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OCCUPATION TIME LIMIT THEOREMS FOR THE VOTER MODEL

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Let $\{\eta_s^\theta(x)\}$, $s \geq 0$, $x \in Z^d$ be the basic voter model starting from product measure with density $\theta(0 < \theta < 1)$. We consider the asymptotic behavior, as $t \to \infty$, of the occupation time field $\{T_t^x\}_{x \in Z^d}$, where $T_t^x = \int_0^t \eta_s^\theta(x)\, ds$. Our main result is that, properly scaled and normalized, the occupation time field has a (weak) limit field as $t \to \infty$, whose covariance structure we compute explicitly. This field is Gaussian in dimensions $d \geq 2$. It is not Gaussian in dimension one, but has an "explicit" representation in terms of a system of coalescing Brownian motions. We also prove that $\lim_{t\to\infty} T_t^x/t = \theta$ a.s. for $d \geq 2$ (the result is false for $d = 1$). A striking feature of the behavior of the occupation time field is its elaborate dimension dependence.

0. Introduction. A prominent theme in contemporary mathematical physics is the dimensional dependence of the critical phenomena for a given system. Empirical evidence and simulation suggest that many physical models, e.g. the Ising model and bond percolation on Z^d, have one mode of behavior in high dimensions and quite different modes in each of the low dimensions. Very roughly, if d is large enough the interaction is negligible, whereas strength of dependence distinguishes each low dimension. For the Ising model "large enough" means $d \geq 5$ (cf. [1]), for (unoriented) percolation $d \geq 7$ [18]. Since rigorous results along these lines are hard to come by, our primary objective in this paper is to illustrate the theme of *critical dimensionality* as evidenced by one of the simplest interacting particle systems, the so-called voter model [5], [12].

Let $S = \{\text{all subsets of } Z^d\}$, $S_0 = \{\text{finite subsets of } Z^d\}$, and for $\eta \in S$ write $\eta(x) = 1_{\{x \in \eta\}}$. The (*basic*) voter model η_t is the S-valued Markov process having flip rates at each $x \in Z^d$ at time t:

$$\eta_t(x) \to 1 - \eta_t(x) \quad \text{at rate} \quad (2d)^{-1}|\{y: |y - x| = 1, \eta_t(y) \neq \eta_t(x)\}|.$$

Clifford and Sudbury [5] showed that the ergodic theory of the voter model is qualitatively different in low dimensions and in high dimensions. Denote by η_t^θ the voter model started in Bernoulli product measure $\mu_\theta (0 \leq \theta \leq 1)$, with $\mu_\theta\{\eta(x) = 1\} \equiv \theta$. If $d = 1$ or 2, then

$$P(\eta_t^\theta \in \cdot) \Rightarrow (1 - \theta)\mu_0 + \theta\mu_1 \quad \text{as} \quad t \to \infty \tag{0.1}$$

($\Rightarrow$ means weak convergence). For $d \geq 3$ on the other hand,

$$P(\eta_t^\theta \in \cdot) \Rightarrow \nu_\theta, \tag{0.2}$$

where ν_θ is an extreme invariant measure with density θ. Holley and Liggett [12] independently discovered the same result, and gave a detailed analysis of the set of invariant measures. In terms of the sample paths, (0.1) indicates clustering while (0.2) indicates stability. Thus the dichotomy (0.1) vs. (0.2) is already an example of critical dimensionality. Concerning the clustering when $d = 1$, see [4]; the macroscopic structure of ν_θ (renormalization limit) when $d \geq 3$ is determined in [3]. See also [13].

To prove (0.1) and (0.2) one uses a duality equation which connects the voter model with a system of coalescing random walks. For our purposes it is expedient to derive this

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connection from Harris' graphical representation [11]. Start with the space-time diagram $Z^d \times [0, \infty)$. For each pair of distinct $x, y \in Z^d$ with $|y - x| = 1$, draw (oriented) arrows from (x, τ_{xy}^n) to (y, τ_{xy}^n) $(n = 1, 2, \cdots)$ where the $\tau_{xy}^n - \tau_{xy}^{n-1}$ are i.i.d. exponential with mean $(2d)^{-1}(\tau_{xy}^0 = 0)$. The resulting random scheme $\mathscr{P}$ is called the percolation substructure. By a *path up from* (x_0, t_0) *to* (x_n, t_n) *in* $\mathscr{P}$ we mean a sequence of space-time points

$$(x_0, t_0), (x_0, t_1), (x_1, t_1), \cdots, (x_{n-1}, t_n), (x_n, t_n),$$

with increasing time coordinates t_k, such that for each k there is an arrow from (x_{k-1}, t_k) to (x_k, t_k) *and* no arrow arrives at x_k at any time $u \in (t_k, t_{k+1})$. See [9] for more details and a picture. Now the voter model η_t^η with initial state $\eta \in S$ can be represented as

$$\begin{aligned} \eta_t^\eta(x) &= 1 \quad \text{if } \exists \text{ a path from } (z, 0) \text{ to } (x, t) \text{ for some } z \in \eta, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Moreover, on $\mathscr{P}_t = \mathscr{P}$ restricted to $Z^d \times [0, t]$ one can consider a dual substructure $\hat{\mathscr{P}}_t$ obtained by reversing time *and* reversing the direction of all arrows. If we define dual processes $(\hat\eta_s^\eta;\ 0 \le s \le t)$ on $\hat{\mathscr{P}}_t$ as above except that no arrow should *leave* x_k in (t_k, t_{k+1}), then this dual system consists of *coalescing random walks*, and we immediately obtain the duality equation

$$P(\eta_t^\eta \cap \Lambda = \varnothing) = P(\hat\eta_t^\Lambda \cap \eta = \varnothing) \quad (\Lambda \in S_0), \tag{0.3}$$

since both sides express the probability of no path in the substructure connecting $\eta \times \{0\}$ and $\Lambda \times \{t\}$. Integrate (0.3) against μ_θ to get

$$P(\eta_t^\theta \cap \Lambda = \varnothing) = E\{(1 - \theta)^{|\hat\eta_t^\Lambda|}\}. \tag{0.4}$$

The results (0.1) and (0.2) follow easily from (0.4) and simple properties of coalescing random walks. Recurrence of the individual walks translates into clustering of the corresponding voter model, while transience corresponds to stability. See [5], [12] or [9]. Note in particular that

$$P(x \in \eta_t^\theta) = \theta \quad \text{for all} \quad x, t, \tag{0.5}$$

i.e. η_t^θ is *density preserving.*

In this paper we study the occupation time functionals of the voter model:

$$T_t^x = \int_0^t \eta_s^\theta(x)\, ds.$$

(For the rest of the paper we fix $\theta \in (0, 1)$ and drop the superscript since we consider only processes starting from nondegenerate μ_θ.) As it turns out, the limit theory for the occupation time fields $(T_t^x;\ x \in Z^d)$ as $t \to \infty$ exhibits a rather elaborate dimension dependence. Here high dimensions means $d \ge 5$, and each of the low dimension cases $d = 1, 2, 3, 4$ is quite distinct. To indicate the spirit of what will follow, let us begin by computing the covariances $\operatorname{Cov}(T_t^0, T_t^x)$. From (0.5), $E[T_t^x] = \theta t$ for all x, t, so

$$\begin{aligned} \operatorname{Cov}(T_t^0, T_t^x) &= \iint_{0 \le r, s \le t} E[(\eta_r(0) - \theta)(\eta_s(x) - \theta)]\, dr\, ds \\ &= \iint_{0 \le r, s \le t} [P(\eta_r(0) = \eta_s(x) = 1) - \theta^2]\, dr\, ds \\ &= \int_{0 \le r, s \le t} [P(\eta_r(0) = \eta_s(x) = 0) - (1 - \theta)^2]\, dr\, ds. \end{aligned} \tag{0.6}$$

