The Annals of Probability
1988, Vol. 16, No. 3, 1174–1183

THE CONTACT PROCESS IN HIGH DIMENSIONS

BY ENRIQUE D. ANDJEL

Instituto de Matemática Pura e Aplicada

We improve the complete convergence theorem for the contact process in high dimensions by enlarging the range of the infection parameter in the hypothesis of that theorem.

1. Introduction. The d-dimensional contact process is a continuous time Markov process evolving in $S = \{\text{all subsets of } \mathbb{Z}^d\}$, where $\mathbb{Z}^d$ is the integer lattice of dimension d. This process represents the random evolution of an infection in the following sense: When the process is in state A the elements of A are infected individuals and the elements of $\mathbb{Z}^d \setminus A$ are healthy individuals. Infected individuals become healthy at rate 1 and healthy individuals are infected at a rate proportional to the number of infected neighbors. To describe these transition rates more precisely, we let λ be a positive real number and for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ we define $|x| = \sum_{i=1}^d |x_i|$ (throughout this paper this will be the norm used for integer lattices in all dimensions). Then

$$A \to A \cup \{x\} \quad \text{at rate } \lambda \sum_{y:\, |x-y|=1} 1_A(y) \quad \text{if } x \notin A,$$

$$A \to A \setminus \{x\} \quad \text{at rate } 1 \quad \text{if } x \in A.$$

The parameter λ represents the speed at which the infection propagates; we will call it the infection parameter. The contact process exhibits a critical phenomenon in the following sense: For each dimension d there exists a critical value $\lambda_c^{(d)}$ of λ such that $0 < \lambda_c^{(d)} < \infty$. This critical value satisfies the following properties: If $\lambda < \lambda_c^{(d)}$, then the process is ergodic and from any initial distribution it converges weakly to $\delta_\varnothing$ (the point mass on the empty set), if $\lambda > \lambda_c^{(d)}$, then the process has an invariant measure $\mu_\lambda \neq \delta_\varnothing$ which is stochastically above any other invariant measure. In this last case we have $P(\xi_t^0 \neq \varnothing \;\forall\, t \geq 0) = \alpha_d(\lambda) > 0$. (See [9] for the proofs of these results and related definitions.) Here, and in the sequel, ξ_t denotes the contact process and the superscript is the initial distribution. (When the initial distribution is a point mass at A we will write superscript A rather than δ_A. Furthermore, if A is a singleton $\{x\}$, then our superscript will be x rather than $\{x\}$.) Elementary coupling arguments show that $\lambda_c^{(d)} \geq \lambda_c^{(d+1)}$ and $\alpha_d(\lambda) \leq \alpha_{d+1}(\lambda)$. It follows from these considerations that for $\lambda > \lambda_c^{(1)}$ and any dimension d, the process is not ergodic. However, Durrett and Griffeath [1] have proved that for any initial distribution μ,

$$\xi_t^\mu \Rightarrow P(\tau^\mu < \infty)\delta_\varnothing + P(\tau^\mu = \infty)\mu_\lambda, \tag{1.1}$$

where $\Rightarrow$ denotes weak convergence, τ^μ is the hitting time of $\varnothing$ starting from μ

Received November 1986; revised May 1987.
AMS 1980 *subject classification*. Primary 60K35.
Key words and phrases. Contact process, complete convergence theorem.

and μ_λ is the upper invariant measure mentioned previously. The proof of this result, known as the complete convergence theorem, can be found in [9] (for $d = 1$) and [1] (for $d > 1$). In fact most of what is known about the one-dimensional contact process is contained in [9, Chapter VI]. For higher dimensions the reader is also referred to [1], [3], [5]–[8] and [11]–[13]. A shorter proof of (1.1) (always assuming $\lambda > \lambda_c^{(1)}$) can be found in [11]. In [12] this is generalized by proving that if (1.1) holds in dimension d for some $\lambda > \lambda_c^{(d)}$, then it also holds in dimension $d' \ge d$ for any $\lambda' \ge \lambda$. Since $\lambda_c^{(d)} \to 0$ as $d \to \infty$ (see [9]), it is natural to ask whether (1.1) holds in dimension $d > 1$ if $\lambda > \lambda_c^{(d)}$. This paper gives an improved version of the complete convergence theorem in high dimensions, but still leaves a range of the parameter for which (1.1) is not proved. Namely we prove

