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TWO-SIDED MARKOV CHAINS

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Let S be a countable set, and a, b two distinct elements not in S. Let Ω be the set of functions ω: Z (integers) $\to S \cup \{a, b\}$ so that (i) $\omega(n) \in S$ for some n, (ii) $\omega(n) = a \Rightarrow \omega(m) = a$ for $m < n$, and (iii) $\omega(n) = b \Rightarrow \omega(m) = b$ for $m > n$. On Ω, let $x(n)$ be the nth coordinate, θ_n the shift operator, and $\alpha = \inf\{n\colon x(n) \in S\}$. A measure P on Ω is Markov if $\forall\ i \in S,\ n \in Z$, $P(x(n) = i) < \infty$, and $P(x(n+1) = i | \sigma(x(k)\colon\ k \le n)) = P(x(n+1) = i | x(n))$ on $\{x(n) \in S\}$. A Markov measure P is called a two-sided Markov chain if there are substochastic matrices on S, p and q, so that $\forall\ i, j \in S$, $n \in Z$, $P(x(n) = i,\ x(n+1) = j) = P(x(n) = i)p(i, j) = P(x(n+1) = j)q(j, i)$. It is shown that if p is a certain kind of irreducible matrix then there exists an integer $d \ge 1$ and a real $\rho > 0$ so that $P \circ \theta_{nd}^{-1} = \rho^n P\ \forall\ n \in Z$. Of particular interest is the generalization of the case $d = 1$. A two-sided Markov chain is called quasistationary if there exists $\rho > 0$ with $P \circ \theta_n^{-1} = \rho^n P$. It is shown that if π is a measure on S and $\rho > 0$, and p is a substochastic matrix on S, with $\pi p \le \rho\pi$, then there exists a quasistationary chain with p as forward transition, $P(x(0) = i) = \pi(i)$, and $P \circ \theta_n^{-1} = \rho^n P$. P is used to prove the Riesz decomposition theorem for π. Finally, it is shown under certain verifiable assumptions, that a quasistationary chain, restricted to $\{\alpha \le 0\}$, is an extended chain, with transition p, which is a certain kind of approximate p-chain in the sense of Hunt.

1. Introduction. Recently there has been some interest in Markov processes with random birth and death. These have been constructed in, for example, [10] and [11], for the general state space, and with the real line as the index set. The point of view in [11] is that such two-sided processes can usefully represent two Markov processes in weak duality in the sense that potential theoretic ideas may be extended via this tool; see, e.g., [1], [6], and [7]. These two-sided processes are invariant under shift in time and, run either forward or backward from a fixed time, they are time-homogeneous Markov, conditioned to be alive at that fixed time.

In this paper we consider exclusively discrete time and space, and the purpose is to generalize the properties discussed in the preceding paragraph. Thus, if S is a countable set, and Z represents the integers, we seek to study processes indexed by Z with state space S and random birth and death which are time homogeneous Markov in both directions. Following, we briefly mention, section by section, the primary set up and results.

To begin, and to motivate, Section 2 gives some of the basic facts concerning the usual one-sided Markov chains. Let $b \notin S$, and Ω_+^b the set of paths from $Z_+ = \{0, 1, 2, \dots\}$ into $S \cup \{b\}$ that start in S and either stay in S for all time or

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are absorbed eventually at the state b; see (2.1). On Ω_+^b, let $x(n)$ stand for the nth coordinate, and θ_n: $\Omega_+^b \to \Omega_+^b$ the usual shift operator. Let $(p(i, j))$ be a substochastic matrix on S, and let P_b^i be the probability on Ω_+^b which makes $(x(n))$ a Markov chain with one-step transition p and so that $P_b^i(x(0) = i) = 1$. One way to describe all Markov chains on Ω_+^b with one-step transition p is as follows: Let P be a measure on Ω_+^b so that $\forall\ i \in S,\ n \in Z_+,\ P(x(n) = i) < \infty$. Then P is a Markov chain with transition $p \Leftrightarrow \forall\ f \in (\sigma(x(n))\colon\ n \geq 0)^+$, and $n \in Z_+,\ P(f \circ \theta_n | x(n)) = P_b^{x(n)} f$ on $\{x(n) \in S\}$. In this case we say that P is *subordinate* to the family $(P_b^i\colon\ i \in S)$; see (2.7). In fact, the family $(P_b^i\colon\ i \in S)$ is *self-subordinate* in the sense that $\forall\ j \in S$, P_b^j is subordinate to $(P_b^i\colon\ i \in S)$.

Now let $a \notin S \cup \{b\}$ and Ω the set of functions from $Z \to S \cup \{a, b\}$ so that there is some time when the function is in S, and to the left the function either stays in S for all sufficiently large negative n, or it is absorbed at a, and a similar behavior occurs to the right relative to absorption at b; see (2.9). (Note: Ω was defined in [9].) On Ω, let $x(n)$ be the nth coordinate and θ_n the shift, $\forall\ n \in Z$. Also, let $\alpha = \inf\{n\colon\ x(n) \in S\}$, $\beta = \sup\{n\colon\ x(n) \in S\}$. Let p, q be substochastic matrices on S, and for each i, let P^i be the probability on Ω with the following properties:

(1) $P^i(x(0) = i) = 1$.
(2) Under P^i, $\sigma(x(n)\colon\ n < 0)$, $\sigma(x(n)\colon\ n > 0)$ are independent.
(3) Under P^i, $(x(k)\colon\ k \in Z_+)$ is a Markov chain with one-step transition p.
(4) Under P^i, $(x(-k);\ k \in Z_+)$ is a Markov chain with one-step transition q.

A measure P on Ω, with $P(x(n) = i) < \infty\ \forall\ i \in S,\ n \in Z$, is a *two-sided Markov chain with forward and backward transitions p and q* if P is *subordinate* to the family $(P^i\colon\ i \in S)$ in the sense that $\forall\ f \in \sigma(x(n)\colon\ n \in Z)^+$, and $\forall\ n \in Z$, $P(f \circ \theta_n | x(n)) = P^{x(n)} f$ on $\{x(n) \in S\}$. This is the appropriate analog of the one-sided situation.

Section 3 concerns itself with the construction of two-sided Markov chains with given p and q as forward and backward transitions. If P is such a chain, and if we let $\pi_n(i) = P(x(n) = i)\ \forall\ i \in S,\ n \in Z$, then it follows that $P(x(n) = i, x(n+1) = j) = \pi_n(i) p(i, j) = \pi_{n+1}(j) q(j, i)\quad \forall\quad i, j \in S,\quad n \in Z$. Theorem (3.1) states that this condition on a family of measures $(\pi_n\colon\ n \in Z)$, and substochastic matrices p, q is sufficient to construct such a P.

In Section 4 we analyze carefully the condition, mentioned above, $\pi_n(i) p(i, j) = \pi_{n+1}(j) q(j, i)$. The main result states that if p is irreducible so that for some $k \geq 1$, $p^k(i, i) > 0\ \forall\ i \in S$, then there exist $d \geq 1$ and a real $\rho > 0$ so that $\forall\ n \in Z$, $P \circ \theta_{nd}^{-1} = \rho^n P$; see (4.2). Taking this as motivation we show that if given $p(i, j)$, $d \geq 1$ (an integer), $\rho > 0$ and measures $\pi_0, \pi_1, \ldots, \pi_{d-1}$ satisfying $\pi_k p \leq \pi_{k+1}$ for $0 \leq k \leq d-2$, $\pi_{d-1} p \leq \rho \pi_0$, and $\pi_n(i) \pi_{m+1}(j) = \pi_m(i) \pi_{n+1}(j)$ whenever $p(i, j) > 0$ and $0 \leq m < n \leq d-1$ (where we define $\pi_d = \rho \pi_0$) then there exists a two-sided Markov chain P having p as forward transition so that $P \circ \theta_{nd}^{-1} = \rho^n P$ and $\forall\ i \in S,\ 0 \leq n \leq d-1,\ P(x(n) = i) = \pi_n(i)$; see (4.9). We end the section with an example where $d = 2$.

In Section 5 we consider the case of $d = 1$ in the preceding paragraph. Thus, if given $p(i, j)$, a measure π, and $\rho > 0$ so that $\pi p \leq \rho \pi$, then there exists a two-sided Markov chain P with forward transition p so that $\forall\ i \in S,\ n \in Z$, $P \circ \theta_n^{-1} = \rho^n P$, and $P(x(0) = i) = \pi(i)$. We call such a P a *quasistationary* chain. Note that P is "almost" stationary, and is stationary if $\rho = 1$. If $\pi p = \rho\pi$, then π is a quasistationary distribution according to [2] and [14], from which we borrow the term. It follows from (5.4), that $\pi p = \rho\pi \Leftrightarrow P(\alpha > -\infty) = 0$, thus relating an analytic condition to a "probabilistic" one. In fact, if we define $\mu(i) = P(x(0) = i, \alpha = -\infty)$, and $\xi(i) = P(x(0) = i, \alpha > -\infty)$, then $\pi = \mu + \xi$ where $\mu p = \rho\mu$ and ξ is a pure ρ-potential; see (5.4). I.e., P is used to give a "path-wise" proof of the Riesz decomposition theorem. This decomposition theorem is used to show that if $\lambda \geq 0$ is an eigenvalue (with a nonnegative left eigenvector) of a nonnegative matrix A on S, then $\lambda \leq \sup\{\Sigma_{j \in S} A(i, j)\colon i \in S\}$. For finite matrices, this was shown in [5] by using the Perron–Frobenius theory.