In order to evaluate the last probability, consider the graphical representation and recall the duality argument leading to (0.4). The only complication is now the dual random walks start at differing times $t - r$, $t - s$ (see Figure 1). Assume $r \le s$ and consider the following

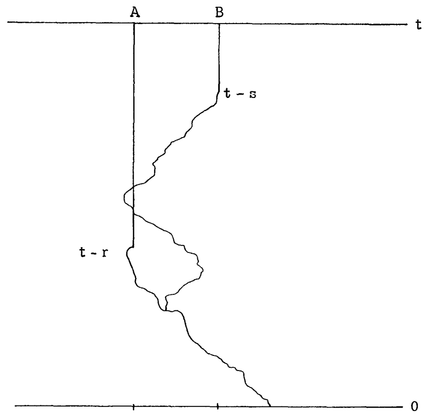

FIG. 1

2-particle coalescing system. Particle A is "frozen" at 0 until time $t - r$ and then begins at a rate 1 simple random walk. Particle B stays at x until time $t - s$ and then executes another rate 1 walk. The two walks are independent, except that if they collide *after* time $t - r$ then they coalesce into one and perform a single rate 1 walk from then on. Letting N be the number of particles in this system at time t, the duality trick yields

$$P(\eta_r(0) = \eta_s(x) = 0) = E[(1 - \theta)^N], \tag{0.7}$$

and hence

$$P(\eta_r(0) = \eta_s(x) = 0) - (1 - \theta)^2 = \theta(1 - \theta)P(N = 1). \tag{0.8}$$

Note that the distance between particle B and particle A is x for time $t - s$, then evolves as a rate 1 random walk until time $t - r$, at which time the difference random walk changes to a rate 2 random walk until the particles collide. Henceforth X_t will denote simple rate 1 random walk, P_x the law of X_t started at x. Introduce the "last exit" variables:

$$L_u^y = \sup\{t \le u : X_t = y\}, \quad (u \ge 0, y \in Z^d),$$

$$L^y = \sup\{t < \infty : X_t = y\},$$

defining the empty sup to be 0 in each case. Then after a simple change of time scale we get

$$P(N = 1) = P_x(L_{s+r}^0 > s - r) = P_0(L_{s+r}^x > s - r). \tag{0.9}$$

By monotonicity

$$P_0(L_s^x > s - r) \le P_0(L_{s+r}^x > s - r) \le P_0(L_{2s}^x > s - r)$$

so we have

$$\frac{2\theta(1 - \theta)}{t} \int_0^t E_0[L_s^x]\, ds \le t^{-1}\mathrm{Cov}(T_t^0, T_t^x) \le \frac{2\theta(1 - \theta)}{t} \int_0^t E_0[L_{2s}^x]\, ds. \tag{0.10}$$

By monotone convergence,

$$E_0[L_s^x] \uparrow E_0[L^x] \quad \text{as} \quad s \to \infty. \tag{0.11}$$

Decomposing according to the time of the last exist from 0, $d \ge 3$ we get

$$E_0[L^x] = \int_0^\infty P_0(L^x > t)\, dt = \int_0^\infty \int_t^\infty p_s(0, x)\, ds\, \gamma_d\, dt,$$

where

$$p_s(x, y) = P_x(X_s = y), \quad \gamma_d = P_e(X_t \ne 0 \ \forall\ t),$$

$e = (1, 0, \cdots, 0)$. (γ_d is the probability of no return in discrete time simple random walk; e.g., $\gamma_3 \approx .659$.) Since $p_s(0, x) = O(s^{-d/2})$,

$$E_0[L^x] = \gamma_d \int_0^\infty s p_s(0, x)\, ds < \infty \quad \text{provided} \quad d \geq 5. \tag{0.12}$$

Finally, combining (0.10), (0.11) and (0.12) we conclude that

$$\lim_{t\to\infty} t^{-1}\mathrm{Cov}(T_t^0, T_t^x) = 2\theta(1-\theta)\, \gamma_d \int_0^\infty s p_s(0, x)\, ds \tag{0.13}$$

in dimensions $d \geq 5$.

The computation (0.13) suggests that in 5 or more dimensions the random field of occupation times, normalized in the usual way, should converge to a limit as $t \to \infty$:

$$\left\{\frac{T_t^x - \theta t}{\sqrt{t}}\right\}_{x\in Z^d} \Rightarrow \xi,$$

where $\xi = \{\xi(x)\}_{x\in Z^d}$ is translation invariant with the above covariance function. We will prove such a limit theorem, and show that ξ is Gaussian. In four or fewer dimensions, as it turns out, one must choose a normalizer $\sigma(t)$ greater than $\sqrt{t}$ to obtain a limiting field. (The covariances of the occupation times grow more rapidly due to the stronger interaction.) Appropriate choices are:

$$\begin{aligned} \sigma(t) &= t && (d = 1),\\ &= t/\sqrt{\ln t} && (d = 2),\\ &= t^{3/4} && (d = 3),\\ &= \sqrt{t \ln t} && (d = 4),\\ &= \sqrt{t} && (d \geq 5). \end{aligned} \tag{0.14}$$

With σ so defined there is a limit field ξ in each dimension such that

$$\left\{\frac{T_t^x - \theta t}{\sigma(t)}\right\} \Rightarrow \xi \quad \text{as} \quad t \to \infty.$$

For $d \leq 4$, however, we find that the limiting covariances are constant. In other words, the limit field is totally correlated. To get a nontrivial covariance structure it is necessary to consider limits of scaled fields:

$$\left\{\frac{T_t^{[f(t)x]} - \theta t}{\sigma(t)}\right\} \Rightarrow \xi \quad \text{as} \quad t \to \infty. \tag{0.15}$$

(For $r \in R^d$ let $[r]$ be the nearest lattice point to r, with some convention in case of ties.) It seems reasonable to expect a "natural" scaling $f(t)$ under which ξ has nontrivial covariances, such that the limit is totally correlated when scaled by g with $\lim_{t\to\infty} g(t)/f(t) = 0$, and such that the limit is a product field when scaled by g with $\lim_{t\to\infty} g(t)/f(t) = \infty$. This scenario holds in dimensions other than 4; the natural scalings are:

$$\begin{aligned} f(t) &= \sqrt{t} && (d \leq 3),\\ &= 1 && (d \geq 5). \end{aligned} \tag{0.16}$$

In dimension 4 there is no natural scaling. Rather, by taking

$$f(t) = t^\alpha \quad (d = 4, \alpha \geq 0), \tag{0.17}$$

one gets a distinct limit for each $\alpha \in [0, ½]$. All are independent mixtures of the totally correlated limit ($\alpha = 0$) and the product limit ($\alpha = ½$).

Having determined the correct normalization and scaling, we are able to compute the covariance function of ξ explicitly in each dimension. Write

$$\mathrm{Cov}(\xi(x), \xi(y)) = \theta(1-\theta)C(y-x).$$

Exact expressions for $C(x)$ dimensions 1, 2 and 3 will be derived in Section 3. Here we simply describe the tail behavior: as $|x| \to \infty$,

$$\begin{aligned} C(x) &\sim c_1 |x|^{-5} e^{-|x|^2/4} \quad (d=1), \\ &\sim c_2 |x|^{-6} e^{-|x|^2/2} \quad (d=2), \\ &\sim c_3 |x|^{-6} e^{-3|x|^2/4} \quad (d=3), \\ &\sim c_d |x|^{4-d} \quad (d \geq 5), \end{aligned} \tag{0.18}$$

where $c_1 = 128/\sqrt{\pi}$, $c_2 = 32$, $c_3 = 64\gamma_3/(3\pi)^{3/2}$ and for $d \geq 5$, $c_d = d^2 \gamma_d \pi^{-d/2} \Gamma(d/2 - 2)$, Γ the gamma function. As mentioned above, in four dimensions $C(x) = C_\alpha(x)$ is constant,

$$C_\alpha(x) = \frac{8\gamma_4}{\pi^2}(1-2\alpha)^+ \quad (d=4), \tag{0.19}$$

and according to (0.13),

$$C(x) = 2\gamma_d \int_0^\infty s p_s(0, x)\, ds \quad (d \geq 5). \tag{0.20}$$

In Sections 4 and 5 we prove more: the limit field ξ is Gaussian in 2 or more dimensions, but *not* dimension 1. To summarize, the main result of the paper is as follows.