THEOREM 1.2. *If $d \ge 3k + 1$ and $\lambda > \lambda_c^{(k)}$, then* (1.1) *holds.*

A key role in the proof of this theorem is played by a lemma due to Griffeath (see [3] for its proof). It states that (1.1) holds if and only if two independent copies ξ_t and $\hat{\xi}_t$ of the contact process satisfy

$$\lim_{t\to\infty} P\left(\xi_t^A \cap \hat{\xi}_t^B = \varnothing, \tau^A > t, \hat{\tau}^B > t\right) = 0 \tag{1.3}$$

for all finite A and B in $S \setminus \{\varnothing\}$.

Irreducibility arguments and the Markov property show that it suffices to prove (1.3) for $A = B = \{0\}$. To prove (1.3) in this case we will need an auxiliary process η_t. It will be a Markov process evolving in the subsets of $\mathbb{Z}^{l_1} \times \Gamma$, where Γ is a connected subset of $\mathbb{Z}^{l_2}$ (Γ is called connected if $\forall\, x, y \in \Gamma$ there exists a finite sequence $x_0, \dots, x_n$ such that $x_0 = x$, $x_n = y$, $x_i \in \Gamma$ for $0 \le i \le n$ and $|x_i - x_{i-1}| = 1$ for $1 \le i \le n$). The η_t process, whose transition rates will be given, is similar to the contact process except for the following rule: If a hyperplane $H_a = \{(x, a) \colon x \in \mathbb{Z}^{l_1}\}$ is such that $\eta_{t^-} \cap H_a \ne \varnothing$, then at time t an element of H_a cannot be infected by elements off H_a. More formally, if $y \in \Gamma$ and $\eta \subset \mathbb{Z}^{l_1} \times \Gamma$, let

$$(\pi\eta)(y) = \begin{cases} 1, & \text{if } \eta \cap H_y \ne \varnothing, \\ 0, & \text{if } \eta \cap H_y = \varnothing. \end{cases}$$

With this notation, the transition rates of η_t are ($x \in Z^{l_1}$, $y \in \Gamma$)

$$A \to A \cup \{(x, y)\} \quad \text{at rate} \begin{cases} \lambda \displaystyle\sum_{\substack{z \in H_y \\ |z-(x,y)|=1}} 1_A(z) & \text{if } (\pi\eta)(y) = 1, \\ \lambda \displaystyle\sum_{\substack{z \in \mathbb{Z}^{l_1}\times\Gamma \\ |z-(x,y)|=1}} 1_A(z) & \text{if } (\pi\eta)(y) = 0, \end{cases}$$

if $(x, y) \notin A$ and

$$A \to A \setminus \{(x, y)\} \quad \text{at rate } 1 \quad \text{if } (x, y) \in A.$$

Finally, if one does not allow infected elements to recover, a third Markov process ζ_t is obtained. For ζ_t the rates are

$$A \to A \cup \{x\} \quad \text{at rate } \lambda \sum_{y:\,|y-x|=1} 1_A(y) \quad \text{if } x \notin A.$$

The ζ_t process was first studied by Richardson [10]. Of the results in [10] we will only need the following: There exists a positive constant L (it depends on λ) such that for all finite $A \in S$, we have

$$\lim_{t\to\infty} P\big(\zeta_t^A \subset B_{Lt}\big) = 1, \tag{1.4}$$

where, as in the rest of the paper, B_r denotes $\{x: |x| \le r\}$.