In the last part of Section 5, we consider the relationship to extended chains as defined in [8] and [9]. An extended chain with transition p is a measure Q on Ω so that $\forall\ i \in S$, the expected number of visits to i is finite, and so that for every finite subset E of S the following holds: starting from the first passage to E, the process is a Markov chain with one-step transition p. [There are a couple other technical conditions which can be found in the proof of (5.14).] Now, if P is a quasistationary chain then $\forall\ i \in S$, $\Sigma_{n \in Z} P(x(n) = i) = \pi(i)\Sigma_{n \in Z}\rho^n = \infty \cdot \pi(i)$, and, unless $P = 0$, P has no hope of being an extended chain. However, P can be modified. The main result, (5.14), states that, modulo those above mentioned technical conditions, if $(x(n))$ is restricted to the set $\{\alpha \leq 0\}$, then it becomes an extended chain with transition p. This provides another method of constructing extended chains (see [9], Theorem 10-9), and will perhaps be useful, in future investigations, for considering Martin boundary problems.

2. Preliminaries and definitions.

2.1. Measures and matrices. Throughout the paper S shall stand for a fixed countable set. A *measure on* S is a function $\pi\colon S \to [0, \infty]$. Such a π corresponds to a measure, in the sense of measure theory, defined on the power set of S via the formula, $\pi(A) = \Sigma_{i \in A}\pi(i)$. Thus, we call π *finite* if $\Sigma_{i \in S}\pi(i) < \infty$, and we call π *σ-finite* if $\pi(i) < \infty\ \forall\ i \in S$.

A $[0, \infty]$-*valued matrix* on S is a function $p\colon S \times S \to [0, \infty]$. (All matrices in this paper shall be $[0, \infty]$-valued.) Such a p is called *substochastic* (resp. *stochastic*) if $\forall\ i \in S$, $\Sigma_{j \in S} p(i, j) \leq 1$ (resp. $= 1$).

As usual, a $[0, \infty]$-valued matrix p can be considered as an operator on measures. Thus, if π is a measure, then πp is the measure defined by $\pi p(j) = \Sigma_{i \in S}\pi(i)p(i, j)$. Also, if p and q are $[0, \infty]$-valued matrices then pq is defined by $pq(i, j) = \Sigma_{s \in S} p(i, s)q(s, j)$.

If p and q are $[0, \infty]$-valued matrices then $p \leq q$ (resp. $< q$) means that $\forall\ i, j$, $p(i, j) \leq q(i, j)$ [resp. $< q(i, j)$]. (Of course, this abuses notation, but is nonetheless standard.) A similar notation goes for inequalities between measures. If a matrix or measure is ever compared to the symbol 0 in an inequality, it is understood that 0 stands for the matrix or measure with every entry equal to 0. E.g., $\pi > 0$ shall mean $\pi(i) > 0$, $\forall\ i$.

2.2. One-sided processes. In this paper Z shall denote the set of integers and $Z_+ = \{n \in Z: n \geq 0\}$. This subsection lists some basic facts about Markov chains indexed by Z_+, the one-sided case. This will motivate definitions in Section 2.3 for two-sided processes.

DEFINITION 2.1. Let $x \notin S$.

(a) $\Omega_+^x = \{$functions ω: $Z_+ \to S \cup \{x\}$: $\omega(0) \in S$, and $\omega(n) = x \Rightarrow \omega(m) = x$ $\forall$ $m > n\}$.
(b) For $n \geq 0$ define $x(n)$: $\Omega_+^x \to S \cup \{x\}$ by $x(n)(\omega) = \omega(n)$.
(c) For $n \geq 0$ define θ_n: $\Omega_+^x \to \Omega_+^x$ by $\theta_n(\omega)(m) = \omega(m+n)$.

DEFINITION 2.2. Let P be a nonnegative measure on $(\Omega_+^x, \sigma(x(n)\colon n \in Z_+))$, and assume that $\forall$ $i \in S$, $n \in Z_+$, $P(x(n) = i) < \infty$. Then P is called *Markov* if $\forall$ $i \in S$ and $n \in Z_+$ we have

$$P(x(n+1) = i|\sigma(x(k)\colon k \leq n)) = P(x(n+1) = i|x(n)) \tag{2.3}$$

on $\{x(n) \in S\}$.

REMARK 2.4. In (2.2), P need not be a probability. Also, (2.3) is to be understood in the sense of conditional expectation which, of course, still has the meaning (via the Radon–Nikodym theorem) as if P were a probability.

DEFINITION 2.5. Let P be a Markov measure on $(\Omega_+^x, \sigma(x(n)\colon n \in Z_+))$. Then P is called a *Markov chain* if there exists a substochastic matrix p on S so that $\forall$ $i, j \in S$, $n \in Z_+$, $P(x(n) = i, x(n+1) = j) = P(x(n) = i)p(i, j)$. In other words, a Markov chain is a Markov measure which is "time homogeneous." In this case we say that P has one-step transition matrix p.

Now, suppose given a substochastic matrix p. Then corresponding to every finite measure π [i.e., $\sum_{i \in S}\pi(i) < \infty$] there exists a Markov chain P_x^π on Ω_+^x with one-step transition matrix p and so that $\forall$ $i \in S$, $P_x^\pi(x(0) = i) = \pi(i)$. If π is point mass at i we write $P_x^\pi = P_x^i$.

DEFINITION 2.6. $(P_x^i\colon i \in S)$ is called the *family generated by* p.

It is easy to check that if P is a Markov chain with one-step transition matrix p, then $\forall$ $f \in \sigma(x(k)\colon k \in Z_+)^+$, $n \in Z_+$, $P(f \circ \theta_n|x(n)) = P_x^{x(n)}f$ on $\{x(n) \in S\}$. We single this property out in

DEFINITION 2.7. Let $(Q^i\colon i \in S)$ be a family of probability measures on $(\Omega_+^x, \sigma(x(n)\colon n \in Z_+))$ so that $\forall$ $i \in S$, $Q^i(x(0) = i) = 1$. Also, let P be a measure on $(\Omega_+^x, \sigma(x(n)\colon n \in Z_+))$ so that $\forall$ $i \in S$, $n \in Z_+$, $P(x(n) = i) < \infty$. Then we say that P is *subordinate to the family* $(Q^i\colon i \in S)$ if $\forall$ $f \in \sigma(x(k)\colon k \in Z_+)^+$, $n \in Z_+$, $P(f \circ \theta_n|x(n)) = Q^{x(n)}f$ on $\{x(n) \in S\}$.

Thus, if $(P_x^i\colon i \in S)$ is the family generated by the matrix p, then any Markov chain on Ω_+^x with p as one-step transition is subordinate to $(P_x^i\colon i \in S)$. Actually, it is not hard to verify that if P is a measure on $(\Omega_+^x, \sigma(x(n)\colon n \in Z_+))$ with $P(x(n) = i) < \infty\ \forall\ i \in S$, $n \in Z_+$, then P is a Markov chain with p as one-step transition if and only if P is subordinate to the family generated by p.

Also, $(P_x^i\colon i \in S)$ is *self-subordinate* in the sense that $\forall\ j \in S$, P_x^j is subordinate to $(P_x^i\colon i \in S)$. In fact, the appropriately interpreted converse of this holds. The elementary proof is omitted.

THEOREM 2.8. *Suppose $(Q^i\colon i \in S)$ is as in* (2.7), *and is self-subordinate in the sense that $\forall\ j \in S$, Q^j is subordinate to $(Q^i\colon i \in S)$. Then $(Q^i\colon i \in S)$ is the family generated by the substochastic matrix q where $q(i, j) = Q^i(x(1) = j)$, $\forall$ $i, j \in S$.*

2.3. Two-sided processes. Fix $a \notin S$, and $b \notin S \cup \{a\}$.

DEFINITION 2.9.

(a) $\Omega = \{$functions $\omega\colon\ Z \to S \cup \{a, b\}$: there exists $n \in Z$ with $x(n) \in S$, $x(k) = a \Rightarrow x(l) = a$ for $l < k$, and $x(k) = b \Rightarrow x(l) = b$ for $l > k\}$.
(b) For $n \in Z$ define $x(n)\colon \Omega \to S \cup \{a, b\}$ by $x(n)(\omega) = \omega(n)$.
(c) For $n \in Z$ define $\theta_n\colon \Omega \to \Omega$ by $\theta_n(\omega)(m) = \omega(m + n)$.
(d) $\alpha \equiv \inf\{n\colon x(n) \in S\}$. (Note: $-\infty \le \alpha < \infty$.)
(e) $\beta \equiv \sup\{n\colon x(n) \in S\}$. (Note: $-\infty < \beta \le \infty$.)

Let p, q be substochastic matrices on S. Let $(P_b^i\colon i \in S)$ [resp. $(P_a^i\colon i \in S)$] be the family on Ω_+^b (resp. Ω_+^a) generated by p (resp. q); see (2.6). Finally, let Φ: $\Omega_+^b \times \Omega_+^a \to \Omega$ be defined by $\Phi(\omega_1, \omega_2) = \omega$ where

$$\omega(n) = \begin{cases} \omega_1(n) & \text{for } n \ge 0, \\ \omega_2(-n) & \text{for } n < 0. \end{cases}$$

DEFINITION 2.10. For each $i \in S$, let $P^i = (P_b^i \otimes P_a^i) \circ \Phi^{-1}$. $(P^i\colon i \in S)$ is called the *family generated by* (p, q).

