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LOCAL LAWS OF THE ITERATED LOGARITHM FOR DIFFUSIONS

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Suppose X_t is a diffusion, reflecting at 0, with speed measure $m(dx)$. We show, under a mild regularity condition on m, that $\limsup_{t\to 0} X_t/h^{-1}(t) = c$, a.s., where c is a nonzero constant and $h(t) = tm[0, t]/\log|\log t|$. The analogue to Chung's law of the iterated logarithm is also obtained.

1. Introduction. Let $\{X(t),\ t \geq 0\}$ be a diffusion process on $[0, \infty)$ on natural scale, reflecting at 0, with speed measure m satisfying

$$0 < m\{I\} < \infty \quad \text{for all bounded open} \quad I \subset [0, \infty). \tag{1.1}$$

Under (1.1) 0 is both exit and entrance in the terminology of Itô and McKean (1974, page 108). The purpose of this paper is to present a local version, at $t = 0$, of Khintchine's and Chung's Laws of the Iterated Logarithm. We impose a regularity condition on m at 0 which, however, is quite mild. For $\beta > 0$ put $m(\beta) = m[0, \beta]$ and

$$m_\beta\{dx\} = m\{\beta dx\}/m(\beta).$$

Then in addition to (1.1) we require that for some nonempty interval $I_0 \subset (0, 1]$ there is a $\delta_0 > 0$, $\beta_0 > 0$ such that

$$m_\beta\{dx\} \geq \delta_0 dx, \quad x \in I_0, \quad \beta \leq \beta_0. \tag{1.2}$$

In other words m must have an absolutely continuous part with a density μ satisfying $\beta\mu(\beta x)/m(\beta) \geq \delta_0$ for all $x \in I_0$. The class of such m includes, but is not restricted to, those of the form $m\{dx\} = x^\alpha L(x)dx$, $\alpha > -1$, for some slowly varying (at 0) L. (Note: As the reader may easily show, if m satisfies (1.2) with $I_0 = [a, b]$, $b < 1$, then, in fact, (1.2) is also satisfied on $I_0 = [a, 1]$, (but with a smaller δ_0, β_0), so for convenience we take $I_0 = [a, 1]$ for some $0 < a < 1$.)

THEOREM 1. *Let h^{-1} be the inverse of the function*

$$h(\beta) = \beta m(\beta)/\log|\log \beta|.$$

Then for some constant c, $0 < c < \infty$,

$$\limsup_{t\to 0+} X(t)/h^{-1}(t) = c \quad a.s.\ on \quad [X(0) = 0].$$

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THEOREM 2. *Let g^{-1} be the inverse of the function*

$$g(\beta) = \beta m(\beta)\log|\log \beta|.$$

Then for some constant c^*, $0 < c^* < \infty$,

$$\liminf_{t\to 0+} X^*(t)/g^{-1}(t) = c^* \quad \textit{a.s. on} \quad [X(0) = 0],$$

where $X^*(t) = \max_{0\le s\le t} X(s)$.

REMARKS.
1. h and g are strictly increasing near 0, so h^{-1} and g^{-1} exist near 0.
2. If $m\{dx\} = \mu(x)dx$ with $0 < \mu(0+) < \infty$, then (1.2) is satisfied and we get the usual iterated log laws:

$$\limsup \frac{X(t)}{(t \log|\log t|)^{1/2}} = c$$

$$\liminf \frac{X^*(t)}{(t/\log|\log t|)^{1/2}} = c^*.$$

In this case these results can be obtained immediately from the classical laws for Brownian motion together with time changes. See McKean (1969), page 57, for an example. One can even show that $c = \sqrt{2}/\mu(0+)$ and $c^* = (\pi/\sqrt{8})/\mu(0+)$.

3. The class of m satisfying (1.2) is not restricted to m of the form $m(dx) = x^\alpha L(x)\,dx$, $\alpha > -1$, L slowly varying. In fact, m can even have atoms. If, however, m is of the above form and L is sufficiently smooth, then we can determine the constants c and c^*. See Theorems 3 and 4 of Section 5.

