[W(T) - W(T - a_T)]/d(T, a_T)$ without making any assumptions whatsoever on a_T (except $0 < a_T \le T$). (See our Corollary 2.1. Also see Theorem 2.2.) The corresponding "Theorem" for lead sums can be shown, by example, to be false without additional assumption(s) on a_T. If we let $c_T = T$ and $b_T = T + a_T$ then, if $a_T \ge 0$ and a_T is continuous, our Corollary 2.2 gives a generalization of the Csörgő-Révész result for lead sums. (Note that, as stated by Csörgő and Révész in the middle of page 735 of [1], their conditions—a_T nondecreasing and a_T/T nonincreasing—imply that a_T is continuous.)

The proof of Theorem 2.2 involves looking at sequences $[W(T_k) - W(T_k - a_{T_k})]/d(T_k, a_{T_k})$ for certain sequences T_k. Our results in Lemma 3.2 and Theorem 2.1 came out of our efforts to deal with the problem of this paper for sequences rather than for the continuous case. If

(2.8a) $b_k \to \infty$ as $k \to \infty$,

(2.8b) $b_{k-1} \le c_k$ for all $k \ge 2$, and

(2.8c) there is an $\varepsilon > 0$ such that $\varepsilon b_k \le c_k \le (1 - \varepsilon) b_k$ for all k,

then condition (2.3c) is exactly what is needed to obtain

(2.9) $$\limsup_{k\to\infty}[W(b_k) - W(c_k)]/d(b_k, b_k - c_k) \ge 1 \quad \text{a.s.}$$

I.e., (2.3c) is necessary and sufficient for (2.9) *in this case* (as can be seen from our proofs). However, we have an example showing that Theorem 2.1 is false if (2.3) is not assumed. In particular, we have a sequence $\{(b_k, c_k)\}$ for which (2.3c) is violated for every ε in (0, 1) and for which $\limsup_{k\to\infty}[W(b_k) - W(c_k)]/d(b_k, b_k - c_k) < 1$ a.s. We have another example where (2.3c) is violated but where (2.9) holds anyway.

3. Proofs.

LEMMA 3.1. *For each fixed $a > 0$, $d(t, a)$ is an increasing function of t for $t \ge a$. For each fixed $t \ge \exp\{e^e\}$, $d(t, a)$ is an increasing function of a for $0 < a \le t$.*

The proof of this lemma is straightforward and omitted.

We now prove the following simplified version of Theorem 2.1 as a lemma.

LEMMA 3.2. *Suppose $0 \le c_k < b_k < \infty$ for $k = 1, 2, \cdots$; that $0 < \varepsilon < 1$; that $c_k \ge \varepsilon b_k$ for all k; that $b_k \to \infty$ as $k \to \infty$; and that* (2.3a) *or* (2.3b) *or* (2.3c) *holds. Then $P\{K(\omega) \supset [-1, 1]\} = 1$.*

PROOF OF LEMMA 3.2. Obviously (2.3b) implies (2.3c). It is also true that (2.3a) implies (2.3b). Thus it suffices to prove this lemma under the assumption that (2.3c) is true.

Let $Z_k = [W(b_k) - W(c_k)]/d(b_k, b_k - c_k)$. To prove that $P\{K(\omega) \supset [-1, 1]\} = 1$ it suffices to prove that for every pair (d, d') with $0 < d < d' < 1$ or $-1 < d < d' < 0$ there is a set K (possibly depending on (d, d')) such that $P\{Z_k \in [d, d']$ for infinitely many k's in $K\} = 1$. The argument for the case $-1 < d < d' < 0$ is like the argument for the case $0 < d < d' < 1$ and will be omitted.

Fix $0 < d < d' < 1$. Note that $c_k = c_k^{\varepsilon}$ for all k. Choose $\lambda > 1/\varepsilon$ and note that

$$b_k \geq \lambda^n \text{ implies } c_k > \lambda^{n-1}, \quad \text{and} \quad c_k \leq \lambda^n \text{ implies } b_k < \lambda^{n+1}. \tag{3.1}$$

For $k \geq 1$ let $I_k = (\lambda^{k-1}, \lambda^k] \cap S'_\varepsilon$. Let $\gamma = (d')^2$ and define the measure σ on the Borel subsets by

$$\sigma(A) = \int_A \frac{1}{t(\log t)^\gamma} I_{[e,\infty)}(t)\, dt. \tag{3.2}$$