Standard techniques allow us to construct, for a fixed dimension $l_1 + l_2$, the three processes on the same probability space in such a way that

$$P\big(\eta_t^A \subset \xi_t^A \subset \zeta_t^A\big) = 1 \tag{1.5}$$

for all $t \ge 0$ and all $A \subset \mathbb{Z}^{l_1} \times \Gamma$. Given $A \subset B$, these techniques also allow us to construct two versions of each of these processes on the same probability space, in such a way that

$$P\big(\eta_t^A \subset \eta_t^B\big) = P\big(\xi_t^A \subset \xi_t^B\big) = P\big(\zeta_t^A \subset \zeta_t^B\big) = 1$$

for all $t \ge 0$.

These techniques can be either graphical or via generators. However graphical methods will make it easier to understand the beginning of Section 2. For this reason we refer the reader to either [4] or [7]. Note that in this way we obtain right continuous versions of these processes.

In the next section we will prove some properties of the η_t process which will be used in Section 3 to prove Theorem 1.2. In both sections $|A|$ will denote the cardinality of A.

2. The process η_t. Suppose Γ is a connected subset of $\mathbb{Z}^{l_2}$, $\lambda > \lambda_c^{(k)}$ and $l_1 \ge k$. Let

$$\alpha = P\big(\xi_t^0 \ne \varnothing \ \forall\, t \ge 0\big),$$

where ξ_t is the contact process on $\mathbb{Z}^{l_1}$ with infection parameter λ. Since $\lambda > \lambda_c^{(k)} \ge \lambda_c^{(l_1)}$, α must be strictly positive. Let the η_t process start from $\{(0, x)\}$. Then, for each $y \in \Gamma$, define a stopping time τ_y and a random variable Z_y:

$$\tau_y = \inf\big\{t \ge 0: \big(\pi\eta_t^{(0,x)}\big)(y) = 1\big\}$$

(by convention $\inf \varnothing = +\infty$),

$$Z_y = \begin{cases} 1, & \text{if } \big(\pi\eta_t^{(0,x)}\big)(y) = 1 \text{ for all } t > \tau_y, \\ 0, & \text{otherwise.} \end{cases}$$

Since Γ is connected, $P(\tau_y < \infty \,|\, (\pi\eta_t^{(0,x)})(x) = 1 \ \forall\, t \ge 0) = 1$ for all $y \in \Gamma$. It follows from this and from the graphical construction of η_t that after conditioning on $\{(\pi\eta_t^{(0,x)})(x) = 1 \ \forall\, t \ge 0\}$, for each $y \in \Gamma$ the random variables τ_y and Z_y are independent and the random variables Z_y, $y \in \Gamma - \{x\}$, are i.i.d. with common distribution given by

$$P(Z_y = 1) = 1 - P(Z_y = 0) = \alpha.$$

Given $z \in \mathbb{Z}^{l_1}$, $x, y \in \Gamma$ and $t \geq 0$, we define a random variable

$$Z(t, y) = \begin{cases} 1, & \text{if } (\pi\eta_s^{(0,x)})(y) = 1 \,\forall\, s \geq t, \\ 0, & \text{otherwise,} \end{cases}$$

a random subset of Γ,

$$F_t(z, x) = \{w \in \Gamma: (\pi\eta_s^{(z,x)})(w) = 1 \text{ for some } s \leq t\},$$

and a distance in Γ,

$$d_\Gamma(x, y) = \inf\{n: \exists\, x_0 = x, x_1, \dots, x_n = y \text{ such that } x_i \in \Gamma \text{ for } 0 \leq i \leq n \text{ and } |x_i - x_{i-1}| = 1 \text{ for } 1 \leq i \leq n\}.$$

Lemma 2.1 gives an estimate for the growth of the "projection on Γ" of the η_t process.