We list some elementary properties without proof.

PROPOSITION 2.11. *Let $(P^i\colon i \in S)$ be the family generated by the pair (p, q). Then for each i, P^i is a probability and $P^i(x(0) = i) = 1$. Moreover, if $k, l \ge 0$, $i_\gamma, j_\delta \in S$ for $0 \le \gamma \le k$ and $0 \le \delta \le l$, and $i_0 = j_0 = i$, then*

$$P^i\big(x(-\gamma) = i_\gamma, x(\delta) = j_\delta\colon 0 \le \gamma \le k, 0 \le \delta \le l\big)$$
$$= \left(\prod_{\gamma=0}^{k-1} q(i_\gamma, i_{\gamma+1})\right)\left(\prod_{\delta=0}^{l-1} p(j_\delta, j_{\delta+1})\right),$$

where we let $\prod_{\gamma=0}^{k-1} q(i_\gamma, i_{\gamma+1}) = 1$ if $k = 0$ and $\prod_{\delta=0}^{l-1} p(j_\delta, j_{\delta+1}) = 1$ if $l = 0$.

DEFINITION 2.12. Let $(T^i\colon i \in S)$ be a family of probabilities on $(\Omega, \sigma(x(k)\colon k \in Z))$, so that $\forall\, i \in S$, $T^i(x(0) = i) = 1$. Also, let P be a measure on $(\Omega, \sigma(x(k)\colon k \in Z))$, so that $\forall\, i \in S$, $n \in Z$, $P(x(n) = i) < \infty$. Then we call P *subordinate to the family* $(T^i\colon i \in S)$ if $\forall\, f \in \sigma(x(k)\colon k \in Z)^+$, $n \in Z$, $P(f \circ \theta_n | x(n)) = T^{x(n)} f$ on $\{x(n) \in S\}$.

The following definition is motivated by the discussion following (2.7).

DEFINITION 2.13. Let P be a measure on $(\Omega, \sigma(x(n)\colon n \in Z))$ so that $\forall\, i \in S$, $n \in Z$, $P(x(n) = i) < \infty$. Then P is called a *two-sided Markov chain* if there exist substochastic matrices p, q so that P is subordinate to the family generated by (p, q). In this case we say that P has *forward transition* p and *backward transition* q.

Now, let P be as in (2.13). For each $n \in Z$ define the measure π_n by $\pi_n(i) = P(x(n) = i)\ \forall\, i \in S$. Fix $i, j \in S$ and $n \in Z$. Then

$$\begin{aligned} P(x(n) = i, x(n+1) = j) &= P\big(\theta_n^{-1}(x(1) = j); x(n) = i\big) \\ &= \pi_n(i) P^i(x(1) = j) = \pi_n(i) p(i, j). \end{aligned}$$

Also,

$$\begin{aligned} P(x(n) = i, x(n+1) = j) &= P\big(\theta_{n+1}^{-1}(x(-1) = i); x(n+1) = j\big) \\ &= \pi_{n+1}(j) P^j(x(-1) = i) = \pi_{n+1}(j) q(j, i). \end{aligned}$$

Thus, we have

$$\pi_n(i) p(i, j) = \pi_{n+1}(j) q(j, i) \quad \forall\, i, j \in S,\ n \in Z. \tag{2.14}$$

Also, if $n \in Z$, $k \geq 0$, $i_\gamma \in S$ for $0 \leq \gamma \leq k$ then

$$\begin{aligned} &P\big(x(n+\gamma) = i_\gamma\colon 0 \leq \gamma \leq k\big) \\ &\quad = P\big(\theta_n^{-1}\big(x(\gamma) = i_\gamma\colon 0 \leq \gamma \leq k\big); x(n) = i_0\big) \\ &\quad = \pi_n(i_0) P^{i_0}\big(x(\gamma) = i_\gamma\colon 0 \leq \gamma \leq k\big) \\ &\quad = \pi_n(i_0) \prod_{\gamma=0}^{k-1} p\big(i_\gamma, i_{\gamma+1}\big). \end{aligned}$$

This easily shows that P is Markov [i.e., $\forall\, i \in S$, $n \in Z$, $P(x(n+1) = i | \sigma(x(k)\colon k \leq n)) = P(x(n+1) = i | x(n))$ on $\{x(n) \in S\}$] and that $\forall\ i \in S$, $n \in Z$, $P(x(n+1) = i | x(n)) = p(x(n), i)$ on $\{x(n) \in S\}$. Similarly, $\forall\ i \in S$, $n \in Z$, $P(x(n-1) = i | \sigma(x(k)\colon k \geq n)) = P(x(n-1) = i | x(n)) = q(x(n), i)$. This is what justifies calling p and q the forward and backward transitions.

Actually, it is easy to check that if P is a measure which is Markov, in the above sense, then P is a two-sided Markov chain with p and q as forward and backward transitions if both quantities in (2.14) are equal to $P(x(n) = i, x(n+1) = j)$.

REMARK 2.15. (a) If P is as in (2.13) then P defines a process which is Markov and time homogeneous in both directions, and as such P is analogous to one-sided Markov chains. However, unlike one-sided chains, the family generated by (p, q), $(P^i\colon i \in S)$, can only be self-subordinate [i.e., $\forall\, j$, P^j is subordinate to $(P^i\colon i \in S)$] in the trivial case where $p = q^T$, and p is deterministic in the sense that $\forall\, i, j$, $p(i, j) = 0$ or 1. (The not too difficult proof is omitted.) In other words, we cannot expect each of the P^is to be a two-sided Markov chain with forward and backward transitions p and q.

(b) The purpose for having two-sided chains defined on two-sided path space is because it is not possible to have both forward and backward transition matrices homogeneous in one-sided chains for arbitrary initial measures. In Section 4 we shall see that there is much freedom possible for two-sided chains with a prescribed "initial" measure.

3. Construction of two-sided Markov chains. Let P be a two-sided Markov chain with forward and backward transitions given by p and q. We have already seen that (2.14) is a necessary condition. It is also sufficient for the construction of two-sided Markov chains. [In the case where p and q are stochastic, (3.1) follows from [3], Theorem B.]

THEOREM 3.1. *Let p,q be substochastic matrices on S, and $(\pi_n\colon n \in Z)$ a family of σ-finite measures on S. Suppose that* (2.14) *holds. Then there exists a two-sided Markov chain P with p and q as forward and backward transitions so that $\forall\, i \in S$, $n \in Z$, $P(x(n) = i) = \pi_n(i)$.*

PROOF. Let $(P^i\colon i \in S)$ be the family generated by (p, q), and $\Omega_n = \{x(n) \in S\}$, $\forall\, n \in Z$. If the P we are seeking exists, then $\forall\, i \in S$, $A \in \sigma(x(k)\colon k \in Z)$, $n \in Z$ we should have

$$P(A; x(n) = i) = P\big(\theta_n^{-1}(\theta_{-n}^{-1}(A)); x(n) = i\big) = \pi_n(i)P^i\big(\theta_{-n}^{-1}(A)\big).$$

Thus, we are led to define $P_n = \sum_{i\in S}\pi_n(i)P^i \circ \theta_{-n}^{-1}$. Note that P_n is concentrated on Ω_n for each n.

Suppose $m < n$, $i, j \in S$. Let $k, l \geq 0$, $i_0 = i$, $j_0 = j$, $s_0 = i$, $s_{n-m} = j$ and $i_0, \dots, i_k, j_0, \dots, j_l, s_0, \dots, s_{n-m} \in S$. Let $A = \{x(m - \gamma) = i_\gamma, x(m + \delta) = s_\delta, x(n + \mu) = j_\mu\colon\ 0 \leq \gamma \leq k, 0 \leq \delta \leq n - m, 0 \leq \mu \leq l\}$. Then $\theta_{-m}^{-1}(A) = \{x(-\gamma) = i_\gamma, x(\delta) = s_\delta, x(n - m + \mu) = j_\mu\colon\ 0 \leq \gamma \leq k, 0 \leq \delta \leq n - m, 0 \leq \mu \leq l\}$, and $\theta_{-n}^{-1}(A) = \{x(-(n - m) - \gamma) = i_\gamma, x(-(n - m) + \delta) = s_\delta, x(\mu) = j_\mu\colon 0 \leq \gamma \leq k, 0 \leq \delta \leq n - m, 0 \leq \mu \leq l\}$.

In the sum for $P_m(A)$, there is only one possibly nonzero term which corresponds to our fixed i. Thus,

$$\begin{aligned} P_m(A) &= \pi_m(i)P^i\big(\theta_{-m}^{-1}(A)\big) \\ &= \pi_m(i)\left(\prod_{\gamma=0}^{k-1} q(i_\gamma, i_{\gamma+1})\right) \\ &\quad\times\left(\prod_{\delta=0}^{n-m-1} p(s_\delta, s_{\delta+1})\right)\left(\prod_{\mu=0}^{l-1} p(j_\mu, j_{\mu+1})\right), \end{aligned}$$

by (2.11), where, as in (2.11), we understand that a product with no terms is 1. But by (2.14)

$$\pi_m(i)\left(\prod_{\delta=0}^{n-m-1} p(s_\delta, s_{\delta+1})\right) = \pi_n(j)\left(\prod_{\delta=0}^{n-m-1} q(s_{\delta+1}, s_\delta)\right),$$

and hence

$$P_m(A) = \pi_n(j)\left(\prod_{\gamma=0}^{k-1} q(i_\gamma, i_{\gamma+1})\right)$$
$$\times\left(\prod_{\delta=0}^{n-m-1} q(s_{\delta+1}, s_\delta)\right)\left(\prod_{\mu=0}^{l-1} p(j_\mu, j_{\mu+1})\right).$$

However, again by (2.11) and the definition of P_n, this is also equal to $P_n(A)$.