THEOREM 1. *With σ defined by* (0.14) *and f by* (0.16)–(0.17), *there is a random field $\xi = \{\xi_x\}_{x \in Z^d}$ such that* (0.15) *holds. ξ has covariance function $\theta(1-\theta)C(x)$ satisfying* (0.18)–(0.20), *and is Gaussian* iff $d \geq 2$.

We also prove a law of large numbers for the occupation time in two or more dimensions.

THEOREM 2. *As $t \to \infty$,*

$$T_t^x / t \to \theta \quad \textit{almost surely if} \quad d \geq 2.$$

(*There is no law of large numbers if $d = 1$.*)

Occupation time strong laws for other particle systems are discussed in [9].

Our proofs will be divided into six main parts. Section 1 contains a derivation of the generalized duality equation we will need, and some preliminary random walk estimates. In Section 2 we compute the variance of T_t^0 in order to determine the appropriate normalizer σ. Section 3 contains the more intricate covariance calculations required to obtain the right scaling f and limiting covariance $C(x)$. The proof of Theorem 1 is completed in Section 4 for $d \geq 2$, and in Section 5 for $d = 1$. Finally, we finish the proof of Theorem 2 in Section 6.

A few references to related work are in order here. There are results similar to some of ours in a very nice paper by Iscoe [14] on weighted occupation times for the measure-valued branching processes introduced by Dawson [7]. In fact, it was Iscoe's work and the intriguing parallelism between Dawson's process and the voter model which motivated the present paper. The parallel only occurs for $d \geq 3$, and only the scaling $f(t) = 1$ is considered in [14]. On the other hand, Iscoe treats a wider class of models. (Our case corresponds to his $\alpha = 2$.) Random walks for which $E_x[L^0] < \infty$ are sometimes called strongly transient. They appear, in various contexts, in the work of Port [19], Jain and

Orey [10], Lawler [15] and Aizenmann [1]. The last two references contain beautiful examples of critical dimension in physical models (self-avoiding random walks and the Ising model respectively). The distinguishing feature in high dimensions ($d \ge 5$) is strong transience.

1. Preliminaries. The key tool needed for our limit theorems is a duality equation which generalizes (0.6). To state it, we introduce the following processes, all defined on a common percolation substructure $\hat{\mathscr{P}}_t$:

$\hat{\eta}_t((x_1, s_1), \cdots, (x_n, s_n))$
= the system of coalescing rate 1 simple random walks starting at sites x_i, the walk from x_i frozen until time s_i, and such that two walks coalesce only after both are unfrozen ($n \ge 1$, $x_i \in Z^d$, $0 \le s_i \le t$);

$X_t(x, s) = \hat{\eta}_t((x, s))$
= a rate 1 simple random walk frozen at starting state x until time s;

$$N_t((x_1, s_1), \cdots, (x_n, s_n)) = |\hat{\eta}_t((x_1, s_1), \cdots, (x_n, s_n))|.$$

We will make extensive use of the following:

DUALITY EQUATION. If $x_i \in Z^d$, $s_i \ge 0$ $(1 \le i \le n)$ and $t \ge s = \max(s_1, \cdots; s_n)$, then

$$P(\eta_{s_i}(x_i) = 1 \ \forall\ 1 \le i \le n) = E[\theta^{N_t((x_1, t-s_1), \cdots (x_n, t-s_n))}]. \tag{1.1}$$

The proof extends the one for (0.6) described in the introduction. Namely, the duality trick shows that

$$P(\eta_{s_i}(x_i) = 0 \ \forall \le i \le n) = E[(1-\theta)^{N_s((x_1, s-s_1), \cdots, (x_n, s-s_n))}],$$

since both sides represents the probability that no path connects the μ_θ-distributed initial state with any of the (x_i, s_i). Since the dynamics of the voter model are symmetric in 0's and 1's we have

$$P(\eta_{s_i}(x_i) = 1 \ \forall\ 1 \le i \le n) = E[\theta^{N_s((x_1, s-s_1), \cdots, (x_n, s-s_n))}].$$

Finally, the more convenient form (1.1) follows from the observation that

$$\hat{\eta}_u((x_1, t_1), \cdots, (x_n, t_n)) =_d \hat{\eta}_{u+v}((x_1, t_1 + v), \cdots, (x_n, t_n + v))$$

provided $u \ge \max(t_1, \cdots, t_n)$.

We also collect here various estimates for rate 1 simple random walk X_t on Z^d which we will need in later sections. Recall that $p_t(x, y) = P_x(X_t = y)$, and introduce the Green's function and hitting times:

$$G_u(x, y) = \int_0^u p_t(x, y)\, dt, \quad G(u) = G_u(0, 0);$$

$$\tau_x = \inf\{t \ge 0 : X_t = x\} \quad (= \infty \quad \text{if} \quad X_t \ne x \ \forall\ t).$$

Remember also that $\gamma_d = P_e(\tau_0 = \infty)$. In the asymptotics which follow $\sim$ has the usual meaning: $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} f(t)/g(t) = 1$.

PROPOSITION 1. *As* $u \to \infty$,

$$p_u(0, 0) \sim \left(\frac{d}{2\pi u}\right)^{d/2}, \tag{1.2}$$

and

$$p_u(0, x) = \left(\frac{d}{2\pi u}\right)^{d/2}\left[\frac{u\varepsilon(x, u)}{|x|^2} + \exp\left\{-\frac{d|x|^2}{2u}\right\}\right], \tag{1.3}$$
$$\text{where} \quad \lim_{u\to\infty}\sup_x|\varepsilon(x, u)| = 0.$$

PROOF. Results P9 and P10 in Chapter II of [20] are easily adapted to our continuous time setting.

PROPOSITION 2. *As* $u \to \infty$,

$$\begin{aligned} G(2u) - G(u) &\sim \alpha_1 u^{1/2} && (d = 1), \\ &\sim \alpha_2 && (d = 2), \\ &\sim \alpha_d u^{-(d-2)/2} && (d \ge 3), \end{aligned} \tag{1.4}$$

where $\alpha_1 = (2 - \sqrt{2})/\sqrt{\pi}$, $\alpha_2 = \ln 2/\pi$, *and for* $d \ge 3$, $\alpha_d = (d/2\pi)^{d/2}(2/(d-2)) \cdot (1 - 2^{-(d-2)/2})$.

PROOF. Use (1.2) in $G(2u) - G(u) = \int_u^{2u} p_t(0, 0)\, dt$.

PROPOSITION 3. *As* $t \to \infty$

$$\begin{aligned} P_e(\tau_0 > t) &\sim \beta_1 t^{-1/2} && (d = 1), \\ &\sim \beta_2/\ln t && (d = 2), \\ &\sim \gamma_d && (d \ge 3), \end{aligned} \tag{1.5}$$

where $\beta_1 = (2/\pi)^{1/2}$ *and* $\beta_2 = \pi$.