Fix n. Let $\mathscr{I}_{n,1} = \{(c_k, b_k] \mid I_n \cap (c_k, b_k] \neq \phi\}$. Because $c_k \geq \varepsilon b_k > b_k/\lambda$ we have

$$\lambda^{n-2} < c_k \text{ and } b_k < \lambda^{n+1} \quad \text{for all} \quad (c_k, b_k] \in \mathscr{I}_{n,1}. \tag{3.3}$$

Because $b_k \to \infty$ (and hence $c_k \to \infty$) as $k \to \infty$, $\mathscr{I}_{n,1}$ is a finite collection. Let $\mathscr{I}_{n,2} = \{J_1, \cdots, J_m\}$ be a minimal subcollection of $\mathscr{I}_{n,1}$ which has the property that $\bigcup_{J \in \mathscr{I}_{n,2}} J = \bigcup_{J \in \mathscr{I}_{n,1}} J$. (I.e., if $\mathscr{I}^*$ is any proper subcollection of $\mathscr{I}_{n,2}$ then $\bigcup_{J \in \mathscr{I}^*} J \neq \bigcup_{J \in \mathscr{I}_{n,1}} J$.) Suppose $J_i = (c_{k_i}, b_{k_i}]$ and that the indices are chosen so that $b_{k_1} \leq \cdots \leq b_{k_m}$. Then because of the minimality of $\mathscr{I}_{n,2}$

$$b_{k_i} < b_{k_{i+1}} \quad \text{for} \quad i = 1, \cdots, m-1, \text{ and} \tag{3.4a}$$

$$b_{k_i} \leq c_{k_{i+2}} \quad \text{for} \quad i = 1, \cdots, m-2. \tag{3.4b}$$

Thus if we define $\mathscr{I}_0 = \{J_i \in \mathscr{I}_{n,2}: i \text{ is odd}\}$ and $\mathscr{I}_e = \{J_i \in \mathscr{I}_{n,2}: i \text{ is even}\}$

$$\mathscr{I}_0 \text{ and } \mathscr{I}_e \text{ are both collections of pairwise disjoints intervals.} \tag{3.5}$$

Let $\mathscr{I}_n$ be either $\mathscr{I}_0$ or $\mathscr{I}_e$, the choice being made so that

$$\textstyle\sum_{j \in \mathscr{I}_n} \sigma(I) \geq \frac{1}{2}\, \sigma(\bigcup_{J \in \mathscr{I}_{n,2}} J) = \frac{1}{2}\, \sigma(\bigcup_{J \in \mathscr{I}_{n,1}} J). \tag{3.6}$$

Now let $K_0 = \{k \mid (c_k, b_k] \in \mathscr{I}_n$ for some odd $n\}$ and let $K_e = \{k \mid (c_k, b_k] \in \mathscr{I}_n$ for some even $n\}$ except that if several indices give the same interval $(c, b]$, at most one of those indices is in K_0 and at most one of those indices is in K_e. Then

$$\{(c_k, b_k] \mid k \in K_0\} = \{(c_k, b_k] \mid (c_k, b_k] \in \mathscr{I}_n \text{ for some odd } n\} \text{ and } \{(c_k, b_k] \mid k \in K_e\}$$

and both pairwise disjoint collections of sets. Define

$$S_0 = \textstyle\sum_{k \in K_0} (c_k, b_k] = \sum_{n \text{ odd}} \sum_{J \in \mathscr{I}_n} J$$

and

$$S_e = \textstyle\sum_{k \in K_e} (c_k, b_k] = \sum_{n \text{ even}} \sum_{J \in \mathscr{I}_n} J.$$

Then by (3.6), the fact that $\sum_{J \in \mathscr{I}_{n,1}} J \supset I_n$, and (2.3c), we have

$$\begin{aligned} \sigma(S_0) + \sigma(S_e) &= \textstyle\sum_{n=1}^{\infty} \sigma(\sum_{J \in \mathscr{I}_n} J) \geq \frac{1}{2} \sum_{n=1}^{\infty} \sigma(\bigcup_{J \in \mathscr{I}_{n,1}} J) \\ &\geq \frac{1}{2} \textstyle\sum_{n=1}^{\infty} \sigma(I_n) = \frac{1}{2} \displaystyle\int_{S'_\varepsilon} \frac{dt}{t(\log t)^\gamma}\, dt = +\infty. \end{aligned}$$

Thus either $\sigma(S_0) = +\infty$, or $\sigma(S_e) = +\infty$, or both.

