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THE SIZE OF AN ANALYTIC FUNCTION AS MEASURED BY LÉVY'S TIME CHANGE[1]

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For f analytic in the unit disc put $\nu(f) = \int_0^\tau |f'(B(s))|^2\, ds$ where τ is the exit time of Brownian motion $B(t)$ from the disc. We prove that $E\Phi(\tau) \le E\Phi(\nu(f))$ for all f satisfying $|f'(0)| = 1$ and a wide class of Φ. In particular, we may take $\Phi(\lambda) = |\lambda|^p$ for $0 < p < \infty$.

Let D denote the unit disc in the complex plane and τ the first exit time from D of standard complex Brownian motion, $B(t)$, started from 0. Lévy observed that for any function f, analytic in D, the process $f(B(A(t)))$, where

$$A^{-1}(t) = \int_0^{\tau\wedge t} |f'(B(s))|^2\, ds,$$

is again a standard Brownian motion up to time $A^{-1}(\tau)$. This result has many applications to the probabilistic study of analytic functions (See, e.g., Davis, 1979.)

To conform with the notation of Davis (1979), set $\nu(f) = \int_0^\tau |f'(B_s)|^2\, ds$. If f is univalent, the random variable $\nu(f)$ has the same distribution as the first exit time of Brownian motion started at $f(0)$ from the range of f. Classical results such as Schwarz's Lemma show that the function $f(z) = z$ is, in a sense, the smallest analytic function in D satisfying $|f'(0)| = 1$. Motivated by this, Davis (1979) asked whether the stochastic inequality

$$P(\nu(f) \ge \lambda) \ge P(\nu(z) \ge \lambda), \quad \lambda > 0, \tag{1}$$

holds for such f. We show below that (1) may fail for very small λ. On the other hand, we do have the following result.

THEOREM. *Let Φ be a nonnegative, nondecreasing function on $\mathbb{R}$ which is either concave or convex, and for which the function $\phi(e^{2x})$ is convex. Then for any analytic function f on D satisfying $|f'(0)| \ge 1$ we have*

$$E\Phi(\nu(z)) \le E\Phi(\nu(f)). \tag{2}$$

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In particular,

(3) $$E\nu(z)^p \le E\nu(f)^p, \quad 0 < p < \infty.$$

PROOF. First consider the case of convex Φ. For any real θ, the process $e^{i\theta}B(t)$ is again a standard Brownian motion; moreover the exit times of these processes from D are independent of θ. Therefore by Jensen's inequality and the subharmonicity of $|f'(z)|^2$ we have

$$\begin{aligned} E\Phi(\nu(f)) &= \frac{1}{2\pi}\int_0^{2\pi} E\Phi\left(\int_0^{\tau} |f'(e^{i\theta}B(s))|^2\,ds\right)d\theta \\ &\ge E\Phi\left(\int_0^{\tau}\frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta}B(s))|^2\,d\theta\,ds\right) \\ &\ge E\Phi(\tau) = E\Phi(\nu(z)). \end{aligned}$$

For concave Φ set $\Psi(x) = \Phi(e^{2x})$. Again by Jensen's inequality and the subharmonicity for all constants b of $\log|bf'(z)|$ we have

$$\begin{aligned} E\Phi(\nu(f)) &\ge \frac{1}{2\pi}\int_0^{2\pi} E\int_0^{\tau} \Phi(\tau\,|f'(e^{i\theta}B(s))|^2)\,\frac{ds}{\tau}\,d\theta \\ &= E\int_0^{\tau}\frac{1}{2\pi}\int_0^{2\pi} \Psi(\log(\tau^{1/2}|f'(e^{i\theta}B(s))|))\,d\theta\,\frac{ds}{\tau} \\ &\ge E\Psi(\log(\tau^{1/2})) = E\Phi(\nu(z)) \end{aligned}$$

and the proof is complete.

To show that (1) fails, take for f the function $\tan^{-1}(z)$ which maps D conformally onto the vertical strip $-\pi/4 < \mathrm{Re}(z) < \pi/4$. Let T denote the exit time of Brownian motion from this strip. Then we have the following well-known estimate in terms of one-dimensional Brownian motion $X(t)$:

$$\begin{aligned} P(T \le \lambda) &= P(\max_{0\le s\le\lambda} |X(s)| \ge \pi/4) \\ &\ge 2P(X(\lambda) \ge \pi/4) \\ &\ge c\exp(-\pi^2/32\lambda) \end{aligned}$$

where c is an absolute constant.

To estimate $P(\tau \le \lambda)$ we replace D with an inscribed regular n-sided polygon and use the rotational invariance of $B(t)$:

$$\begin{aligned} P(\tau \le \lambda) &\le nP(\max_{0\le s\le\lambda} X_s \ge \cos(\pi/n)) \\ &\le c(n)\exp\{-\cos^2(\pi/n)/2\lambda\}, \end{aligned}$$

where $c(n)$ depends only on n. After fixing a large enough n, it is clear that the inequality

$$P(\tau \leq \lambda) \leq P(T \leq \lambda)$$

holds for small λ. Thus (1) fails for such λ and the chosen f.

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