LEMMA 2.1. *Let $N(t)$ be a Poisson process of parameter $\lambda\alpha$. Then for all $x, y \in \Gamma$ and all $t \geq 0$,*

$$P(y \in F_t(0, x) | (\pi\eta_s^{(0,x)})(x) = 1 \,\forall\, s \geq 0) \geq 1 - P(N(t) < d_\Gamma(x, y)).$$

PROOF. Let $x_0, x_1, \dots, x_n$ be as in the definition of $d_\Gamma(x, y)$. Then define a process $M(s)$ by

$$M(s) = \begin{cases} 0, & \text{if } \tau_{x_n} \leq s, \\ n - \sup\{0 \leq i \leq n: Z(s, x_i) = 1\}, & \text{otherwise,} \end{cases}$$

where, by convention, $\sup \varnothing = -\infty$. Of course, $M(s)$ is not Markovian but it does not increase and it decreases by jumps greater than or equal to 1. Suppose $1 \leq M(s) = k < \infty$; then $Z(s, x_{n-k}) = 1$ and $\tau_{x_n} > s$. Therefore, at rate greater than or equal to λ, an entirely healthy hyperplane among the H_{x_i}'s for $n - k < i \leq n$ gets an infected point. Having in mind the graphical construction of η_t, one can verify the property: Given that an entirely healthy hyperplane H_{x_i} gets a point infected at time t, then with probability α and independently of both η_s for $s \leq t$ and the random variables $Z(s, y)$ for $s \leq t$ and $y \in \Gamma \setminus \{x_i\}$, the hyperplane H_{x_i} will have at least one infected point at all times later than t. This shows that if $N(t)$ is a Poisson process of parameter $\lambda\alpha$, then

$$\begin{aligned} P(M(t) = 0 | (\pi\eta_s^{(0,x)})(x) = 1 \,\forall\, s \geq 0) &\geq P(N(t) \geq n) \\ &= 1 - P(N(t) < d_\Gamma(x, y)) \end{aligned}$$

and the lemma is proved. □

In the remaining part of this section we will use the notation: If $x \in \mathbb{Z}^{l_2}$ and $r \in [0, \infty)$, then

$$x + B_r = \{y \in \mathbb{Z}^{l_2}: |x - y| \leq r\}.$$

COROLLARY 2.2. *Let $\Gamma = \mathbb{Z}^{l_2}$ and suppose $r < \lambda\alpha$. Then for all $(y, z) \in \mathbb{Z}^{l_1} \times \mathbb{Z}^{l_2}$,*

$$\lim_t P\big(F_t(y, z) \supset z + B_{rt} | \big(\pi\eta_s^{(y,z)}\big)(z) = 1 \,\forall\, s \geq 0\big) = 1.$$

PROOF. By Lemma 2.1 and the translation invariance of the η_t process, the left-hand side is bounded below by

$$1 - \lim_t \sum_{y \in B_{rt}} P(N(t) < |y|) \geq 1 - \lim_t (2rt + 1)^{l_2} P(N(t) < rt) = 1. \qquad \square$$

In Proposition 2.3 and Corollary 2.4 η_t and $\hat{\eta}_t$ are independent copies of the η process on $\mathbb{Z}^{l_1} \times \mathbb{Z}^{l_2}$.

PROPOSITION 2.3. *Suppose $l_1 \geq k$, $l_2 \in \mathbb{N}$ and $\lambda > \lambda_c^{(k)}$. For $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{Z}^{l_1} \times \mathbb{Z}^{l_2}$, let*

$$A_t^{u,v} = \big\{z \in \mathbb{Z}^{l_2} \colon (\pi\eta_t^u)(z) = 1 \text{ and } (\pi\hat{\eta}_t^v)(z) = 1\big\}.$$