If $\sigma(S_0) = +\infty$ let $K = K_0$ and $S = S_0$; otherwise let $K = K_e$ and $S = S_e$. In either case

$$\int_S \frac{dt}{t(\log t)^\gamma} = +\infty. \tag{3.7}$$

Since $\{(c_k, d_k]: k \in K\}$ is a collection of pairwise disjoint sets, $\{Z_k: k \in K\}$ is an independent collection of random variables. Hence, to prove that

$$P\{Z_k \in [d, d'] \text{ for infinitely many } k\text{'s in } K\} = 1$$

we need only prove that

$$\sum_{k\in K} P\{Z_k \in [d, d']\} = +\infty.$$

Note that, because $c_k \leq \varepsilon\, b_k$,

$$\frac{b_k - c_k}{b_k(\log b_k)^\gamma} = \int_{c_k}^{b_k} \frac{dt}{b_k(\log b_k)^\gamma} \geq \frac{c_k(\log c_k)^\gamma}{b_k(\log b_k)^\gamma}\int_{c_k}^{b_k} \frac{dt}{t(\log t)^\gamma} \gtrsim \varepsilon \int_{c_k}^{b_k} \frac{dt}{t(\log t)^\gamma}. \tag{3.8}$$

Let Z denote a unit normal random variable and let

$$D_k = [d\{2\log[(b_k \log b_k)/(b_k - c_k)]\}^{1/2}, d'\{2\log[(b_k\log b_k)/(b_k - c_k)]\}^{1/2}].$$

We have for fixed $0 < d < d' < 1$

$$\begin{aligned}\sum_{k\in K} P\{Z_k \in (d, d')\} &= \sum_{k\in K} P\{Z \in D_k\}\\ &\geq C\sum_{k\in K}\left[\log\left(\frac{b_k\log b_k}{b_k - c_k}\right)\right]^{1/2} \exp\left\{-(d')^2\log\left(\frac{b_k\log b_k}{b_k - c_k}\right)\right\}\\ &\geq C\sum_{k\in K}\left(\frac{b_k - c_k}{b_k\log b_k}\right)^{(d')^2} \geq C\sum_{k\in K}\frac{b_k - c_k}{b_k(\log b_k)^\gamma}\\ &\geq C\sum_{k\in K}\int_{c_k}^{b_k}\frac{dt}{t(\log t)^\gamma} = C\int_S \frac{dt}{t(\log t)^\gamma} = \infty.\end{aligned}$$

This completes the proof of Lemma 3.2.

PROOF OF THEOREM 2.1. As was the case with Lemma 3.2, it suffices to prove this theorem under the assumption that (2.3c) holds.

We will argue that it suffices to consider $\{(c_k, b_k)\}$ such that $b_k \to \infty$ as $k \to \infty$. That will enable us to use Lemma 3.2 in our proof. Fix $\lambda > 1$. Let $I_1 = [0, \lambda]$ and for $n > 1$ let $I_n = (\lambda^{n-1}, \lambda^n]$. For $\gamma \leq 1$ define, for Borel sets A,

$$\sigma_\gamma(A) = \int_A \frac{1}{t(\log t)^\gamma} I_{[e,\infty)}(t)\, dt.$$

For each positive integer n there exists a finite subset S_n of the positive integers such that $k \in S_n$ implies $(c_k^\varepsilon, b_k] \cap I_n \neq \phi$ and such that

$$\sigma_{1/2}[I_n \cap (S_\varepsilon - \cup_{k\in S_n}(c_k^\varepsilon, b_k])] < 2^{-n}. \tag{3.9}$$

Define $S^* = \cup_{n=1}^\infty S_n$ and $R = (\cup_{n\in S^*}(c_n, b_n]) \cap S'_\varepsilon$. Then $\sigma_{1/2}(S'_\varepsilon - R) < 1$ so $\sigma_\gamma(S'_\varepsilon - R) < 1$ for all γ in $[1/2, 1)$. Thus $\int_R (dt/t(\log t)^\gamma) = +\infty$ for all γ in $[1/2, 1)$ and hence for all γ in $(0, 1)$. In addition, $b_k < \lambda^n$ for only finitely many k's in S^* (it is possible only for $k \in \cup_{j=1}^{n-1} S_n$) so $\lim_{k\to\infty, k\in S^*} b_k = \infty$. To prove that $P\{K(\omega) \supset [-1, 1]\}$ it suffices to prove the same result for some subsequence. We shall do so for the k's in S^* but, in order to reduce notational problems, will simply assume that $b_k \to \infty$ in the original theorem statement instead. Call the original ε of the theorem ε_0. For each ε in $(0, \varepsilon_0]$ we write