PROOF. If Y_n denotes discrete time simple random walk, τ its hitting time for 0,

$$P_e(\tau_0 > t) = \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!} P_e(\tau > n). \tag{1.6}$$

From [20, pages 167, 381], as $n \to \infty$

$$\begin{aligned} P_e(\tau > n) &\sim \left(\frac{2}{\pi n}\right)^{1/2} && (d = 1) \\ &\sim \pi/\ln n && (d = 2). \end{aligned} \tag{1.7}$$

For $d \ge 3$,

$$\lim_{t\to\infty} P_e(\tau_0 > t) = P_e(\tau_0 = \infty) = P_e(\tau = \infty) = \gamma_d > 0. \tag{1.8}$$

Use (1.6), (1.7) and (1.8) to show (1.5).

2. Variance calculations. In Section 0 we saw that $\mathrm{Var}(T_t^0) \sim Ct$ if $d \ge 5$. Here we show that in each dimension d the variance of T_t^0 grows asymptotically like $\sigma^2(t)$, where σ is given by (0.14). Our computations are based on the formula

$$\mathrm{Cov}(T_t^0, T_t^x) = 2\theta(1-\theta) \iint_{0 \le r \le s \le t} P_0(L_{s+r}^x > s - r)\, dr\, ds, \tag{2.1}$$

obtained by combining (0.6), (0.8) and (0.9). We will use results from the last section to estimate the right side. To begin, a "last time at 0" decomposition like the one leading to (0.12) yields

$$P_0(L_{s+r}^x > s - r) = p_{s+r}(0, x) + \int_{s-r}^{s+r} p_u(0, x) P_e(\tau_0 > s + r - u)\, du.$$

It is easy to check, using (1.2) and the inequality $p_u(0, x) \le p_u(0, 0)$ that

$$\lim_{t\to\infty} \sigma^{-2}(t) \iint_{0\le r\le s\le t} p_{s+r}(0, x)\, dr\, ds = 0.$$

Consequently,

$$\frac{\operatorname{Cov}(T_t^0, T_t^x)}{\sigma^2(t)} \sim \frac{2\theta(1-\theta)}{\sigma^2(t)} \int_0^t dr \int_r^t ds \int_{s-r}^{s+r} p_u(0, x) P_e(\tau_0 > s + r - u)\, du.$$

The change of variables

$$\begin{pmatrix} v \\ w \\ r \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \\ u \end{pmatrix}$$

and integration over the variable r yields

$$\text{(2.2)} \qquad \frac{\operatorname{Cov}(T_t^0, T_t^x)}{\sigma^2(t)} \sim \frac{4\theta(1-\theta)}{\sigma^2(t)} \int_0^t dv \int_0^{t-v} dw\, P_e(\tau_0 > 2w)[G_{2v}(0, x) - G_v(0, x)].$$

Using (2.2), we now show that as $t \to \infty$,

$$\text{(2.3)} \qquad \begin{aligned} V(t) = [\theta(1-\theta)\sigma^2(t)]^{-1}\operatorname{Var}(T_t^0) &\to 2 - \sqrt{2} && (d = 1), \\ &\to 2 \ln 2 && (d = 2), \\ &\to 8 \cdot 3^{1/2} \cdot \pi^{-3/2}(2^{1/2} - 1)\gamma_3 && (d = 3), \\ &\to 8\pi^{-2}\gamma_4 && (d = 4). \end{aligned}$$

The case $d \ge 5$ has already been dealt with. The low dimension cases will be handled individually, with the aid of (2.2), (1.4) and (1.5).

$d = 1$. By taking M sufficiently large one can approximate $V(t)$ arbitrarily closely by

$$V(t) \approx 4t^{-2} \int_M^{t-M} dv \int_M^{t-v} dw \beta_1 (2w)^{-1/2} \alpha_1 v^{1/2} \sim 4\sqrt{2}\, \alpha_1 \beta_1 t^{-2} \int_M^{t-M} v^{1/2}(t - v)^{1/2}\, dv$$

$$\sim 4\sqrt{2}\, \alpha_1 \beta_1 \int_0^1 s^{1/2}(1 - s)^{1/2}\, ds,$$

as $t \to \infty$.

$d = 2$. In the same way we have

$$V(t) \approx 4\alpha_2\beta_2 t^{-2} \ln t \int_M^{t-M} \int_M^{t-v} (\ln 2w)^{-1}\, dw$$

$$= 4\alpha_2\beta_2 t^{-2} \ln t \int_M^{t-M} \left[\frac{t - v}{\ln(2t - 2v)} - \frac{M}{\ln 2M} + \int_M^{t-v} \frac{dw}{(\ln 2w)^2}\right],$$

this last from integration by parts. The contribution from the second and third terms in brackets is negligible as $t \to \infty$. A change of variables and bounded convergence give

$$t^{-2} \ln t \int_M^{t-M} \frac{t - v}{\ln(2t - 2v)}\, dv = \ln t \int_{M/t}^{1-M/t} \frac{1 - s}{\ln(2t - 2st)}\, ds \sim \int_0^1 (1 - s) = \frac{1}{2}.$$

Thus $V(t) \sim 2\alpha_2\beta_2$.

$d = 3$. For $d \geq 3$, $P_e(\tau_0 > 2v) \sim \gamma_3$ as $v \to \infty$. So as $t \to \infty$,

$$V(t) \sim 4\alpha_3\gamma_3 t^{-3/2} \int_0^t dv v^{-1/2} \int_0^{t-v} dw = 4\alpha_3\gamma_3 t^{-3/2} \int_0^t (tv^{-1/2} - v^{1/2})\, dv \sim \frac{16}{3}\alpha_3\gamma_3.$$

$d = 4$. This case is similar to the last:

$$V(t) \sim 4\alpha_4\gamma_4(t \ln t)^{-1} \int_0^t dv v^{-1} \int_0^{t-v} dw \sim 4\alpha_4\gamma_4.$$

Upon substituting the explicit values of the constants α_d and β_d given in Section 1, we obtain (2.3) as desired.

3. Covariance calculations. In this section we show that under the scaling $f(t)$ given by (0.16)–(0.17), the normalized occupation times have nontrivial covariances. Specifically, we prove that

$$\lim_{t\to\infty} \frac{\operatorname{Cov}(T_t^0, T_t^{[xf(t)]})}{\theta(1-\theta)\sigma^2(t)} = C(x), \tag{3.1}$$

with $C(x)$ satisfying (0.18)–(0.19).

Using (2.2), let us rewrite the left side of (3.1) (asymptotically in t) as

$$\sim 4\sigma^{-2}(t)\left[\int_0^{2t} du p_u(0, [x\sqrt{t}]) \int_{u/2}^t dv \int_0^{t-v} P_e(\tau_0 > 2w)\, dw \right.$$

$$\left. - \int_0^t du p_u(0, [x\sqrt{t}]) \int_u^t du \int_0^{t-v} P_e(\tau_0 > 2w)\, dw\right]$$

$$= 4\sigma^{-2}(t)[I_1(t) - I_2(t)].$$

For $d = 1, 2, 3$, we put $f(t) = \sqrt{t}$ and use (1.3) and (1.5) to estimate I_1, I_2. (For brevity we will omit details which show that the error terms are negligible.) We assume below that $x \neq 0$, since the variances were obtained in the last section.