Then there exist constants $K > 0$ and $b > 0$ such that

$$\lim_t \left[\inf_{\substack{u,v \\ |u-v| < bt}} P\big(|A_t^{u,v}| > Kt^{l_2} | (\pi\eta_s^u)(u_2) = (\pi\hat{\eta}_s^v)(v_2) = 1 \,\forall\, s \geq 0\big) \right] = 1.$$

PROOF. First note that if $0 < b < r$, then there exists a constant $K' > 0$ such that

$$\inf_{\substack{u,v \in \mathbb{Z}^{l_1} \times \mathbb{Z}^{l_2} \\ |u-v| < bt}} |\{u_2 + B_{rt}\} \cap \{v_2 + B_{rt}\}| > K't^{l_2}$$

for t large enough. Therefore, the proposition follows from Corollary 2.2, the independence of the processes η_t and $\hat{\eta}_t$, the weak law of large numbers and our remark concerning the joint distribution of the random variables Z_y (see again the beginning of this section). $\square$

COROLLARY 2.4. *Let b be the constant in the conclusion of Proposition 2.3. Then*

$$\liminf_t \left[\inf_{\substack{u,v \\ |u-v| < bt}} P\big(\exists\, y \in \mathbb{Z}^{l_2} \colon (\pi\eta_t^u)(y) = (\pi\hat{\eta}_t^v)(y) = 1\big) \right] \geq \alpha^2.$$

PROOF. Condition the probability on the left-hand side of the inequality to be proved on $\{(\pi\eta_s^u)(u_2) = (\pi\hat{\eta}_s^v)(v_2) = 1 \,\forall\, s \geq 0\}$, note that this event has probability α^2 and then apply Proposition 2.3. $\square$

3. The contact process. We start this section with a lemma which can be proved easily from the contents of [6, Section 9]. Details of this are left to the reader.

LEMMA 3.1. *If $\lambda > \lambda_c^{(d)}$, then the d-dimensional contact process satisfies*

(a)
$$\lim_n \inf_{A:|A|=n} P(\tau^A = \infty) = 1$$

and

(b)
$$P\big(|\xi_t^0| \to \infty | \tau = \infty\big) = 1.$$

In the sequel for a fixed $t_0 \geq 0$, ${}_{t_0}\xi_t$ will represent a Markov process evolving as ξ_t when $t \in [0, t_0]$ and as η_t when $t \in (t_0, \infty)$. Again standard techniques allow us to construct the processes η_t, ${}_{t_0}\xi_t$ and ξ_t on the same probability space and in such a way that

$$P\big(\eta_t^A \subset {}_{t_0}\xi_t^A \subset \xi_t^A\big) = 1 \quad \forall\, t \geq 0 \,\forall A.$$

It will be assumed that this construction has been used whenever a statement involves two of these processes.

LEMMA 3.2. *Suppose $l_1 \geq k$, $\lambda > \lambda_c^{(k)}$ and $\varepsilon > 0$. Consider the ${}_{t_0}\xi$ processes on $\mathbb{Z}^{l_1} \times \mathbb{Z}^{l_2}$ and let τ be the hitting time of $\varnothing$ for the contact process in dimension $l_1 + l_2$ starting from $\{0\}$. Then there exists a $t_0 \geq 0$ such that*

$$P\Big(\exists\, x \in \mathbb{Z}^{l_2} \text{ such that } \big(\pi_{t_0}\xi_t^0\big)(x) = 1 \,\forall\, t \geq t_0 | \tau = \infty\Big) \geq 1 - \varepsilon.$$

PROOF. By part (b) of Lemma 3.1 $|\xi_t| \to \infty$ a.s. on $\{\tau = \infty\}$. Hence,

$$\sup\Big\{|\{x: (\pi\xi_t)(x) = 1\}|, \sup_x \sum_{y \in \mathbb{Z}^{l_1}} \xi_t(y, x)\Big\} \to \infty \quad \text{a.s. on } \{\tau = \infty\}.$$

The lemma now follows from part (a) of Lemma 3.1. □

In the following proposition ξ_t and $\hat{\xi}_t$ are two independent copies of the $2k$-dimensional contact process, both in the same probability space. The η processes appearing in the proof are the ones evolving on subsets of $\mathbb{Z}^k \times \Gamma$, for various choices of subsets Γ of $\mathbb{Z}^k$. As before η and $\hat{\eta}$ are two independent copies of the η process.