Our results are closely related to an open problem posed in 1965 by Itô and McKean (Itô and McKean, 1974, page 161). The paper by Knight (1973) is also related.

NOTATION. We will suppose throughout that X is equal to a time change of Brownian motion. That is, let $B = \{B(t), t \ge 0; P_x, x \in R\}$ be a standard Brownian motion on $(-\infty, \infty)$ with probabilities $P_x(\cdot) = P(\cdot \mid B(0) = x)$. Let $L(t, x)$ be the standard local time functional and put $A(t) = \int_0^\infty L(t, x)m\{dx\}$, $A^{-1}(t) = \inf\{s: A(s) > t\}$. Then

$$X(t) = B(A^{-1}(t)), \quad t \ge 0. \tag{1.3}$$

See Itô and McKean (1974), Section 5. Note that A^{-1} is discontinuous since A is flat when B is negative, but $A^{-1}(t)$ is strictly increasing and continuous at points $t = A(s)$ with $B(s) > 0$ because of (1.1).

We put

$$D_\beta = \inf\{t: B(t) = \beta\}$$

$$D_\beta^+ = \inf\{t: |B(t)| = \beta\} = D_\beta \wedge D_{-\beta}.$$

Note that for $\beta > 0$,

$$A(D_\beta) = \inf\{t: X(t) = \beta\}.$$

Note further that since $\{|B(t)|, t \ge 0; P_0\}$ is equivalent to

$$\{B(C^{-1}(t)), t \ge 0; P_0\}, \quad C(t) = 2\int_0^\infty L(t, x)\, dx,$$

we have

$$D_\beta^+ =_d 2\int_0^\beta L(D_\beta, x), \quad \beta > 0$$

where $=_d$ means equality in distribution with respect to P_0.

We occasionally suppress the 0 in P_0, E_0. For a process, Y, $[Y_s \le f(s), s \downarrow 0]$ is the set of ω for which there is an $s_0(\omega) > 0$ such that $Y_s(\omega) \le f(s)$ for all $0 < s \le s_0(\omega)$. i.o. means infinitely often. w.p.1 means the same as a.s.

2. Probability estimates for $A(D_\beta)$. In this section we establish the following:

PROPOSITION 1. *Assume m satisfies* (1.1). *Then there are positive constants $c_1 > 0$, $c_2 < \infty$ such that*

$$P[A(D_\beta) \ge z] \le \exp(-c_1 z/\beta m(\beta)), \quad z/\beta m(\beta) \ge 1, \tag{2.1}$$

$$P[A(D_\beta) \le z] \ge \exp(-c_2 \beta m(\beta)/z), \quad \beta m(\beta)/z \ge 1. \tag{2.2}$$

PROPOSITION 2. *Assume m satisfies* (1.1) *and* (1.2). (i) *For any q, $0 \le q < 1$, there is a constant $c_3 > 0$ such that*

$$P[A(D_\beta) - A(D_{\beta q}) \ge z] \ge \exp(-c_3 z/\beta m(\beta)) \tag{2.3}$$

for all β sufficiently small, $z/\beta m(\beta) \ge 1$. (ii) *There is a constant $c_4 < \infty$ such that for β sufficiently small*

$$P[A(D_\beta) \le z] \le \exp(-c_4 \beta m(\beta)/z), \quad \beta m(\beta)/z \ge 1. \tag{2.4}$$

PROOF OF PROPOSITION 1. From the Brownian scaling property one sees that

$$\{L(D_\beta, x), 0 \le x \le \beta, P_0\} =_d \{\beta L(D_1, x/\beta), 0 \le x \le \beta; P_0\}. \tag{2.5}$$

Hence

$$A(D_\beta) =_d \beta m(\beta) \int_0^1 L(D_1, x) m_\beta \{dx\}. \tag{2.6}$$

Applying the Ray-Knight theorem, (Itô and McKean, 1974, page 66, 11d), the Kolmogorov inequality for submartingales, and the fact that $m_\beta[0, 1] = 1$, we get

$$\begin{aligned} P[A(D_\beta) \ge z] &= P\left[\int_0^1 Z(1-x)^2 m_\beta\{dx\} \ge 2z/\beta m(\beta)\right] \\ &\le P[\max_{0\le s\le 1} \exp(kZ(s)^2) \ge \exp(2kz/\beta m(\beta))] \\ &\le (E\, e^{kZ(1)^2}) \exp(-2kz/\beta m(\beta)) \end{aligned}$$

where $\{Z(s); 0 \le s \le 1\}$ is the radial part of a 2-dimensional Brownian motion. For $k > 0$ sufficiently small the last written expectation is finite and (2.1) follows.