$d = 1$. As $t \to \infty$,

$$t^{-2}I_1(t) \sim \frac{4}{3\sqrt{2}\,\pi t^2}\int_0^{2t} u^{-1/2}e^{-|x|^2t/2u}\left(t - \frac{u}{2}\right)^{3/2} du$$

$$= \frac{128}{3\pi}|x|^{-5}e^{-|x|^2/4}\int_0^\infty s^{3/2}\left(\frac{|x|^2}{4s + |x|^2}\right)^3 e^{-s}\, ds$$

after the change of variables $s = |x|^2t/2u - |x|^2/4$, and similarly,

$$t^{-2}I_2(t) \sim \frac{16}{3\pi}|x|^{-5}\, e^{-|x|^2/2}\int_0^\infty s^{3/2}\left(\frac{|x|^2}{2s + |x|^2}\right)^3 e^{-s}\, ds.$$

So $C(x)$ exists, and by dominated convergence,

$$\lim_{|x|\to\infty} |x|^5 e^{|x|^2/4} C(x) = \frac{512}{3\pi}\int_0^\infty s^{3/2}\, e^{-s}\, ds = \frac{128}{\sqrt{\pi}}.$$

***d* = 2.** The calculation is similar but more involved. Two integrations by parts followed by a change of variables yield

$$\sigma^{-2}(t)I_1(t) \approx \frac{\beta_2 \ln t}{t^2}\int_0^{2t-M} du p_u(0, [x\sqrt{t}])\int_{u/2}^{t} dv \int_M^{t-v}\frac{dw}{\ln 2w}$$

$$\sim \frac{\beta_2 \ln t}{t^2}\int_0^{2t-M} du p_u(0, [x\sqrt{t}])\int_{u/2}^{t}\frac{t-v}{\ln(2t-2v)}\,dv$$

$$\sim \frac{\beta_2 \ln t}{2\pi t^2}\int_0^{2t-M} u^{-1}\frac{\left(t-\dfrac{u}{2}\right)^2}{\ln(2t-u)}e^{-|x|^2 t/u}\,du$$

$$= 4\,|x|^{-6}e^{-|x|^2/2}\int_{|x|^2M/(4t-2M)}^{\infty}\frac{\ln t}{\ln\left[\dfrac{4st}{2s+|x|^2}\right]}\left(\frac{|x|^2}{2s+|x|^2}\right)^3 s^2 e^{-s}\,ds,$$

as $t \to \infty$. By dominated convergence, we get

$$\sigma^{-2}(t)I_1(t) \sim 4\,|x|^{-6}e^{-|x|^2/2}\int_0^{\infty} s^2\left(\frac{|x|^2}{2s+|x|^2}\right)^3 e^{-s}\,ds \quad \text{as} \quad t\to\infty.$$

In the same way,

$$\sigma^{-2}(t)I_2(t) \sim \frac{1}{2}\,|x|^{-6}e^{-|x|^2}\int_0^{\infty} s^2\left(\frac{|x|^2}{s+|x|^2}\right)^3 e^{-s}\,ds \quad \text{as} \quad t\to\infty.$$

Hence $C(x)$ exists, and

$$\lim_{|x|\to\infty}|x|^6 e^{|x|^2/2}C(x) = 16\int_0^{\infty} s^2 e^{-s}\,ds = 32.$$

***d* = 3.** Since $P_e(\tau_0 > t) \sim \gamma_3$.

$$\sigma^{-2}(t)I_1(t) \sim \frac{\gamma_3}{2t^{3/2}}\int_0^{2t} p_u(0, [x\sqrt{t}])\left(t-\frac{u}{2}\right)^2 du$$

$$\sim \frac{\gamma_3}{2t^{3/2}}\left(\frac{3}{2\pi}\right)^{3/2}\int_0^{2t} u^{-3/2}\left(t-\frac{u}{2}\right)^2 e^{-3|x|^2t/2u}\,du$$

$$= \frac{24\gamma_3}{\pi^{3/2}}\,|x|^{-6}e^{-3|x|^2/4}\int_0^{\infty} s^2\left(\frac{|x|^2}{4s+3|x|^2}\right)^{5/2} e^{-s}\,ds$$

as $t \to \infty$. Similarly,

$$\sigma^{-2}(t)I_2(t) \sim \frac{6\gamma_3}{\pi^{3/2}\sqrt{2}}\,|x|^{-6}e^{-3|x|^2/2}\int_0^{\infty} s^2\left(\frac{|x|^2}{2s+3|x|^2}\right)^{5/2} e^{-s}\,ds$$

as $t \to \infty$. So $C(x)$ exists and

$$\lim_{|x|\to\infty}|x|^6 e^{3|x|^2/4}C(x) = \frac{96\gamma_3}{\pi^{3/2}}\int_0^{\infty} s^2 3^{-5/2}e^{-s}\,ds = 64\gamma_3/(3\pi)^{3/2}.$$

To finish the verification of (0.18), let us now determine the tail behavior of $C(x)$ for $d \geq 5$. In light of (0.20), it suffices to check that

$$\lim_{|x|\to\infty} |x|^{d-4} \int_0^\infty u p_u(0, x)\, du = \left(\frac{d}{2\pi}\right)^{d/2} \int_0^\infty s^{-d/2+1} e^{-d/2s}\, ds. \tag{3.2}$$

A change of variables shows that the right side equals c_d as specified below (0.18). By the local central limit theorem, if $|x|$ is large and $\delta < 1$ we should have

$$|x|^{d-4} \int_0^\infty u p_u(0, x)\, dx \approx |x|^{d-4} \left(\frac{d}{2\pi}\right)^{d/2} \int_{\delta|x|^2}^{\delta^{-1}|x|^2} u^{-d/2+1} e^{-d|x|^2/2u}\, du$$

$$= \left(\frac{d}{2\pi}\right)^{d/2} \int_\delta^{\delta^{-1}} s^{-d/2+1} e^{-d/2s}\, ds,$$

giving the desired result as $\delta \to 0$. This approximation is made rigorous as follows. First, it is easy to see that

$$\lim_{\delta\to 0} |x|^{d-4} \int_0^{\delta|x|^2} u p_u(0, x)\, du = 0 \quad \text{uniformly in} \quad x, \tag{3.3}$$

and that

$$\lim_{\delta\to\infty} |x|^{d-4} \int_{\delta^{-1}|x|^2}^\infty u p_u(0, x)\, du = 0 \quad \text{uniformly in} \quad x. \tag{3.4}$$

In addition, over the range from $\delta|x|^2$ to $\delta^{-1}|x|^2$ one needs a refinement of (1.3):

$$p_u(0, x) = (2\pi u)^{-d/2}\left[\frac{\varepsilon(x, u)u^d}{|x|^{2d}} + d^{d/2}\exp\left(-\frac{d|x|^2}{2u}\right)\right], \tag{3.5}$$

where $\lim_{u\to\infty}\sup_x|\varepsilon(x, u)| = 0$. See [17]. Using (3.5) it is not hard to check that for any $0 < \delta < 1$,

$$\lim_{|x|\to\infty} |x|^{d-4} \int_{\delta|x|^2}^{\delta^{-1}|x|^2} \left[u p_u(0, x) - \left(\frac{d}{2\pi}\right)^{d/2} u^{-d/2+1} e^{-d|x|^2/2u}\right] du = 0. \tag{3.6}$$

Together, (3.3), (3.4) and (3.6) justify (3.2) as desired.

The final covariance computation is (0.19) for $d = 4$. From (1.3) if $y \neq 0$,

$$G_{2v}(0, y) - G_v(0, y) \sim \int_v^{2v} \left(\frac{2}{\pi u}\right)^2 \exp\left(-\frac{2|y|^2}{u}\right) du$$

$$= \frac{2}{\pi^2|y|^2}\left(e^{-|y|^2/v} - e^{-2|y|^2/v}\right)$$

as $v \to \infty$. Thus, by (2.2),

$$\sigma^{-2}(t)\mathrm{Cov}(T_t^0, T_t^y) \sim \frac{8\theta(1-\theta)\gamma_4}{\pi^2|y|^2}\frac{1}{t\ln t}\int_0^t dv \int_0^{t-v} dw\left[e^{-|y|^2/v} - e^{-2|y|^2/v}\right]$$

$$= \frac{8\theta(1-\theta)\gamma_4}{\pi^2|y|^2}\frac{1}{t\ln t}\int_0^t (t-v)\left[e^{-|y|^2/v} - e^{-2|y|^2/v}\right] dv.$$

Consequently, setting $y = [t^\alpha x]$, as $t \to \infty$.