$$[W(b_k) - W(c_k)]/d(b_k, b_k - c_k) = U_k^\varepsilon R_k^\varepsilon + V_k^\varepsilon S_k^\varepsilon \tag{3.10}$$

where

$$U_k^\varepsilon = [W(b_k) - W(c_k^\varepsilon)]/d(b_k, b_k - c_k^\varepsilon), \tag{3.10a}$$

$$R_k^\varepsilon = d(b_k, b_k - c_k^\varepsilon)/d(b_k, b_k - c_k), \tag{3.10b}$$

$$V_k^\varepsilon = [W(c_k^\varepsilon) - W(c_k)]/d(c_k^\varepsilon, \max\{1, c_k^\varepsilon - c_k\}), \text{ and} \tag{3.10c}$$

$$S_k^\varepsilon = \begin{cases} 0 & \text{if } c_k = c_k^\varepsilon \\ d(c_k^\varepsilon, \max\{1, c_k^\varepsilon - c_k\})/d(b_k, b_k - c_k) & \text{otherwise.} \end{cases} \tag{3.10d}$$

Now if (2.3c) is satisfied for some epsilon, in particular for ε_0, it is satisfied for every smaller epsilon. Thus the conditions for Lemma 3.2 are satisfied and

(3.11) the set of limit points (as $k \to \infty$) of $\{U_k^\varepsilon\}$ contains $[-1, 1]$.

Using Lemma 3.1 we see that for all b_k large enough

$$1 = d(b_k, b_k - c_k)/d(b_k, b_k - c_k) \ge R_k^\varepsilon \ge d(b_k, b_k(1-\varepsilon))/d(b_k, b_k - 0)$$
$$= \{(1-\varepsilon)\,[\log(1/(1-\varepsilon)) + \log\log b_k]/\log\log b_k\}^{1/2}$$

so that

$$\liminf_{k\to\infty} R_k^\varepsilon \ge (1-\varepsilon)^{1/2} \quad \text{and} \quad \limsup_{k\to\infty} R_k^\varepsilon \le 1. \tag{3.12}$$

Since $c_k^\varepsilon \ge \varepsilon b_k \to \infty$, it follows immediately from (3.9b) of Theorem 3.1 of [2] that

$$\limsup_{k\to\infty} |V_k^\varepsilon| \le 1 \text{ a.s.} \tag{3.13}$$

Again using Lemma 3.1 we get

$$0 \le S_k^\varepsilon \le \frac{d(\varepsilon b_k, \varepsilon b_k)}{d(b_k, (1-\varepsilon)b_k)} = \left\{\left(\frac{\varepsilon}{1-\varepsilon}\right)\left(\frac{\log\log b_k}{\log(1/(1-\varepsilon)) + \log\log b_k}\right)\right\}^{1/2}$$

so that

$$0 \le S_k^\varepsilon \quad \text{for all} \quad k \quad \text{and} \quad \limsup_{k\to\infty} S_k^\varepsilon \le (\varepsilon/(1-\varepsilon))^{1/2}. \tag{3.14}$$

Putting (3.10) through (3.14) together we see that, with probability one, if $-1 \le x \le 1$ then there is a limit point of $\{[W(b_k) - W(c_k)]/d(b_k, b_k - c_k)\}$ within a distance of $|x|(1 - \sqrt{1-\varepsilon}) + \sqrt{\varepsilon/(1-\varepsilon)}$ from x. Since $\varepsilon > 0$ was arbitrary (as long as $\varepsilon \le \varepsilon_0$), this proves that $P\{K(\omega) \supset [-1, 1]\} = 1$.

That $P\{K(\omega) = [-1, 1]\} = 1$ under our additional assumptions follows immediately from Theorem 3.1 of [2].

PROOF OF THEOREM 2.2. To prove that $P\{L(\omega) \supset [-1, 1]\} = 1$ it suffices to prove that $P\{K(\omega) \supset [-1, 1]\} = 1$ for some appropriate sequence $(b_k, c_k) = (t_k, t_k - a_{t_k})$.