PROOF OF (2.2). Computing as above we obtain

$$P[A(D_\beta) \le z] \ge P[\max_{0\le s\le 1} Z(s)^2 \le 2z/\beta m(\beta)]$$
$$\ge P[D_\alpha^+ \ge 1]^2, \quad \alpha = (z/\beta m(\beta))^{1/2},$$

where $D_\alpha^+ =_d \inf\{t: |B_t| = \alpha\}$. To get the last inequality, recall that Z^2 may be thought of as the sum of squares of two independent 1-dimensional Brownian motions. Now (2.2) is evident since

$$P[D_\alpha^+ \ge 1] \ge \exp(-\pi^2/8\alpha^2)$$

for all α sufficiently small. See Feller (1971), page 342.

PROOF OF PROPOSITION 2. Let a be as in (1.2). We may suppose $a \le q \le 1$. Then

$$\begin{aligned} A(D_\beta) - A(D_{\beta q}) & \\ &= \int_{\beta q}^{\beta} L(D_\beta, x)m\{dx\} + \int_0^{\beta q} (L(D_\beta, x) - L(D_{\beta q}, x))m\{dx\} \\ &\ge \int_{\beta q}^{\beta} L(D_\beta, x)m\{dx\} =_d \beta m(\beta) \int_q^1 L(D_1, y)m_\beta\{dy\} \\ &\ge \delta_0 \beta m(\beta) \int_q^1 L(D_1, y)\,dy =_d \delta_0\beta m(\beta) \int_0^{1-q} L(D_{1-q}, y)\,dy \\ &=_d \delta_0 \beta m(\beta)(1-q)^2 \int_0^1 L(D_1, y)\,dy =_d \tfrac{1}{2}\delta_0(1-q)^2 \beta m(\beta) D_1^+. \end{aligned}$$

Recall that $=_d$ means equality in distribution with respect to P_0, Wiener measure on paths starting at 0. In the third distributional equality from the last we have used the strong Markov property, the translation invariance property, and the fact that local time at $y > q$ is zero until time D_q. It follows that

$$P[A(D_\beta) - A(D_{\beta q}) \ge z] \ge P[D_1^+ \ge \lambda] \ge \exp(-\pi^2\lambda/8)$$

for $\lambda = [z/\beta m(\beta)][\tfrac{1}{2}\delta_0(1-q)^2]^{-1}$ sufficiently large. Thus we obtain (2.3).

PROOF OF (2.4). As in the preceding calculation we obtain

$$P[A(D_\beta) \le z] \le P\left[2\int_a^1 L(D_1, x)\,dx \le 2z/\beta m(\beta)\delta_0\right] = P[D_1^+ \le \varepsilon],$$

where $\varepsilon = (z/\beta m(\beta))(\delta_0(1-a)^2/2)^{-1}$. But for small ε we have

$$P[D_1^+ \le \varepsilon] \le 4P[B_1 > \varepsilon^{-1/2}] = O(\exp(-1/2\varepsilon)),$$

and (2.4) follows.