$$\sigma^{-2}(t)C_\alpha(x) \sim \frac{8\gamma_4}{\pi^2}\frac{1}{\ln t}\int_0^t \left(1 - \frac{v}{t}\right)[\exp(-t^{2\alpha}|x|^2/v) - \exp(-2t^{2\alpha}|x|^2/v)]\frac{1}{t^{2\alpha}|x|^2}\,dv$$

$$= \frac{8\gamma_4}{\pi^2}\frac{1}{\ln t}\int_0^{t^{1-2\alpha}/|x|^2}\left(1 - \frac{|x|^2 u}{t^{1-2\alpha}}\right)e^{-1/u}(1 - e^{-1/u})\,du.$$

If $\alpha \geq ½$ this last expression is asymptotically 0. If $0 \leq \alpha < ½$ we find that

$$\sigma^{-2}(t)C_\alpha(x) \sim \frac{8\gamma_4}{\pi^2}\frac{1}{\ln t}\int_M^{t^{1-2\alpha}/|x|^2}\left(1 - \frac{|x|^2 u}{t^{1-2\alpha}}\right)u^{-1}\,du = \frac{8\gamma_4}{\pi^2}\frac{1}{\ln t}\int_M^{t^{1-2\alpha}/|x|^2}\left(u^{-1} - \frac{|x|^2}{t^{1-2\alpha}}\right)du$$

$$\sim \frac{8\gamma_4}{\pi^2}\left[\frac{\ln(t^{1-2\alpha}/|x|^2)}{\ln t}\right] \sim \frac{8\gamma_4}{\pi^2}(1 - 2\alpha)$$

as claimed. Further details are left to the reader.

REMARK. It seems appropriate here to point out that we could have considered generalized random fields (as in [3]) instead of the fields $\{T_t^{[xf(t)]}\}_{x\in Z^d}$. That is, let $\mathscr{S}^d$ be the Schwartz space of rapidly decreasing functions on R^d and define

$$T_t^\varphi = \sum_{x\in Z^d}\varphi(x)(T_t^x - t\theta), \quad \varphi \in \mathscr{S}^d$$

and

$$\varphi_r(y) = \varphi(y/f(r)), \quad y \in \mathbb{R}^d.$$

Instead of Theorem 1 we could prove that the fields $\{T_t^\varphi\}_{\varphi\in\mathscr{S}^d}$ converge weakly as $t \to \infty$ to a generalized random field $\{\xi(\varphi)\}_{\varphi\in\mathscr{S}^d}$ whose covariance structure can be computed using the results of this section. Since the proper choice of $f(t)$ for $d \geq 5$ is $f(t) \equiv 1$, we have chosen to present our results in the "lattice" setting.

4. The proof of Theorem 1 ($d \geq 2$). The argument for (0.15) is modelled after the proof of Theorem 3 in [3]. We use the method of semi-invariants, referring the reader to [3] and the additional references given there for background on this approach. For fixed $N \geq 1$, and $x_1, \cdots, x_N \in Z^d$, let

$$S_m(t) = \text{the } m\text{th semi-invariant of } \sum_{i=1}^N \frac{T_t^{[x_i f(t)]} - t\theta}{\sigma(t)}.$$

To show that the limit variable ξ exists, and is Gaussian with covariance function $C(x)$, it suffices to prove that

$$\lim_{t\to\infty} S_m(t) = 0 \quad \text{for all} \quad m \geq 3.$$

($S_1(t) \equiv 0$ since $ET_t^x = t\theta$, and $S_2(t) \to \sum_{i,j=1}^N C(x_j - x_i)$ by the results of the last section.) Given m random variables $Y_1, \cdots, Y_m$ (distinct or not), let $u_m(Y_1, \cdots, Y_m)$ denote the mth order Ursell function of the Y_i (cf. [3]). Some straightforward combinatorics yields

$$S_m(t) = \sigma^{-m}(t)\sum_{i_1,\cdots,i_m=1}^N \int_0^t ds_1 \cdots \int_0^t ds_m u_m(\eta_{s_1}(z_{i_1}), \cdots, \eta_{s_m}(z_{i_m})),$$

where $z_i = [x_i f(t)]$, so to prove the theorem it suffices to show that

$$\sup_{y_i\in Z^d}\sigma^{-m}(t)\int_0^t ds_1 \cdots \int_0^t ds_m u_m(\eta_{s_1}(y_1), \cdots, \eta_{s_m}(y_m)) \to 0 \tag{4.1}$$

as $t \to \infty$.

The first step in the demonstration of (4.1) is a straightforward generalization of Proposition 2 in [3] to our "space-time setting", so we omit the proof.

LEMMA. *There is an absolute constant K_m such that if $y_i \in Z^d$, $s_i \ge 0\,(1 \le i \le m)$ and $t \ge \max(s_1, \cdots, s_m)$, then*

$$u_m(\eta_{s_1}(y_1), \cdots, \eta_{s_m}(y_m)) \le K_m P(N_t((y_1, t - s_1), \cdots, (y_m, t - s_m)) = 1).$$

To simplify notation, observe that

$$\int_0^t ds_1 \cdots \int_0^t ds_m P(N_t((y_1, t - s_1), \cdots, (y_m, t - s_m)) = 1)$$
$$= \int_0^t ds_1 \cdots \int_0^t ds_m P(N_t((y_1, s_1), \cdots, (y_m, s_m)) = 1).$$

In view of this and the Lemma, it remains to prove that

$$\text{(4.2)} \qquad \sup_{y_i \in Z^d} \sigma^{-m}(t) \int_0^t ds_1 \cdots \int_0^t ds_m P(N_t((y_1, s_1), \cdots, (y_m, s_m)) = 1) \to 0$$

for $m \ge 3$. Fortunately this is somewhat simpler than the analogous task in [3]. Write

$$\rho_t((y_1, s_1), \cdots, (y_m, s_m)) = P(N_t((y_1, s_1), \cdots, (y_m, s_m)) = 1),$$
$$\Gamma_t(x, s; x_1, \cdots, x_k) = \int_0^t ds_1 \cdots \int_0^t ds_k \rho_t((x, s), (x_1, s_1), \cdots, (x_k, s_k)),$$

and

$$\Phi(t) = \Gamma_t(0, 0; 0).$$

We will show that there are constants $C(k) < \infty$, $k \ge 1$, such that

$$\text{(4.3)} \qquad \sup_{s \ge 0, x, x_1, \cdots, x_k} \Gamma_t(x, s; x_1, x_2, \cdots, x_k) \le C(k)\Phi(t)^k,$$

and that

$$\text{(4.4)} \qquad \begin{aligned} \Phi(t) &= O(t/\ln t) && (d = 2) \\ &= O(t^{1/2}) && (d = 3) \\ &= O(\ln t) && (d = 4) \\ &= O(1) && (d \ge 5). \end{aligned}$$

Together, (4.3) and (4.4) imply (4.2). (Put $k = m - 1$, $(x, s) = (y_m, s_m)$.)
To prove (4.3) we will need two observations:

$$\text{(4.5)} \qquad \rho_t((0, 0), (0, s)) \text{ decreases in } s,$$

and

$$\text{(4.6)} \qquad \begin{aligned} \Gamma_t(x_0, s_0; x_1, x_2, \cdots, x_k) \le \Gamma_t(x_0, 0; x_1, \cdots, x_k) + \Gamma_t(x_1, 0; x_0, x_2, \cdots, x_k) \\ + \cdots + \Gamma_t(x_k, 0; x_0, x_1, \cdots, x_{k-1}). \end{aligned}$$