Thus, whenever A has the above specified form, then $P_m(A) = P_n(A)$. This proves that $P_m|_{\Omega_m \cap \Omega_n} = P_n|_{\Omega_m \cap \Omega_n}$.

Let $n_1, n_2, \dots$ be an enumeration of Z. Let $\Omega'_1 = \Omega_{n_1}$, and for $k \geq 2$, let $\Omega'_k = \Omega_{n_k} - (\bigcup_{l=1}^{k-1}\Omega_{n_l})$. Then $\Omega'_1, \Omega'_2, \dots,$ forms a partition of Ω. For each $k \geq 1$, let $P'_k = P_{n_k}$, and finally, define P by $P(A) = \sum_{k=1}^{\infty} P'_k(A \cap \Omega'_k)$. We now will verify that P has the desired properties.

Fix n and choose k_0 so that $n_{k_0} = n$. Then if $A \subset \Omega_n$ we have $P(A) = \sum_{k=1}^{k_0} P'_k(A \cap \Omega'_k)$. But for $1 \leq k \leq k_0$, $A \cap \Omega'_k \subset \Omega_{n_k} \cap \Omega_n$ and hence $P'_k(A \cap \Omega'_k) = P_n(A \cap \Omega'_k)$. Since $\Omega_n = \bigcup_{k=1}^{k_0} \Omega'_k$, it follows that $P(A) = P_n(A)$. Thus $P|_{\Omega_n} = P_n|_{\Omega_n}$.

Thus, $\forall\, i \in S$, $n \in Z$ we have $P(\theta_n^{-1}(A); x(n) = i) = P_n(\theta_n^{-1}(A); x(n) = i) = \pi_n(i)P^i(\theta_{-n}^{-1}(\theta_n^{-1}(A))) = \pi_n(i)P^i(A)\ \forall\, A$. This proves that P is subordinate to $(P^i\colon i \in S)$, and, letting $A = \Omega$, $P(x(n) = i) = \pi_n(i)\ \forall\, i \in S$, $n \in Z$, completing the proof. □

4. Necessary conditions and generalizations.

4.1. Case of irreducible forward transitions. Let P be a two-sided Markov chain with forward and backward transitions p and q, and for each $n \in Z$ let π_n be the "distribution" of $x(n)$ under P, i.e., $\forall\, i \in S$, $\pi_n(i) = P(x(n) = i)$. By the remark in (2.15a) it can happen that the measures π_n are all point masses at distinct points and are, in this sense, all independent of one another. However, if one stipulates that p is irreducible [see (4.1)] and that there exists $k \geq 1$, an integer, so that the diagonal entries of p^k [also see (4.1)] are strictly positive, then there is a type of periodic dependence among the π_n; see (4.2).

DEFINITION 4.1. (a) Let p be a substochastic matrix on S. p^0 is the identity matrix, i.e., $p^0(i, j) = 1_{\{i=j\}}$. Also, if $k \geq 1$, then $p^k = pp^{k-1}$; see Section 2.1. That is p^k is the kth power of p.

(b) A substochastic matrix p is called *irreducible* if $\forall\, i, j \in S$ there exists $k \geq 1$ with $p^k(i, j) > 0$.

THEOREM 4.2. *Let P be a two-sided Markov chain with forward transition p. Further suppose p is irreducible and that there exists $k \geq 1$ so that $p^k(i, i) > 0$ $\forall$ $i \in S$. Let d be the greatest common divisor of the set of $k \geq 1$ so that $p^k(i, i) > 0$ $\forall$ $i \in S$. Then there exists a real number $\rho > 0$ so that $\forall$ $n \in Z$, $P \circ \theta_{nd}^{-1} = \rho^n P$.*

PROOF. If $P = 0$ the conclusion is obvious. Thus, we shall suppose that $P \neq 0$.

Fix $i, j \in S$. Then by (2.14), letting q be the backward transition for P,

$$\begin{aligned}
\forall\ n \in Z, \qquad \pi_n(i)p^2(i, j) &= \sum_{s \in S} \pi_n(i)p(i, s)p(s, j) \\
&= \sum_{s \in S} \pi_{n+1}(s)q(s, i)p(s, j) \\
&= \sum_{s \in S} \pi_{n+2}(j)q(s, i)q(j, s) \\
&= \pi_{n+2}(j)q^2(j, i).
\end{aligned}$$

Continuing by induction it follows that

$$\begin{aligned}
&\forall\ i, j \in S, \qquad n \in Z, \qquad k \in Z_+, \\
&\pi_n(i)p^k(i, j) = \pi_{n+k}(j)q^k(j, i).
\end{aligned} \tag{4.3}$$

Now, since $P \neq 0$ then there exist $i_0 \in S$, $n_0 \in Z$ so that $\pi_{n_0}(i_0) > 0$. Let $i \in S$ and $k \geq 1$ so that $p^k(i_0, i) > 0$. By (4.3) we have, $0 < \pi_{n_0}(i_0)p^k(i_0, i) \leq \pi_{n_0+k}(i)$. Thus, $\forall$ $i \in S$ there exists $n \in Z$ with $\pi_n(i) > 0$.

Let $T = \{k \geq 1\colon\ p^k(i, i) > 0\ \forall\ i \in S\}$. By hypothesis, $T \neq \varnothing$. Let $k, l \in T$, and $i \in S$. Then $p^{k+l}(i, i) \geq p^k(i, i)p^l(i, i) > 0$, and hence $k + l \in T$. In other words, T is closed under addition. By [9] Lemma 1-66, there exists $K \geq 1$ so that $\forall$ $i \in S$, $k \geq K$, $p^{kd}(i, i) > 0$.

Fix $i \in S$. By the paragraph following (4.3), there exists $n \in Z$ with $\pi_n(i) > 0$. By (4.3), $\pi_n(i)p^{Kd}(i, i) = \pi_{n+Kd}(i)q^{Kd}(i, i)$ and thus, $q^{Kd}(i, i) > 0$. By a similar argument it follows that $q^{(K+1)d}(i, i) > 0$. Define

$$\rho_i = \left(q^{Kd}(i, i)p^{(K+1)d}(i, i)\right)\left(q^{(K+1)d}(i, i)p^{Kd}(i, i)\right)^{-1}. \tag{4.4}$$

Now, $\forall$ $i \in S$, $n \in Z$ we have $\pi_n(i)p^{(K+1)d}(i, i) = \pi_{n+(K+1)d}(i)q^{(K+1)d}(i, i)$ and $\pi_{n+d}(i)p^{Kd}(i, i) = \pi_{n+(K+1)d}(i)q^{Kd}(i, i)$. This implies that $\forall$ $i \in S$, $n \in Z$, $\pi_n(i) > 0 \Leftrightarrow \pi_{n+d}(i) > 0$ and

$$\forall\ i \in S, \qquad n \in Z, \qquad \pi_{n+d}(i) = \rho_i \pi_n(i). \tag{4.5}$$

Fix $i, j \in S$. By the paragraph following (4.3) and the hypothesis, there exist $m \in Z$, $n \geq 1$ so that $\pi_m(i), p^n(i, j) > 0$. Thus $\rho_i \pi_{m-d}(i)p^n(i, j) = \pi_m(i)p^n(i, j) = \pi_{m+n}(j)q^n(j, i) = \rho_j \pi_{m+n-d}(j)q^n(j, i)$. But since $\pi_{m-d}(i)p^n(i, j) = \pi_{m+n-d}(j)q^n(j, i)$ then it follows that $\rho_i = \rho_j$.

Thus there exists $\rho > 0$ so that $\pi_{n+d}(i) = \rho\pi_n(i)$ $\forall$ $i \in S$, $n \in Z$. A straightforward monotone class argument completes the proof. □

REMARK 4.6. (a) Let p be as in the preceding theorem, and T be as in the proof. For $i \in S$, let $T_i = \{k \geq 1: p^k(i, i) > 0\}$. The same argument used for T gives that each T_i is closed under addition. Also, $T = \bigcap_{i \in S} T_i$. Let d_i be the greatest common divisor of the elements of T_i. Since p is irreducible, it follows that $d_i = d_j \ \forall\ i, j \in S$, and the common value, which we denote by d', is called the period of p; see [5] and [13].

Let K be as in the proof of (4.2), and $i \in S$. Then Kd, $Kd + d \in T \subset T_i$, and hence $d' | Kd$ and $d' | Kd + d$. Thus, $d' | d$. Thus, d is a multiple of the period d'.

(b) Let p be irreducible and for each $i \in S$, let T_i be as in (a) above. By [9] Lemma 1-66, $\forall\ i \in S$ there exists $K_i \geq 1$ so that $k \geq K_i \Rightarrow kd' \in T_i$, where d' is also as in (a) above. Thus, if there exists an integer K so that $K_i \leq K \ \forall\ i \in S$, then $Kd' \in T_i$, $\forall\ i \in S$, i.e., $T = \bigcap_{i \in S} T_i \neq \varnothing$. As a result, it follows that $T \neq \varnothing$ if, for example, S is finite.