PROPOSITION 3.3. *Suppose $\lambda > \lambda_c^{(k)}$ and $a > 0$. Then there exist constants $M > 0$, $d = d(a) > 0$ and t_0 such that*

$$\inf_{\substack{x,y \in \mathbb{Z}^{2k} \\ |x-y| \leq at}} P\big(\xi_{dt}^x \cap \hat{\xi}_{dt}^y \neq \varnothing\big) \geq \frac{M}{t^k}$$

for all $t \geq t_0$.

PROOF. The statement to be proved is equivalent to: For some $d > 0$,

$$\liminf_t \left[t^k \inf_{\substack{x,y \in \mathbb{Z}^{2k} \\ |x-y| \leq at}} P\big(\xi_{dt}^x \cap \hat{\xi}_{dt}^y \neq \varnothing\big)\right] > 0. \tag{3.4}$$

By Corollary 2.4, (1.4), (1.5) and the Markov property, we may assume (changing the value of d) that k of the coordinates of x are equal to the corresponding coordinates of y. Due to the invariance of the contact process under translations and under permutation of coordinates, we may also assume that these equal coordinates are the first k and that they are equal to 0. Hence the proposition will be proved if we show that $\forall a > 0$ there is a $d > 0$ (not necessarily the same as in the statement of the proposition) such that

$$\liminf_t \left[\inf_{\substack{x, y \in \mathbb{Z}^k \\ |x-y| \leq at}} t^k P\left(\xi_{dt}^{(0,x)} \cap \hat{\xi}_{dt}^{(0,y)} \neq \varnothing \right) \right] > 0.$$

Again using the Markov property, we can see that this is implied by

$$\liminf_t \left[\inf_{\substack{x, y \in \mathbb{Z}^k \\ |x-y| \leq at \\ |x-y| \in 2\mathbb{Z}}} t^k P\left(\xi_{dt-1}^{(0,x)} \cap \hat{\xi}_{dt-1}^{(0,y)} \neq \varnothing \right) \right] > 0,$$

where $2\mathbb{Z}$ is the set of even integers.

By (1.5) this in turn is implied by

$$\liminf_t \left[\inf_{\Gamma} t^k P\left(\eta_{dt-1}^{(0,x_0)} \cap \hat{\eta}_{dt-1}^{(0,x_{2n})} \neq \varnothing \right) \right] > 0, \tag{3.5}$$

where we have adopted the following conventions which will hold throughout the proof of this proposition: Once t is fixed, Γ runs over all subsets of Z^k of the form $\{x_0, \dots, x_{2n}\}$ with $n \leq at/2$ and $|x_i - x_j| = |i - j|$ for all $i, j \in \{0, \dots, 2n\}$, and once Γ is fixed the η_t and $\hat{\eta}_t$ processes are the ones evolving on subsets of $\mathbb{Z}^k \times \Gamma$. Now take $d > a/2\lambda\alpha$ and write

$$\begin{aligned} &P\left(\left(\pi \eta_s^{(0,x_0)} \right)(x_n) = 1 \text{ for some } s \leq dt - 1 \right) \\ &\quad \geq P\left[\left(\pi \eta_s^{(0,x_0)} \right)(x_n) = 1 \text{ for some } s \leq dt - 1 \middle| \left(\pi \eta_s^{(0,x_0)} \right)(x_0) = 1 \, \forall s \geq 0 \right] \alpha \\ &\quad \geq \left[1 - P(N(dt-1) < n) \right] \alpha \geq \left[1 - P\left(N(dt-1) < \frac{at}{2} \right) \right] \alpha, \end{aligned}$$