Claim (4.5) follows from the remarks leading to (0.9). For (4.6), break the region of integration $\{0 \le s_i \le t; 1 \le i \le k\}$ into $k + 1$ parts: $\{0 \le s_0 \le s_i \le t; 1 \le i \le k\}$, $\{0 \le s_1 \le s_i \le t; i = 0, 2 \le i \le k\}$, $\cdots$, $\{0 \le s_k \le s_i \le t; 0 \le i \le k - 1\}$. On the first region we have

$$\int_{s_0}^t ds_1 \int_{s_0}^t ds_2 \cdots \int_{s_0}^t ds_k \rho_t((x_0, s_0), (x_1, s_1), \cdots, (x_k, s_k))$$
$$\le \int_{s_0}^t ds_1 \int_{s_0}^t ds_2 \cdots \int_{s_0}^t ds_k \rho_t((x_0, 0), (x_1, s_1 - s_0), \cdots, (x_k, s_k - s_0))$$
$$\le \int_0^t ds_1 \cdots \int_0^t ds_k \rho_t((x_0, 0), (x_1, s_1), \cdots, (x_k, s_k))$$
$$= \Gamma_t(x_0, 0; x_1, \cdots, x_k).$$

The second region is typical of the rest:

$$\int_0^{s_0} ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_1}^t ds_k \rho_t((x_0, s_0), (x_1, s_1), \cdots, (x_k, s_k))$$
$$\le \int_0^{s_0} ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_1}^t ds_k \rho_t((x_0, s_0 - s_1), (x_1, 0), (x_2, s_2 - s_1), \cdots, (x_k, s_k - s_1))$$
$$\le \int_0^t ds_1 \int_0^t ds_2 \cdots \int_0^t ds_k \rho_t((x_1, 0), (x_0, s_1), (x_2, s_2), \cdots, (x_k, s_k))$$
$$= \Gamma_t(x_1, 0; x_0, x_2, \cdots, x_k).$$

The remaining regions are handled similarly; summing we get (4.6).

Let us now check (4.3) for $k = 1$. If $0 \le s_0 \le s_1 \le t$,

$$\rho_t((x_0, s_0), (x_1, s_1)) = \int_{s_1 - s_0}^{2t - s_0 - s_1} p_u(0, x_0 - x_1) P_e(\tau_0 > 2t - s_0 - s_1 - u)\, du + p_{2t - s_0 - s_1}(0, x_0 - x_1)$$

$$\le \int_{s_1 - s_0}^{2t - s_0 - s_1} p_u(0, 0) P_e(\tau_0 > 2t - s_0 - s_1 - u) du + p_{2t - s_0 - s_1}(0, 0) \tag{4.7}$$

$$= \rho_t((0, s_0), (0, s_1)).$$

Similarly the inequality holds for $0 \le s_1 \le s_0 \le t$, so using (4.6) we find that

$$\Gamma_t(x_0, s_0; x_1) \le \Gamma_t(0, s_0; 0) \le 2\Phi(t).$$

Thus (4.3) holds for $k = 1$ with $C(1) = 2$.

We proceed by induction, assume (4.3) for $k \le m - 1$, and consider the case $k = m$. Fix $x_0, x_1, \cdots, x_m \in Z^d$, $s_0, s_1, \cdots, s_m \ge 0$. Given a subset $\pi = \{i_1, \cdots, i_l\}$ of $\{0, 1, \cdots, m\}$, let

$$\tau(\pi) = \min\{t : t \ge s_{i_j} \quad \text{for all} \quad 1 \le j \le l \quad \text{and} \quad N_t((x_{i_1}, s_{i_1}), \cdots, (x_{i_l}, s_{i_l})) = 1\}.$$

Roughly, $\tau(\pi)$ is the hitting time of cardinality 1 for the π subsystem of coalescing walks. Now let $\{\pi, \pi^c\}$ be a nontrivial partition of $\{0, 1, \cdots, m\}$, and write

$$\rho_t((x_0, s_0), (x_1, s_1), \cdots, (x_m, s_m)) = \textstyle\sum_{\{\pi, \pi^c\}} P(A_t(\pi)), \tag{4.8}$$

where

$$A_t(\pi) = \{N_t((x_0, s_0), \cdots, (x_m, s_m)) = 1 \quad \text{and}$$
$$\tau(\{i, j\}) = \tau(\{0, 1, \cdots, m\}) \quad \text{for all} \quad i \in \pi, j \in \pi^c\}.$$

The decomposition (4.8) is made by considering the "ancestors" of the particles in the last collision. For $\pi \subset \{0, 1, \cdots, m\}$, $y \in Z^d$, $u \geq 0$, abbreviate

$$\hat{\eta}_t(\pi) = \hat{\eta}_t((x_i, s_i)_{i\in\pi}),\ \rho_t(\pi, (y, u)) = \rho_t((x_i, s_i)_{i\in\pi}, (y, u)).$$

Without loss of generality we may assume $|\pi| > 1$, in which case we claim that

$$P(A_t(\pi)) \leq \textstyle\sum_y \displaystyle\int_0^t P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\})\rho_t(\pi^c, (y, u)). \tag{4.9}$$

In fact, one can represent

$$\begin{aligned} P(A_t(\pi)) = \textstyle\sum_{y,z} \displaystyle\int_0^t \int_0^t & P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\}, \tilde{\tau}(\pi^c) \in dv, \tilde{\eta}_v(\pi^c) = \{z\}, \\ & \hat{\eta}_s(\pi) \cap \hat{\eta}_s(\pi^c) = \varnothing \quad \text{for} \quad 0 \leq s \leq \max\{u, v\}, \\ & \text{and} \quad \hat{\eta}_s(\pi) = \tilde{\eta}_s(\pi^c) \quad \text{for some} \quad s \in [\max\{u, v\}, t]), \end{aligned}$$

where $\tilde{\eta}.(\pi^c)$ is a coalescing system which is *independent* of $\eta.(\pi)$, $\tilde{\tau}(\pi^c)$ the corresponding hitting time. Constructions of this sort are described in [9], for example. Furthermore, the right side is majorized by

$$\begin{aligned} \textstyle\sum_{y,z} \displaystyle\int_0^t \int_0^t & P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\}, \tilde{\tau}(\pi^c) \in dv, \tilde{\eta}_v(\pi^c) = \{z\}, \\ & \text{and} \quad \tilde{\eta}_s(\pi^c) = Y_s(y, u) \quad \text{for some} \quad s \in [\max\{u, v\}, t]), \end{aligned}$$

where $Y.(y, u)$ is a random walk starting from y at time u which is *independent* of *both* $\hat{\eta}.(\pi)$ and $\tilde{\eta}.(\pi^c)$. The last expression equals

$$\begin{aligned} \textstyle\sum_{y,z} \displaystyle\int_0^t \int_0^t & P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\}) P(\tilde{\tau}(\pi^c) \in dv, \tilde{\eta}_v(\pi^c) = \{z\}, \\ & \text{and} \quad \tilde{\eta}_s(\pi_1) = Y_s(y, u) \quad \text{for some} \quad s \in [\max\{u, v\}, t]) \\ \leq \textstyle\sum_{y,z} \displaystyle\int_0^t \int_0^t & P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\}) P(\tau(\pi^c) \in dv, \hat{\eta}_v(\pi^c) = \{z\}, \\ & \text{and} \quad N_t((x_i, s_i)_{i\in\pi^c}, (y, u)) = 1). \end{aligned}$$

Summing on z and integrating over v we get (4.9), and so

$$\begin{aligned} & \Gamma_t(x_0, s_0; x_1, \cdots, x_m) \\ & \leq \textstyle\sum_{\{\pi,\pi^c\}} \displaystyle\int_0^t ds_1 \cdots \int_0^t ds_m \textstyle\sum_y \displaystyle\int_0^t P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\})\rho_t(\pi^c, (y, u)). \end{aligned} \tag{4.10}$$

From now on assume $s_0 = 0$. If $\{\pi, \pi^c\} \neq \{\{0\}, \{1, 2, \cdots, m\}\}$, we may assume that $0 \in \pi$, in which case the corresponding term in the above sum is

$$\leq \int_{i\in\pi-\{0\}} ds_i \textstyle\sum_y \displaystyle\int_0^t P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\}) \times \int_{j\in\pi^c} ds_j \rho_t(\pi^c, (y, u)).$$