Although (2.14) is a sufficient condition for the existence of a two-sided Markov chain, it is somewhat intractable. However, for p and d as in (4.2), (2.14) need only be checked for $0 \leq n \leq d - 1$. This leads one to suspect that there exist more tractable sufficient conditions than (2.14). We explore this in the next subsection.

4.2. A generalization of the irreducible case. Let P be as in (4.2). If $m \neq n$ then, upon eliminating $q(j, i)$ in the equations: $\pi_n(i)p(i, j) = \pi_{n+1}(j)q(j, i)$ and $\pi_m(i)p(i, j) = \pi_{m+1}(j)q(j, i)$, we find that $\forall\ i, j \in S$ such that $p(i, j) \neq 0$, $\pi_n(i)\pi_{m+1}(j) = \pi_m(i)\pi_{n+1}(j)$. However, in view of the result of (4.2) this condition is equivalent to

$$\text{If } 0 \leq m < n \leq d - 1, \text{ then } \forall\ i, j \in S$$
$$\pi_n(i)\pi_{m+1}(j) = \pi_m(i)\pi_{n+1}(j) \quad \text{provided } p(i, j) \neq 0. \tag{4.7}$$

Also, (2.14) $\Rightarrow$

$$\text{If } 0 \leq n \leq d - 1, \text{ then } \pi_n p \leq \pi_{n+1}. \tag{4.8}$$

These lead to new sufficient conditions for the construction of two-sided Markov chains.

THEOREM 4.9. *Suppose p is a substochastic matrix, $d \geq 1$ is an integer, $\rho > 0$ is a real number, and π_n is a σ-finite measure for $0 \leq n \leq d - 1$. Suppose further that* (4.7) *and* (4.8) *hold, where we define $\pi_d = \rho\pi_0$. Then there exists a two-sided Markov chain P having p as forward transition and such that*:

(i) $\forall\ 0 \leq n \leq d - 1$, $i \in S$, $P(x(n) = i) = \pi_n(i)$,
(ii) $\forall\ n \in Z$, $P \circ \theta_{nd}^{-1} = \rho^n P$.

PROOF. Fix $i, j \in S$. If there exists $0 \leq n \leq d - 1$ with $\pi_{n+1}(j) > 0$, then let $q(j, i) = (\pi_{n+1}(j))^{-1}\pi_n(i)p(i, j)$, otherwise let $q(j, i) = 0$. By (4.7), q is well defined.

Fix $j \in S$. If $\pi_{n+1}(j) = 0 \ \forall\ 0 \le n \le d-1$, then $\sum_{i \in S} q(j,i) = 0$. Next, suppose there exists $0 \le n \le d-1$ with $\pi_{n+1}(j) > 0$. Then

$$\sum_{i \in S} q(j,i) = \sum_{i \in S} \left(\pi_{n+1}(j)\right)^{-1} \pi_n(i) p(i,j) = \left(\pi_{n+1}(j)\right)^{-1} \pi_n p(j) \le 1$$

by (4.8). Thus, it follows that q is substochastic.

Extend $\pi_0, \pi_1, \dots, \pi_{d-1}$ to $(\pi_n: n \in Z)$ via the formula, $\pi_{n+d} = \rho \pi_n \ \forall\ n \in Z$. Let $0 \le n \le d-1$, and $i, j \in S$. If $\pi_{n+1}(j) > 0$ then, by definition of q, $\pi_n(i)p(i,j) = \pi_{n+1}(j)q(j,i)$. Whereas, if $\pi_{n+1}(j) = 0$, then by (4.8), $\pi_n(i)p(i,j) \le \pi_{n+1}(j) = 0$, and once again $\pi_n(i)p(i,j) = \pi_{n+1}(j)q(j,i)$. Thus, (2.14) holds for $0 \le n \le d-1$. It follows that (2.14) holds $\forall\ n \in Z$ by multiplying by appropriate powers of ρ. Now apply (3.1), and an easy monotone class argument. □

REMARK 4.10. The method of proof in (4.9) actually shows that if:

(a) $\forall\ n \ne m,\ p(i,j) > 0 \Rightarrow \pi_n(i)\pi_{m+1}(j) = \pi_m(i)\pi_{n+1}(j)$ and
(b) $\forall\ n,\ \pi_n p \le \pi_{n+1}$,

then there exists a two-sided Markov chain with forward transition p and so that π_n is the distribution of $x(n)$, $\forall\ n$. Even though this is a condition which does not involve q, it does not in general present much simplification over (2.14). However, (4.9) does at least give a condition involving a finite number of measures.

EXAMPLE 4.11. Let $S = \{1,2,3\}$ and

$$p = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that p is irreducible with $d = 2$. With $d = 2$, conditions (4.7) and (4.8) become:

(I) $\frac{1}{2}(\pi_0(2) + \pi_0(3)) \le \pi_1(1)$.
(II) $\frac{1}{2}\pi_0(1) \le \pi_1(2)$ and $\frac{1}{2}\pi_0(1) \le \pi_1(3)$.
(III) $\frac{1}{2}(\pi_1(2) + \pi_1(3)) \le \rho\pi_0(1)$.
(IV) $\frac{1}{2}\pi_1(1) \le \rho\pi_0(2)$, $\frac{1}{2}\pi_1(1) \le \rho\pi_0(3)$.
(V) $\pi_1(2)\pi_1(1) = \rho\pi_0(2)\pi_0(1)$.
(VI) $\pi_1(3)\pi_1(1) = \rho\pi_0(3)\pi_0(1)$.

This amounts to eight inequalities in the seven unknowns, $\pi_0(1)$, $\pi_0(2)$, $\pi_0(3)$, $\pi_1(1)$, $\pi_1(2)$, $\pi_1(3)$, ρ. There is naturally some degree of freedom in their selection.

For example if we set $\pi_0(1) = 1$, and change each inequality to an equality then $\pi_0(2) = \pi_0(3) = \pi_1(1)$, $\pi_1(2) = \pi_1(3) = \frac{1}{2}$, $\rho = \frac{1}{2}$. Specifically, if we write π_0, π_1 as row vectors then we can let $\pi_0 = (1,0,0)$, $\pi_1 = (0, \frac{1}{2}, \frac{1}{2})$, $\rho = \frac{1}{2}$. In this case we have

$$q = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

As another example, let $\pi_0 = (1, \frac{1}{2}, \frac{1}{2})$, $\pi_1 = (1, 1, 1)$, $\rho = 2$,

$$q = \frac{1}{4}\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

5. Quasistationary chains.

5.1. Definition. In (4.9), $d = 1$ means that (4.7) is vacuous, and (4.8) becomes $\pi_0 p \le \rho \pi_0$. Moreover, in the case where $d = 1$ and $\rho = 1$, then the resulting P is stationary (i.e., $P \circ \theta_n^{-1} = P \ \forall\ n \in Z$). [Actually the class of measures P corresponding to $d = 1$ and $\rho = 1$ in (4.9) is the class of measures which are Markov and stationary.]

DEFINITION 5.1. Let P be a two-sided Markov chain. P is called *quasistationary* if there exists a real number $\rho > 0$ so that $P \circ \theta_n^{-1} = \rho^n P \ \forall\ n \in Z$.

From now on we will call the collection (P, p, q, π, ρ) *quasistationary* if P is a quasistationary chain with forward and backward transitions p and q, $P \circ \theta_n^{-1} = \rho^n P \ \forall\ n \in Z$, and $\pi(i) = P(x(0) = i) \ \forall\ i \in S$.

REMARK 5.2. (a) In [2] a *quasistationary distribution* is a stationary conditional distribution of a finite state Markov chain $y(n)$. I.e., if $y(n)$ is started with this distribution, then $\forall\ n \ge 0$, the distribution of $y(n)$, given y has not been absorbed by time n, does not depend on n; see [2]—Section 4 for details. It is shown that if π is quasistationary, and q is the transition matrix, then there exists $\rho > 0$ so that $\pi q = \rho \pi$. Of course, if $\rho = 1$, then π is called stationary. Thus, the difference between a distribution π being stationary or quasistationary manifests itself in the issue of whether $\rho = 1$ or not in the equation, $\pi q = \rho \pi$. Also, see [14].

In the analogy to this, the difference between a two-sided Markov chain being stationary or quasistationary manifests itself in the issue of whether $\rho = 1$ or not in the inequality $\pi p \le \rho \pi$ where $\forall\ i$, $\pi(i) = P(x(0) = i)$.

(b) The following theorem follows directly from (4.9) and (2.14):

THEOREM 5.3. *Let p be a substochastic matrix and π a σ-finite measure.*

(A) *If there exists $\rho > 0$ with $\pi p \le \rho \pi$, then there exist P, q so that (P, p, q, π, ρ) is quasistationary.*

(B) *Conversely, if there exist P, q, ρ so that (P, p, q, π, ρ) is quasistationary, then $\pi p \le \rho \pi$.*

(c) Let P be a two-sided Markov chain as in (4.2). If d is one then P is a quasistationary chain by (4.2).