3. Proof of Theorem 1.

STEP 1. Let h be strictly increasing near 0 with $h(0+) = 0$. Then

$$P[X(t) \geq h^{-1}(t), \text{ i.o. } t \downarrow 0] = P[A(D_\beta) \leq h(\beta) \text{ i.o. } \beta \downarrow 0]. \tag{3.1}$$

PROOF. Each side of (3.1) is either 0 or 1 by Blumenthal's $0-1$ law. Suppose first that R.S. (3.1) = 1. But for $\beta > 0$, $\beta = X(A(D_\beta))$. (Recall that $X = B \circ A^{-1}$ and that, because m charges every open interval on $(0, \infty)$, $A^{-1}(s)$ is continuous and strictly increasing at times $s = A(D_\beta)$.) It follows that $P[h^{-1}(A(D_\beta)) \leq X(A(D_\beta))$ i.o. $\beta \downarrow 0] = 1 \Rightarrow$ L.S. (3.1) = 1. Now suppose R.S. (3.1) = 0. Then $P[A(D_\beta) > h(\beta), \beta \downarrow 0] = 1 \Rightarrow P[\beta > B^*(A^{-1}(h(\beta))), \beta \downarrow 0] = 1 \Rightarrow P[X^*(t) < h^{-1}(t), t \downarrow 0] = 1 \Rightarrow$ L.S. (3.1) = 0.

From now on, $h(\beta) = \beta m(\beta)/\log \log(1/\beta)$.

STEP 2. Put $\beta_n = \exp(-n^\gamma)$, $\gamma > 1$. Then

$$P[A(D_{\beta_n}) = O(h(\beta_n)) \text{ i.o.}] = 1. \tag{3.2}$$

PROOF. First note that,

$$\frac{h(\beta_n)}{\beta_{n+1} m(\beta_{n+1})} \geq \frac{(\beta_n/\beta_{n+1})}{\log n^\gamma} \geq \exp(cn^{\gamma-1})$$

for some $c > 0$ and all $n \geq 2$. Writing $W_n = A(D_{\beta_n})$ and applying (2.1) we have for any $\varepsilon > 0$

$$\textstyle\sum_{n=2}^\infty P[W_{n+1} > \varepsilon h(\beta_n)] \leq \sum_{n=2}^\infty \exp(-c_1 \varepsilon \exp(cn^{\gamma-1})) < \infty,$$

and thus

$$\limsup W_{n+1}/h(\beta_n) = 0 \quad \text{a.s.} \tag{3.3}$$

From (2.2) we obtain

$$\sum P[W_n - W_{n+1} < kh(\beta_n)] \geq \sum P[W_n < kh(\beta_n)]$$
$$\geq \sum \exp(-(c_2\gamma/k)\log n) = \infty$$

for $k > \gamma c_2$. By the strong Markov property of Brownian motion and the fact that $B(D_\beta) = \beta$, the process $\beta \to A(D_\beta)$, the first passage time process for X, has independent increments. Hence, by Borel-Cantelli,

$$P[W_n - W_{n-1} < kh(\beta_n) \text{ i.o.}] = 1. \tag{3.4}$$

Clearly (3.3) and (3.4) imply (3.2).

REMARK. Note that what the preceding actually shows is that

$$\liminf_{\beta \to 0} A(D_\beta)/h(\beta) \leq c_2 \quad \text{a.s.}$$

with c_2 as in (2.2). We have not used (1.2).

STEP 3. For some $q < 1$ and all $k > c$ sufficiently small we have

$$\sum_{n=1}^{\infty} P[A(D_\beta) \le kh(\beta) \text{ for some } \beta \in [q^{n+1}, q^n)] < \infty. \tag{3.5}$$

PROOF. Writing p_n for the last written probability and applying (2.4) we obtain

$$p_n \le P[A(D_{q^{n+1}}) \le kh(q^n)] \le \exp\left[-k^{-1}c_4 q \log(nr) \cdot \frac{m(q^{n+1})}{m(q^n)}\right],$$

$r = \log(1/q) > 0$. But for $a < q < 1$, a as in (1.2), we have

$$m(q^{n+1}) \ge m[q^n a, q^n q] \ge \delta_0(q - a)m(q^n).$$

Hence

$$p_n = O(n^{-\lambda}), \quad \lambda = c_4\delta_0 q(q - a)/k,$$

and (3.5) follows for k sufficiently small.