Fix ε in (0, 1) and let $C(t) = \max\{t - a_t, \varepsilon t\}$. For each positive integer n let $\mathscr{I}_{n,1} = \{(C(t), t]: t \in [n, n+1]\}$. Each interval I in $\mathscr{I}_{n,1}$ can be expressed in the form $(a - b, a + b]$ for some a and b. Then $\mathscr{I}_{n,2} = \{(a - b, a + 3b/2): (a - b, a + b] \in \mathscr{I}_{n,1}\}$ is an open cover of $[n, n+1]$ so has some finite subcover $\mathscr{I}_{n,3}$. Suppose $\mathscr{I}_{n,3}$ has k_n elements. If we define $\alpha_m = \sum_{n=1}^m k_n$ and properly index the J's we can then write $\mathscr{I}_{n,3} = \{J_k: \alpha_{n-1} < k \le \alpha_n\}$. Let $(c_k, t_k]$ with $c_k = \max\{t_k - a_{t_k}, \varepsilon t_k\}$ be the member of $\mathscr{I}_{n,1}$ corresponding to J_K. Setting $b_k = t_k$, the sequence $\{(c_k, b_k)\}$ satisfies $0 \le c_k < b_k < \infty$ for $k = 1, 2, \cdots$ and $c_k \ge \varepsilon b_k$ for all k. "Without loss of generality" we assume that $\{J_1, \cdots, J_{k_n}\}$ is a minimal open cover of $[1, N]$ ordered so that the right endpoint of J_i is less than the right endpoint of J_k if $i < k$. As in the proof of Lemma 3.2, if $\mathscr{I}_0 = \{k: 1 \le k \le \alpha_n, k \text{ odd}\}$ and $\mathscr{I}_e = \{k: 1 \le k \le \alpha_n, k \text{ even}\}$, then $\{J_k: k \in \mathscr{I}_0\}$ and $\{J_k: k \in \mathscr{I}_e\}$ are pairwise disjoint collections of open sets. Let $\mathscr{I} = \mathscr{I}_0$ if $\mu([1, N] \cap \bigcup_{k\in\mathscr{I}_0} J_k) > n/2$. Otherwise let $\mathscr{I} = \mathscr{I}_e$. Then, since $\sum_{k\in\mathscr{I}}(c_k, t_k] \subset (0, n+1]$,

$$\mu([0, n+1] \cap S_\varepsilon) \ge \mu\{[0, n+1] \cap \textstyle\sum_{k\in\mathscr{I}}(c_k, t_k]\}$$
$$\ge \mu\{[0, n+1] \cap \textstyle\bigcup_{k\in\mathscr{I}} J_k\} - \sum_{k\in\mathscr{I}} \mu(J_k - (c_k, t_k])$$
$$\ge \frac{n}{2} - \textstyle\sum_{k\in\mathscr{I}}(t_k - c_k)/5 \ge \dfrac{n}{2} - (n+1)/5 = [3(n+1) - 5]/10.$$

It follows that $\liminf_{T\to\infty} T^{-1}\mu([0, T] \cap S_\varepsilon) \geq 3/10$. An application of Theorem 2.1 shows that $P\{K(\omega) \supset [-1, 1]\} = 1$ so that $P\{L(\omega) \supset [-1, 1]\} = 1$. As before, it follows immediately from Theorem 3.1 of [2] that $P\{L(\omega) = [-1, 1]\} = 1$ under our additional assumptions.

PROOF OF COROLLARY 2.1. This is an immediate consequence of the first part of Theorem 2.2.

PROOF OF COROLLARY 2.2. Suppose $t_b = \max\{t: b_t = b\}$ and $S = \{t_b: b \geq b_0\}$. Define $a_b^* = b - a_{t_b}$. Then from Corollary 2.1

$$\begin{aligned} &\limsup_{t\to\infty}[W(b_t) - W(a_t)]/d(b_t, b_t - a_t) \\ &\quad\geq \limsup_{t\to\infty, t\in S}[W(b_t) - W(a_t)]/d(b_t, b_t - a_t) \\ &\quad= \limsup_{b\to\infty}[W(b) - W(b - a_b^*)]/d(b, a_b^*) \geq 1 \text{ a.s.} \end{aligned}$$

PROOF OF COROLLARY 2.3. This is an immediate consequence of Theorem 2.1.

REFERENCES

[1] CSÖRGŐ, M. and RÉVÉSZ, P. (1979). How big are the increments of a Wiener process? *Ann. Probab.* **7** 731–737.
[2] HANSON, D. L. and RUSSO, RALPH P. (1982). Some results on increments of the Wiener process with applications to lag sums of I.I.D. random variables. *Ann. Probab.* **11** 609–623.
[3] LAI, TZE LEUNG (1973). On Strassen-type laws of the iterated logarithm for delayed averages of the Wiener process. *Bull. Inst. Math. Acad. Sinica* **1** 29–39.
[4] LAI, TZE LEUNG (1974). Limit theorems for delayed sums. *Ann. Probab.* **2** 432–440.
[5] STOUT, WILLIAM F. (1974). *Almost Sure Convergence.* Academic, New York.

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