(d) Let P be a Markov measure in the sense that $\forall\ i \in S$, $n \in Z$, $P(x(n) = i) < \infty$, and $P(x(n+1) = i | \sigma(x(k)\colon k \le n)) = P(x(n+1) = i | x(n))$ on $\{x(n) \in S\}$. It is not hard to check (we omit details) that *P is a quasistationary chain if and only if there exists $\rho > 0$ so that $\forall\ n \in Z$, $P \circ \theta_n^{-1} = \rho^n P$.*

Also, note that if P is as in (4.9), then $\forall\ n \in Z$, $P \circ \theta_{nd}^{-1} = \rho^n P$, and we might call such a P "quasiperiodic" if $d > 1$.

5.2. Riesz decomposition theorem for nonnegative matrices with bounded row sums. Let A be a $[0,\infty]$-valued matrix on S, and assume that there exists a real number $\rho > 0$ so that ρA is substochastic. Also, suppose π is a σ-finite measure on S which is A-*excessive*, i.e., $\pi A \le \pi$.

Let $p = \rho A$. Then $\pi p \le \rho\pi$. According to (5.3A) there exist P, q so that (P, p, q, π, ρ) is quasistationary. Define the measures μ and ξ by $\mu(i) = P(x(0) = i, \alpha = -\infty)$ and $\xi(i) = P(x(0) = i, \alpha > -\infty)$.

Also, let $\zeta = \pi - \pi A$, and $N_A = \sum_{k=0}^{\infty} A^k$.

RIESZ DECOMPOSITION THEOREM 5.4.

(a) $\pi = \mu + \xi$.
(b) $\xi = \zeta N_A$.
(c) $\mu A = \mu$.
(d) $\xi A^k \to 0$ *as* $k \to \infty$, *pointwise.*
(e) *If there exist measures* μ', ξ' *with* $\pi = \mu' + \xi'$, $\mu' A = \mu'$, *and* $\xi' A^k \to 0$ *as* $k \to \infty$, *then* $\mu' = \mu$ *and* $\xi' = \xi$.
(f) $\pi A = \pi \Leftrightarrow P(\alpha > -\infty) = 0$.
(g) *The following statements are equivalent:*
   (i) $\pi A^k \to 0$ *as* $k \to \infty$.
   (ii) $\pi = \zeta N_A$,
   (iii) $\pi = \zeta' N_A$, *for some measure* ζ'.
   (iv) $P(\alpha = -\infty) = 0$.

PROOF.

(a) Obvious, since $\pi(i) = P(x(0) = i)\ \forall\, i \in S$.
(b) Fix $k \ge 0$, $i \in S$. Then

$$\begin{aligned}
P(x(0) = i, \alpha = -k) &= P(x(-k-1) = a, x(-k) \in S, x(0) = i) \\
&= P(x(-k) \in S, x(0) = i) \\
&\quad - P(x(-k-1) \in S, x(-k) \in S, x(0) = i) \\
&= \sum_{j\in S} P(x(-k) = j, x(0) = i) \\
&\quad - \sum_{j\in S}\sum_{s\in S} P(x(-k-1) = s, x(-k) = j, x(0) = i) \\
&= \sum_{j\in S} \rho^{-k}\pi(j)p^k(j,i) \\
&\quad - \sum_{j\in S}\sum_{s\in S} \rho^{-k-1}\pi(s)p(s,j)p^k(j,i) \\
&= \sum_{j\in S} \pi(j)A^k(j,i) - \sum_{j\in S}\sum_{s\in S} \pi(s)A(s,j)A^k(j,i) \\
&= \sum_{j\in S} \zeta(j)A^k(j,i) \\
&= \zeta A^k(i).
\end{aligned}$$

Thus,

$$\xi(i) = P(x(0) = i, \alpha > -\infty)$$
$$= \sum_{k=0}^{\infty} P(x(0) = i, \alpha = -k)$$
$$= \zeta N_A(i).$$

(c) $\pi A = \mu A + \xi A = \mu A + \sum_{k=1}^{\infty} \zeta A^k = \mu A + \xi - \zeta$. Thus, $\mu A = \pi A + \zeta - \xi = \pi - \xi = \mu$.

(d) $\xi A^k = \sum_{l=k}^{\infty} \zeta A^l \to 0$ as $k \to \infty$ since $\zeta N_A(i) = \xi(i) \le \pi(i) < \infty \ \forall \ i$.

(e) From (a), (c), (d) it is clear that $\mu = \lim_{k\to\infty} \pi A^k$. Similarly $\mu' = \lim_{k\to\infty} \pi A^k$, and $\mu' = \mu$. This, in turn, $\Rightarrow \xi' = \xi$.

(f) Suppose $\pi A = \pi$. Then $\pi p = \rho \pi$. Thus, $\forall \ i \in S$, $n \in Z$, $P(x(n) = i, \alpha > -\infty) = P(\theta_n^{-1}(x(0) = i, \alpha > -\infty)) = \rho^n \xi(i)$. But by (e), $\xi = 0$, and thus, $P(\alpha > -\infty) = 0$. On the other hand, if $P(\alpha > -\infty) = 0$, then $\pi = \mu$ and $\pi A = \pi$.

(g) (i) $\Rightarrow$ (ii): If (i), then $\pi = \xi$, by (e), and (ii) follows from (b).
(ii) $\Rightarrow$ (iii): Obvious.
(iii) $\Rightarrow$ (iv): The same argument as in (d) shows that (iii) $\Rightarrow \pi A^k \to 0$ as $k \to \infty$. By (e), $\mu = 0$. But, then, $\forall \ i \in S$, $n \in Z$, $P(x(n) = i, \alpha = -\infty) = P(\theta_n^{-1}(x(0) = i, \alpha = -\infty)) = \rho^n \mu(i) = 0$, which $\Rightarrow P(\alpha = -\infty) = 0$, or (iv).
(iv) $\Rightarrow$ (i): If (iv), then $\pi = \xi$, and (i) follows from (d). □

REMARK 5.5. (a) In the proof of (e), i.e., the uniqueness of the decomposition, it is shown that $\mu = \lim_{k\to\infty} \pi A^k$. If we started by defining μ in this way, and letting $\xi = \pi - \mu$, then it is not difficult to prove all the results of (5.3), except for the statements concerning P, directly. Thus, the preceding proof may be considered a "probabilistic" counterpart of a purely analytic proof. Thus, the gain here is insight, and a method which might well lead to nontrivial results for an analogous investigation for continuous time and space.

(b) In terms of p we have that $\mu p = \rho \mu$ and $\rho^{-k} \xi p^k \to 0$ as $k \to \infty$. Similarly, if we define $\mu_1(i) = P(x(0) = i, \beta = \infty)$, and $\xi_1(i) = P(x(0) = i, \beta < \infty)$, then $\mu_1 q = \rho^{-1} \mu_1$ and $\rho^k \xi_1 q^k \to 0$ as $k \to \infty$, which yields a decomposition of π relative to q. Similar to (f) we have that ($\pi q = \rho^{-1}\pi \Leftrightarrow P(\beta < \infty) = 0$) and ($\rho^k \pi q^k \to 0$ as $k \to \infty \Leftrightarrow P(\beta = \infty) = 0$). The conditions $P(\alpha = -\infty) = 0$, $P(\alpha > -\infty) = 0$, $P(\beta = \infty) = 0$, or $P(\beta < \infty) = 0$ can all be interpreted in terms of paths; e.g., $P(\alpha = -\infty) = 0$ means that $x(n)$ always gets absorbed to the left at a.

(c) Suppose given (P, p, q, π, ρ) is quasistationary so that $\pi p = \rho \pi$ and $\pi q = \rho^{-1} \pi$. (Here, we do not assume that this arises from a matrix A as in the beginning of this subsection.) Here is another way to see that $P(\alpha > -\infty) = 0$: $\forall$ $i \in S$, $n \in Z$,

$$P(x(n-1) = a, x(n) = i) = P(x(n) = i) - \sum_{j \in S} P(x(n-1) = j, x(n) = i)$$
$$= \rho^n \pi(i) - \sum_{j \in S} \rho^{n-1} \pi(j) p(j, i)$$
$$= \rho^{n-1}(\rho \pi(i) - \pi p(i)) = 0.$$

Similarly, it follows that $P(\beta < \infty) = 0$. Thus, if $\pi \neq 0$ is finite then $P(x(0) \in S) = P(\Omega) = P(x(1) \in S) = \rho P(x(0) \in S)$, and thus, $\rho = 1$. In this case, $P(x(0) \in S)^{-1}P$ is an example of a Markov random field indexed by Z; see [9].

(d) In [4] the idea of using the sets $\{\alpha = -\infty\}$ and $\{\alpha > -\infty\}$ is also used in giving a "continuous" version of the Riesz decomposition. The two-sided process there is similar to that used in [1], [6], and [11].

By (5.5c), if (P, p, q, π, ρ) is quasistationary, $\pi \neq 0$ is finite, $\pi p = \rho\pi$ and $\pi q = \rho^{-1}\pi$, then $\rho = 1$. Conversely, what can be expected if $\rho = 1$?