By the induction hypothesis each such term is

$$\leq C(|\pi^c|)\Phi(t)^{|\pi^c|} \int_{i\in\pi-\{0\}} ds_i \rho_t(\pi) \leq C(|\pi|-1)C(|\pi^c|)\Phi(t)^m.$$

Moreover, the $\{\{0\}, \{1, 2, \cdots, m\}\}$ term is (taking $\pi = \{1, 2, \cdots, m\}$)

$$\int_{i\in\pi} ds_i \sum_y \int_0^t P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\})\rho_t((x_0, 0), (y, u))$$

$$\leq \int_{i\in\pi} ds_i \sum_y \int_0^t P(\tau(\pi) \in du, \hat{\eta}_u(\pi) = \{y\})\rho_t((0, 0), (0, s_1)),$$

applying (4.7), the fact that $u \geq s_1$ and (4.5). The last expression is

$$= \int ds_1 \rho_t((0, 0), (0, s_1)) \int_0^t ds_2 \cdots \int_0^t ds_m \rho_t(\pi)$$

$$\leq C(m-1)\Phi(t)^{m-1}\cdot\Phi(t) = C(m-1)\Phi(t)^m,$$

again by induction. To summarize, assuming (4.3) for $k \leq m - 1$ we have shown that

$$\Gamma_t(x, 0; x_1, \cdots, x_m) \leq \sum_{\{\pi,\pi^c\}} C(|\pi|-1)C(|\pi^c|)\Phi(t)^m.$$

In light of (4.6), then, (4.3) holds for $k = m$ with

$$C(m) = (m+1)\sum_{j=1}^{m-1}\binom{m+1}{j} C(j)C(m-j) \quad (C(1) = 2).$$

Finally, for (4.4), decompose as in Section 2 to get

$$\Phi(t) \sim \int_0^t du \int_u^{2t-u} p_v(0, 0)P_e(\tau_0 > 2t - u - v)\, dv$$

$$= \int_0^t P_e(\tau_0 > 2s)[G(2t - 2s) - G(t - s)]\, ds.$$

By (1.4) and (1.5), as $t \to \infty$,

$$\Phi(t) = O\left(\int_e^t \frac{ds}{\ln s}\right) \qquad (d = 2),$$

$$= O\left(\int_0^{t-1} (t-s)^{-(d-2)/2}\, ds\right) \qquad (d \geq 3).$$

Thus (4.4) holds, and the proof of Theorem 1 in dimensions $d \geq 2$ is complete.

REMARK. In five or more dimensions, since $\mathrm{Var}(T_t^0) = O(t)$, the central limit theorem for T_t^0 follows from a generalization of the Newman-Wright invariance principle [16] for associated variables. See [6].

5. The proof of Theorem 1 ($d = 1$). To complete the argument for Theorem 1, it remains only to show that in one dimension

(5.1) $$(t^{-1}T_t^{[\sqrt{t}x]})_{x\in Z} \Rightarrow \xi_0 \quad \text{as} \quad t \to \infty.$$

The limit ξ_0 is clearly non-Gaussian since it lives on $[0, 1]^Z$, and the centered limit ξ in (0.15) is given by $\xi(x) = \xi_0(x) - \theta$. (5.1) is an application of a beautiful invariance principle for coalescing Brownian motions due to Arratia [2]. Here we will simply describe the limit ξ_0, referring the reader to [2] for the proof of convergence. Coalescing Brownian motions on R, as the name suggests, is an infinite particle system of Brownian motions moving independently on the real line, except that whenever two particles collide they coalesce into one. The exotic feature of this system c_t is that it starts with a particle located at every $x \in R$. Letting $c_t(x)$ denote the position of the particle started at x after time t, $c_0(x) = x$ $\forall x \in R$. By any time $t > 0$, however, the process has collapsed to a discrete countable set of occupied positions:

$$\forall x \in R, \quad t > 0 \ \exists \ i \in Z, \quad x_i(t) \in R: \quad c_t(x) = x_i(t).$$

Thus the sample paths of c_t comprise a sort of inverted Brownian tree with binary branching which "explodes" as $t \downarrow 0$. If one rescales the voter model and embeds it in R via

$$\eta_s^t(x) = \eta_{st}([x\sqrt{t}]) \quad (0 \le s \le 1),$$

then Arratia has shown that as $t \to \infty$, η^t converges weakly to a limit Ξ which can be constructed from c_s, $0 \le s \le 1$. Informally, the construction goes as follows. The Brownian tree formed by c_s divides $R \times [0, 1]$ into countably many regions. Independently label each region 1 with probability θ, 0 with probability $1 - \theta$. The connected components of 0's and 1's obtained in this manner form the limiting space-time field which describes the scaled cluster structure of the voter model. Namely, Ξ is given by

$$\Xi_s(x) = 1 \quad \text{if} \quad (x, s) \text{ belongs to a component of 1's,}$$
$$= 0 \quad \text{otherwise.}$$

For the field of occupation time functionals we get

$$(t^{-1}T_t^{[x\sqrt{t}]})_{x\in Z} = \left(t^{-1}\int_0^t \eta_u([x\sqrt{t}])\,du\right)_{x\in Z} = \left(\int_0^1 \eta_s^t(x)\,ds\right)_{x\in Z}$$
$$\to \left(\int_0^1 \Xi_s(x)\,ds\right)_{x\in Z} \quad \text{as} \quad t \to \infty.$$

Thus

$$(\xi_0(x)) = \left(\int_0^1 \Xi_s(x)\,ds\right)$$

is the desired representation. The connection between the voter model and coalescing Brownian motions lies in the fact that the borders between extant regions of the form $\{y \in Z: y \in \eta_t^x\}$ execute coalescing random walks. Again, see [2].

6. The proof of Theorem 2. Our law of large numbers follows from the semi-invariant estimates of the last section, Chebyshev's inequality and the Borel-Cantelli lemma. The case $d = 2$ is the hardest; to incorporate it we use a nice technique from Etemadi [8]. Since

$$E[(T_t^0 - \theta t)^4] = S_4(t) + 3S_2^2(t),$$

for $d \ge 2$ the estimates (4.3) and (4.4) yield

$$E[(T_t^0 - \theta t)^4] = O\left(\frac{t^4}{(\ln t)^2}\right).$$

Hence for any $r > 1$,

$$P(|r^{-n}T_{r^n} - \theta| > \varepsilon) \le \varepsilon^{-4}r^{-4n}[(T_{r^n}^0 - \theta r^n)^4] = O(n^{-2}).$$

By Borel-Cantelli,

$$r^{-n}T_{r^n} \to \theta \quad \text{a.s.} \quad n \to \infty.$$

Since T_t is increasing, if $r^n \le t \le r^{n+1}$ then

$$r^{-1}(r^{-n}T_{r^n}) \le t^{-1}T_t \le r(r^{-(n+1)}T_{r^{n+1}}).$$

Thus

$$P\left(\frac{\theta}{r} \le \liminf_{t\to\infty} \frac{T_t}{t} \le \limsup_{t\to\infty} \frac{T_t}{t} \le \theta r\right) = 1$$

for each $r > 1$. Let $r \downarrow 1$ through the rationals to finish the proof.

REMARKS. The strong law for $d = 2$ is rather curious in light of (0.1). The discussion of the previous section shows that there is no law of large numbers in one dimension. It seems most likely that Theorems 1 and 2 will continue to hold when the voter model starts from any measure in the domain of attraction of ν_θ, in particular for the stationary process $\eta_t^{\nu_\theta}$. It should at least be possible to identify a large class of "regular" initial states to which the limit laws apply.

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