Suppose $k > 0$, and let $k_1 = \min(k/2, 1)$, $k_2 = \max(2k, 1)$. Then

$$h(k_1\beta) \le kh(\beta) \le h(k_2\beta) \tag{3.6}$$

for all β sufficiently small. From Steps 2 and 3 it follows that for some constants $0 < k_1 < k_2 < \infty$, we have $P[A(D_\beta) \le h(k_2\beta) \text{ i.o.}, \beta \downarrow 0] = 1$ but

$$P[A(D_\beta) \le h(k_1\beta) \text{ i.o.}, \beta \downarrow 0] = 0.$$

Hence by Step 1,

$$P[X(t) \ge h^{-1}(t)/k_2 \text{ i.o.}, t \downarrow 0] = 1$$

and

$$P[X(t) \ge h^{-1}(t)/k_1 \text{ i.o.}, t \downarrow 0] = 0.$$

But w.p.1 $\limsup X(t)/h^{-1}(t)$ must be a constant. This completes the proof.

4. Proof of Theorem 2.

As the proof of Theorem 2 is so similar to that of Theorem 1, we will leave many of the details to the reader.

STEP 1. Let g be strictly increasing, $g(0+) = 0$. Then

$$\begin{aligned} P[X^*(t) \le g^{-1}(t) \text{ i.o.}, t \downarrow 0] &= P[A(D_\beta) \ge g(\beta) \text{ i.o.}, \beta \downarrow 0] \\ &= 0 \quad \text{or} \quad 1. \end{aligned} \tag{4.1}$$

PROOF. This is left to the reader.

From now on, $g(\beta) = \beta m(\beta) \log\log(1/\beta)$.

STEP 2. Fix q, $a < q < 1$. Then for all $k > 0$ sufficiently small

$$P[A(D_{q^n}) \ge kg(q^n) \text{ i.o.}] = 1. \tag{4.2}$$

PROOF. Put $W_n = A(D_{q^n})$. Then $W_n \geq W_n - W_{n+1}$, and (see proof of (3.4)) it suffices to show that

$$\sum P[W_n - W_{n+1} > kg(q^n)] = \infty \tag{4.3}$$

for $k > 0$ small. But

$$P[W_n - W_{n+1} > kg(q^n)] \geq \exp(-c_3 kg(q^n)/q^n m(q^n))$$
$$= \exp(-c_3 k \log nr), \quad r = |\log q|$$

by (2.3). This yields (4.3) for small k.

STEP 3. For k sufficiently large.

$$\textstyle\sum_{n=1}^{\infty} P[A(D_\beta) \geq kg(\beta) \text{ for some } \beta \in [q^{n+1}, q^n)] < \infty. \tag{4.4}$$

PROOF. If p_n denotes the last written probability, then by (2.1)

$$p_n \leq P[A(D_{q^n}) > kg(q^{n+1})] \leq \exp\left[\frac{-c_1 kg(q^{n+1})}{q^n m(q^n)}\right]$$
$$\leq \exp(-c_1 kq(q-a)\delta_0 \log(n+1)r).$$

and (4.4) is clear for k large. (For the last inequality see proof of (3.5).)

STEP 4. From Step 3 it follows that

$$P[A(D_\beta) \geq kg(\beta) \text{ i.o.}, \beta \downarrow 0] = 0$$

for k large. This, Step 2, Step 1 and the analogue for g of (3.6) yield the conclusion.

5. Refinements. In this section we will obtain, under stronger hypotheses, more precise results than those stated in Theorems 1 and 2. Our proof is entirely different as well.

Let X be as before but with speed measure

$$m\{dx\} = x^\alpha L(x)dx, \quad x > 0, \quad \alpha > -1, \tag{5.1}$$

where L is continuously differentiable on (0, 1] and satisfies

$$\lim_{x\to 0+} \frac{xL'(x)}{L(x)} = 0. \tag{5.2}$$

It follows from (5.2) that L is slowly varying at 0:

$$\lim_{x\to 0+} \frac{L(cx)}{L(x)} = 1 \quad \text{for all} \quad c > 0. \tag{5.3}$$