THEOREM 5.6. *Let $(P, p, q, \pi, 1)$ be quasistationary. If π is finite, then $\pi p = \pi \Leftrightarrow \pi q = \pi$.*

PROOF. By (5.1) and (2.14) it follows that $\forall\ i, j \in S$, $\pi(i)p(i, j) = \pi(j)q(j, i)$. (I.e., p and q are in *duality* relative to π; see [9].) Thus,

$$\sum_{j\in S} \pi q(j) = \sum_{j\in S}\sum_{i\in S} \pi(i)q(i, j) = \sum_{i\in S}\sum_{j\in S} \pi(j)p(j, i).$$

Now, suppose $\pi p = \pi$. Then we have $\sum_{j\in S}\pi q(j) = \sum_{i\in S}\pi(i) = \sum_{j\in S}\pi(j) < \infty$. Thus, $\sum_{j\in S}(\pi(j) - \pi q(j)) = 0$. But since $\pi q(j) \leq \pi(j)\ \forall\ j \in S$, then $\pi q(j) = \pi(j)\ \forall\ j \in S$, and $\pi q = \pi$. Similarly $\pi q = \pi \Rightarrow \pi p = \pi$. □

EXAMPLE 5.7. The following shows the importance of the condition that π be finite in the previous theorem.

Let $S = Z$, and define p by $p(i, j) = 1$ if $j = i + 1$ and $i \neq -1$, and $p(i, j) = 0$ otherwise. Also define π on S by $\pi(i) = 1$ if $i < 0$, $\pi(i) = 0$ otherwise. We now verify that $\pi p = \pi$.

Observe that if $j \neq 0$, then $\pi p(j) = \pi(j - 1)$, and $\pi p(0) = 0$. Thus, if $j < 0$ we have $\pi p(j) = \pi(j)$, and $\pi p(0) = \pi(0)$, and if $j > 0$ we have $\pi p(j) = \pi(j)$. Hence $\pi p = \pi$.

For $i, j \in S$ let $q(j, i) = \pi(j)^{-1}\pi(i)p(i, j)$ if $j < 0$ and $q(j, i) = 0$ otherwise. According to the proof of (4.9), where we let $d = 1$, and $\rho = 1$, then q is substochastic and $\pi(i)p(i, j) = \pi(j)q(j, i)\ \forall\ i, j \in S$. By (4.9) there exists a stationary chain P having p as forward transition and q as backward transition.

However, $\pi q(-1) = \sum_{i\in S}\pi(i)q(i, -1) = \sum_{i\in S}\pi(-1)p(-1, i) = 0 < 1 = \pi(-1)$. Thus, $\pi q \neq \pi$. This example was suggested by J. Glover.

Here is an interesting application of the decomposition theorem concerning the eigenvalues of a nonnegative matrix.

COROLLARY 5.8. *Let A and ρ be as in the beginning of this subsection, $\pi \neq 0$ a finite measure, and $\lambda > 0$ a real number so that $\pi A = \lambda\pi$. Then $\lambda\rho \leq 1$.*

PROOF. Let $A' = \lambda^{-1}A$. Then $\sup_{i\in S}\sum_{j\in S}A'(i, j) \leq \lambda^{-1}\rho^{-1}$. Let $\pi = \mu + \xi$ be the decomposition of π according to A'; see (5.4). Since $\pi A' = \pi$, it follows that $\xi = 0$. But if $\rho\lambda > 1$, then $\forall\ i \in S$, $\pi(A')^k(i) = \sum_{j\in S}\pi(j)(A')^k(j, i) \leq \lambda^{-k}\rho^{-k}(\sum_{j\in S}\pi(j)) \to 0$ as $k \to \infty$, and thus, $\mu = 0$. This contradicts the fact that $\pi \neq 0$. □

REMARK 5.9. This generalizes a fact proven for finite nonnegative matrices in [5]; i.e., that the largest positive eigenvalue is less than or equal to the maximum row sum. The method of proof there uses the theorem of Perron and Frobenius.

EXAMPLE 5.10. This example shows the importance of the condition that π be finite in (5.8).

Let $S = \{1, 2, 3, \dots\}$. We loosely describe a matrix p as follows, leaving the precise definition to the reader:

CASE 1. Let j be the kth odd number. Then $p(i, j) = 0$ unless $i = i_1$, or i_2, where (i_1, i_2) is the kth consecutive pair of odd numbers, in which case $p(i, j) = 1$. [Here $(1, 3)$ is the first pair, $(5, 7)$, the second, etc.]

CASE 2. Let j be the kth even number. Then $p(i, j) = 0$ unless $i = i_1$, i_2, or i_3, where (i_1, i_2, i_3) is the kth consecutive triple of even numbers, in which case $p(i, j) = 1$. [Here $(2, 4, 6)$ is the first triple, $(8, 10, 12)$, the second, etc.]

Let π, π' be defined by $\pi(i) = 1_{\{i \text{ is odd}\}}$, $\pi'(i) = 1_{\{i \text{ is even}\}}$. Then we have $\pi p = 2\pi$, and $\pi' p = 3\pi'$.

This example also shows that not only is it possible to have $\rho > 1$ with $\pi p = \rho\pi$, when π is infinite, but also, there can be more than one such ρ.

5.3. Relationship with extended chains. As mentioned in the introduction the space Ω is defined in [8] and [9], so as to introduce the notion of an extended chain; see [9], Definition 10-5. An extended chain is a certain kind of measure $Q \neq 0$ on $(\Omega, \sigma(x(n)\colon n \in Z))$, which displays a type of Markov property, and such that the occupation measure, defined by $\sum_{n \in Z} Q(x(n) = j)$, $\forall\ j \in S$, is σ-finite.

Now, if (P, p, q, π, ρ) is quasistationary then $\forall\ j$ it follows that

$$\sum_{n \in Z} P(x(n) = j) = \pi(j) \sum_{n \in Z} \rho^n = \infty \cdot \pi(j).$$

Thus, the only way P can have a σ-finite occupation measure is when $P = 0$. Based on this, it follows that no quasistationary chain is an extended chain. The purpose of this subsection is to show how we may modify a quasistationary chain in order to obtain an extended chain.

Now, suppose (P, p, q, π, ρ) is quasistationary, π is finite and $\rho > 1$. Let $A = \rho^{-1} p$. Then $\forall\ i \in S$, $\pi A^k(i) = \rho^{-k} \pi p^k(i) \leq \rho^{-k}(\sum_{j \in S} \pi(j)) \to 0$ as $k \to \infty$. By (5.4g), $P(\alpha = -\infty) = 0$. That is, in this case $(x(n)\colon n \in Z)$ always has a starting time.

Throughout the rest of this subsection we assume that (P, p, q, π, ρ) is quasistationary with:

(a) $\rho > 1$ and $P \neq 0$,
(b) $P(\alpha = -\infty) = 0$.

Of course, by (5.4g) (b) is the same as requiring that $\rho^{-k} \pi p^k \to 0$ as $k \to \infty$. Let $\eta = \rho\pi - \pi p$, and $N = \sum_{k=0}^{\infty} p^k$.

PROPOSITION 5.11. $\forall\, i \in S$, $P(x(\alpha) = i, \alpha \le 0) = (\rho - 1)^{-1}\eta(i)$.

PROOF. Fix $n \in Z$. Then

$$\begin{aligned} P(x(n) = i, \alpha = n) &= P(x(n-1) = a, x(n) = i) \\ &= P(x(n) = i) - \sum_{j \in S} P(x(n-1) = j, x(n) = i) \\ &= \rho^n \pi(i) - \sum_{j \in S} \rho^{n-1}\pi(j)p(j,i) \\ &= \rho^{n-1}\eta(i). \end{aligned}$$

The result follows by summing over $n \le 0$. □

PROPOSITION 5.12. *Fix* $i \in S$. *If* $n < 0$, *then* $P(x(n) = i, \alpha \le 0) = \rho^n \pi(i)$, *while if* $n \ge 0$, *then* $P(x(n) = i, \alpha \le 0) = \pi p^n(i)$.

PROOF. If $n < 0$, then $P(x(n) = i, \alpha \le 0) = P(x(n) = i) = \rho^n\pi(i)$.

Next, let $n \ge 0$, $k \ge 0$. Then

$$\begin{aligned} P(x(n) = i, \alpha = -k) &= \sum_{j \in S} P(x(n) = i, x(-k-1) = a, x(-k) = j) \\ &= \sum_{j \in S} \Big[P(x(-k) = j, x(n) = i) \\ &\qquad - \sum_{s \in S} P(x(-k-1) = s, x(-k) = j, x(n) = i) \Big] \\ &= \sum_{j \in S} \Big[\rho^{-k}\pi(j)p^{k+n}(j,i) \\ &\qquad - \sum_{s \in S} \rho^{-k-1}\pi(s)p(s,j)p^{k+n}(j,i) \Big] \\ &= \sum_{j \in S} \rho^{-k-1}\eta(j)p^{k+n}(j,i) \\ &= \rho^{-k-1}\eta p^{k+n}(i) \\ &= \rho^{-1}\eta A^k p^n(i) = \zeta A^k p^n(i), \end{aligned}$$

where $A = \rho^{-1}p$ and $\zeta = \pi - \pi A$.

But since $P(\alpha = -\infty) = 0$, then $\pi = \zeta N_A$, by (5.4g). Thus,

$$\begin{aligned} P(x(n) = i, \alpha \le 0) &= \sum_{k=0}^{\infty} P(x(n) = i, \alpha = -k) \\ &= \sum_{k=0}^{\infty} \zeta A^k p^n(i) = \pi p^n(i). \;\square \end{aligned}$$

We now begin construction of what will be, under a few more hypotheses, an extended chain. Let Ψ: $\{\alpha \le 0\} \to \Omega$ be the canonical injection, and define $Q = (P|_{\{\alpha \le 0\}}) \circ \Psi^{-1}$. Obviously, Q is concentrated on $\{-\infty < \alpha \le 0\}$.

Recall the definition of Ω_+^b; see (2.1). Define Γ: $\{\alpha \in Z\} \to \Omega_+^b$ by $\Gamma(\omega) = \omega'$ where $\omega'(k) = \omega(\alpha + k)$ for $k \in Z_+$. Finally, let $Q' = (Q|_{\{\alpha \in Z\}}) \circ \Gamma^{-1}$.

PROPOSITION 5.13. *Q' is a Markov chain with p as one-step transition, and such that $\forall\ i \in S$, $Q'(x(0) = i) = (\rho - 1)^{-1}\eta(i)$.*

PROOF. Fix $i \in S$. Then $Q'(x(0) = i) = Q(x(\alpha) = i, \alpha \in Z) = P(x(\alpha) = i, \alpha \le 0) = (\rho - 1)^{-1}\eta(i)$, by (5.11).

Next, let $k \ge 0$, and $i_\gamma \in S$ for $0 \le \gamma \le k$. Then

$$\begin{aligned} Q'\big(x(\gamma) = i_\gamma\colon 0 \le \gamma \le k\big) &= Q\big(x(\alpha + \gamma) = i_\gamma\colon 0 \le \gamma \le k;\ \alpha \in Z\big) \\ &= P\big(x(\alpha + \gamma) = i_\gamma\colon 0 \le \gamma \le k;\ \alpha \le 0\big) \\ &= \sum_{l=0}^{\infty} P\big(x(-l + \gamma) = i_\gamma\colon 0 \le \gamma \le k;\ \alpha = -l\big). \end{aligned}$$

But if $l \ge 0$, then

$$\begin{aligned} &P\big(x(-l + \gamma) = i_\gamma\colon 0 \le \gamma \le k;\ \alpha = -l\big) \\ &\quad = P\big(x(-l - 1) = a;\ x(-l + \gamma) = i_\gamma\colon 0 \le \gamma \le k\big) \\ &\quad = P\big(x(-l + \gamma) = i_\gamma\colon 0 \le \gamma \le k\big) \\ &\qquad - P\big(x(-l - 1) \in S;\ x(-l + \gamma) = i_\gamma\colon 0 \le \gamma \le k\big) \\ &\quad = \rho^{-l}\pi(i_0)\prod_{\gamma=0}^{k-1} p(i_\gamma, i_{\gamma+1}) - \sum_{i \in S} \rho^{-l-1}\pi(i)p(i, i_0)\prod_{\gamma=0}^{k-1} p(i_\gamma, i_{\gamma+1}) \\ &\quad = \rho^{-l-1}\eta(i_0)\prod_{\gamma=0}^{k-1} p(i_\gamma, i_{\gamma+1}), \end{aligned}$$

where, as before, a product with no terms is one.

It now follows by summing over $l \ge 0$ that

$$Q'\big(x(\gamma) = i_\gamma\colon 0 \le \gamma \le k\big) = (\rho - 1)^{-1}\eta(i_0)\prod_{\gamma=0}^{k-1} p(i_\gamma, i_{\gamma+1}),$$

and the proof is complete. □

It will now be convenient to define the matrix H. $\forall\ i, j \in S$, $H(i, j) = P_b^i(x(n) = j$ for some $n \ge 0)$; see (2.6). Also, see [9], page 95.

THEOREM 5.14. *If $\eta H > 0$, and $\pi N < \infty$, then Q is an extended chain with transition matrix p.*

PROOF. According to Definition 10-5 of [9] the following four conditions must be met:

(a) $\forall\ i \in S$, there exists n so that $Q(x(n) = i) > 0$.
(b) $\forall\ i \in S$, $\inf\{n \in Z\colon x(n) = i\} > -\infty$, Q a.e.
(c) Let E be a finite subset of S, and $\alpha_E = \inf\{n \in Z\colon x(n) \in E\}$.

Define Γ_E: $\{\alpha_E \in Z\} \to \Omega_+^b$ by $\Gamma_E(\omega) = \omega'$ where $\omega'(k) = \omega(\alpha_E + k)$ $\forall$ $k \geq 0$. Also, let $Q'_E = (Q|_{\{\alpha_E \in Z\}}) \circ \Gamma_E^{-1}$. Then the condition is that Q'_E be a Markov chain with one-step transition p and finite initial measure.

(d) $\forall$ $i \in S$, $\nu(i) < \infty$, where $\nu(i) = \sum_{n \in Z} Q(x(n) = i)$.

PROOF OF (a). Fix $i \in S$. Since $\eta H(i) > 0$, there exists $j \in S$ with $\eta(j)H(j, i) > 0$. According to (5.13), and the definition of H, there exists $k \geq 0$ so that $Q'(x(0) = j, x(k) = i) > 0$. But then $Q'(x(k) = i) > 0$ and thus, $Q(x(\alpha + k) = i, \alpha \in Z) > 0$, and (a) follows.

PROOF OF (b). This follows from the fact that $\alpha > -\infty$, Q a.e.

PROOF OF (c). Fix $i \in E$, where E is a fixed finite subset of S. Then

$$\begin{aligned} Q'_E(x(0) = i) &= Q(x(\alpha_E) = i, \alpha_E \in Z) \\ &\leq \sum_{n \in Z} Q(x(n) = i) \\ &= \sum_{n \in Z} P(x(n) = i, \alpha \leq 0) \\ &= \sum_{k=1}^{\infty} \rho^{-k}\pi(i) + \sum_{k=0}^{\infty} \pi p^k(i), \end{aligned}$$

by (5.12), which is equal to $(\rho - 1)^{-1}\pi(i) + \pi N(i)$, and this is finite by hypothesis. As E is finite, this shows that Q'_E has a finite initial measure. The fact that Q'_E is a Markov chain with one-step transition p follows from the strong Markov property of Q' applied to the first passage time to the set E; see (5.13).

PROOF OF (d). As above, (5.12) implies that $\nu(i) = (\rho - 1)^{-1}\pi(i) + \pi N(i) < \infty$ by hypothesis. □

REMARK 5.15. As in (d) of the preceding proof, let ν be the *occupation measure* of Q, i.e., $\nu(i) = \sum_{n \in Z} Q(x(n) = i)$. By the above proof, $\nu = (\rho - 1)^{-1}\pi + \pi N$.

Thus,

$$\begin{aligned} \nu p &= (\rho - 1)^{-1}\pi p + \sum_{k=1}^{\infty} \pi p^k \\ &\leq (\rho - 1)^{-1}\rho\pi + \sum_{k=1}^{\infty} \pi p^k \\ &= (\rho - 1)^{-1}\pi + \sum_{k=0}^{\infty} \pi p^k \\ &= \nu. \end{aligned}$$

I.e., ν is *excessive* for p, in accord with [9], Proposition 10-8.

Also, $(\rho - 1)^{-1}\eta N = (\rho - 1)^{-1}(\rho\pi N - \sum_{k=1}^{\infty}\pi p^k) = (\rho - 1)^{-1}((\rho - 1))\pi N + \pi) = (\rho - 1)^{-1}\pi + \pi N = \nu$. This is in accord with [9], Proposition 10-6, since $(\rho - 1)^{-1}\eta$ is the initial measure of Q'; see (5.13).

We conclude this subsection by examining the conditions $\eta H > 0$ and $\pi N < \infty$, of (5.14). From now on, let us assume that p is irreducible. Since $\pi \neq 0$, then there exists $i \in S$ with $\pi(i) > 0$. Fix $j \in S$, and choose $k \geq 1$ with $p^k(i, j) > 0$. Then, by (4.3), $\pi(i)p^k(i, j) = \rho^k\pi(j)q^k(j, i)$, and as such $\pi(j) > 0$. Hence $\pi > 0$.

Next suppose there exists $i \in S$ so that $\pi N(i) = \infty$. Then if $j \in S$ and $k \geq 1$ so that $p^k(i, j) > 0$, we have that

$$\begin{aligned}
\infty &= \pi N(i)p^k(i, j) \\
&= \sum_{s \in S}\sum_{l=0}^{\infty} \pi(s)p^l(s, i)p^k(i, j) \\
&\leq \sum_{s \in S}\sum_{l=0}^{\infty} \pi(s)p^{l+k}(s, j) \\
&= \sum_{l=k}^{\infty} \pi p^l(j) \leq \pi N(j).
\end{aligned}$$

Thus either $\pi N < \infty$ or $\pi N = \infty$. The condition that $\pi N < \infty$ corresponds to a type of transience assumption.

Finally, since $\eta H \geq \eta$, then $\eta > 0 \Rightarrow \eta H > 0$. Also, $\eta > 0$ means that $\pi p < \rho\pi$. Thus, the condition $\pi p \leq \rho\pi$ can be modified to $\pi p < \rho'\pi$ by simply choosing $\rho' > \rho$. Thus, if we are given a finite $\pi > 0$, p irreducible, and $\rho > 0$ so that $\pi N < \infty$ and $\pi p \leq \rho\pi$, then it is possible to construct an extended chain as outlined in this section by changing ρ to ρ' where $\rho' > \max(\rho, 1)$. (Note: Clearly, the condition $\pi N < \infty$ would follow if $\rho < 1$.)

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