Define u by

$$u(x) = \frac{1}{4}\left(\int_0^x \sqrt{s^\alpha L(s)}\, dx\right)^2 \tag{5.4}$$

and let Y be given by

$$Y_t = u(X_t). \tag{5.5}$$

Calculating the local generator of Y, we get

$$\begin{aligned} G^Y f(y) &= \lim_{t\to 0}[E_x f\circ u(X_t) - f\circ u(x)]/t = (G^X(f\circ u))(x) \\ &= yf''(y) + \left(\frac{\alpha+1}{\alpha+2} + \gamma(y)\right)f'(y), \quad x = u^{-1}(y), \quad y > 0, \end{aligned} \tag{5.6}$$

where γ is a continuous function which satisfies

$$\lim_{y\to 0+}\gamma(y) = 0. \tag{5.7}$$

Since $u \geq 0$, $P[Y_t \geq 0 \text{ for all } t] = 1$, and it is then standard that there exists a Brownian motion B_t such that Y_t satisfies

$$dY_t = 2(Y_t \vee 0)^{1/2}dB_t + (\nu + \gamma(Y_t))dt, \quad Y_0 = u(x_0) \tag{5.8}$$

for $t \leq T_1 = \inf\{s\colon Y_s \geq 1\}$, where $\nu = (\alpha+1)/(\alpha+2)$. See Ikeda and Watanabe (1982).

We now have

THEOREM 3. *Suppose* (5.1) *and* (5.2) *hold. Then*

$$\limsup_{t\to 0} \frac{X_t}{u^{-1}(2t\log|\log t|)} = 1, \quad a.s. \tag{5.9}$$

PROOF. For $|\beta| < \nu = (\alpha+1)/(\alpha+2)$, let Z_t^β be the solution to

$$dZ_t^\beta = 2(Z_t^\beta \vee 0)^{1/2}dB_t + (\nu+\beta)dt, \quad Z_0^\beta = 0, \tag{5.10}$$

where B_t is the Brownian motion of (5.8). By Ikeda and Watanabe (1981), pages 221–225, the solution to (5.10) is unique, $Z_t^\beta \geq 0$ for all t, and $(Z_t^\beta)^{1/2}$ is a Bessel process of index $\nu + \beta$ reflecting at 0. By a localization argument and a comparison theorem for the solutions of stochastic differential equations (Ikeda and Watanabe, 1981, page 352).

$$Z_t^{-\varepsilon} \leq Y_t \leq Z_t^{\varepsilon}, \quad \text{a.s.},$$

for t sufficiently small, $|\varepsilon| < (\alpha+1)/(\alpha+2)$. (5.9) follows since u is strictly increasing and

$$\limsup_{t\to 0} Z_t^\beta/2t\log|\log t| = 1$$

for all β sufficiently small by Shiga and Watanabe (1973).

Recall that * means maximum.

THEOREM 4. *Suppose* (5.1) *and* (5.2) *hold. Let* $\nu = (\alpha+1)/(\alpha+2)$, *let* ρ_ν *be the first positive zero of the Bessel function* $J_{\nu/2-1}$, *and let* $d_\nu = \rho_\nu^2/2$. *Then*

$$\liminf_{t\to 0} \frac{X_t^*}{u^{-1}(d_\nu t/\log|\log t|)} = 1, \quad a.s. \tag{5.11}$$

PROOF. By Wichura (1973), if $(Z_t^0)^{1/2}$ is a Bessel process of index ν, reflecting at 0, then

$$\liminf_{t\to 0} \frac{(Z_t^0)^*}{d_\nu t/\log|\log t|} = 1, \quad \text{a.s.} \tag{5.12}$$

(An alternate way to obtain (5.12) is to use the proof of Theorem 2, paying careful attention to the constants. The needed estimates follow from Bass and Erickson (1983), Remark 3.3, by a scale change.)

The proof of Theorem 4 now follows the proof of Theorem 3, using the fact that $\nu \to d_\nu$ is continuous.

It would be interesting to find the local law of the iterated logarithm for the process Y of (5.8) when $\nu = 0$ and γ satisfies (5.7). This corresponds to the case where $\alpha = -1$ in (5.1) with $m(0, 1) < \infty$.

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