where the second inequality comes from Lemma 2.1 and $N(t)$ is as in Lemma 2.1. From the previous inequalities we conclude that there is a t_1 which depends on a but not on Γ or n such that

$$P\left(\left(\pi \eta_s^{(0,x_0)} \right)(x_n) = 1 \quad \text{for some } s \leq dt - 1 \right) > \frac{\alpha}{2}$$

for all $t \geq t_1$. From the independence of τ_{x_n} and Z_{x_n} (see the beginning of Section 2), it now follows that for $t \geq t_1$,

$$P\left(\left(\pi \eta_{dt-1}^{(0,x_0)} \right)(x_n) = 1 \right) > \frac{\alpha^2}{2}.$$

The same argument applies to $(\pi\hat{\eta}_{dt-1}^{(0,x_{2n})})(x_n)$. Hence,

$$P\Big((\pi\eta_{dt-1}^{(0,x_0)})(x_n) = (\pi\hat{\eta}_{dt-1}^{(0,x_{2n})})(x_n) = 1\Big) > \frac{\alpha^4}{4}$$

for all $t \geq t_1$ and all n and Γ as specified previously. This implies that the left-hand side of (3.5) is bounded below by

$$\liminf_t \Big[\inf_\Gamma t^k P\big(\eta_{dt-1}^{(0,x_0)} \cap \hat{\eta}_{dt-1}^{(0,x_{2n})} \neq \varnothing \,|\, C_{t,r}\big)\Big]\frac{\alpha^4}{4}, \tag{3.6}$$

where $C_{t,r}$ is the event $\{(\pi\eta_{dt-1}^{(0,x_0)})(x_n) = (\pi\hat{\eta}_{dt-1}^{(0,x_{2n})})(x_n) = 1\}$. For $y \in \mathbb{Z}^{2k}$ let $y + B_r = \{z \in \mathbb{Z}^{2k} : |z - y| \leq r\}$. It follows from (1.4), (1.5) and $n \leq at/2$ that there exists R such that

$$\lim_t \Big[\inf_\Gamma P\big(\eta_{dt-1}^{(0,x_0)} \subset (0,x_n) + B_{Rt} \text{ and } \hat{\eta}_{dt-1}^{(0,x_{2n})} \subset (0,x_n) + B_{Rt}\big)\Big] = 1. \tag{3.7}$$

Fix t and let $y_1, \ldots, y_{U(t)}$ $[U(t) = (2[Rt] + 1)^k$, [ ] denoting integer part] be an ordering of the elements of $\{y \in Z^k : |y| \leq Rt\}$. Then define

$$Y_i = \eta_{dt-1}^{(0,x_0)}(y_i, x_n), \qquad 1 \leq i \leq U(t),$$

and

$$\hat{Y}_i = \hat{\eta}_{dt-1}^{(0,x_{2n})}(y_i, x_n), \qquad 1 \leq i \leq U(t).$$

Now note that Γ is taken in such a way that the distance from x_n to x_0 is n and there is only one connected path in Γ joining x_0 and x_n. The same comment applies to x_{2n} and x_n. It follows from this and the properties of the η process that the random variables $Y_1, \ldots, Y_{U(t)}$ have the same joint distribution as $\hat{Y}_1, \ldots, \hat{Y}_{U(t)}$. The same happens to the distribution of these random variables after conditioning them on $C_{t,r}$. Note also that after this conditioning the Y_i's remain independent of the $\hat{Y}_i$'s.

Define $i_0 = \inf\{i : Y_i = 1\}$ and $\hat{i}_0 = \inf\{i : \hat{Y}_i = 1\}$ (by convention $\inf \varnothing = \Delta$). It now follows from (3.7) that for t large enough,

$$\sum_{i=1}^{U(t)} P(i_0 = i | C_{t,r}) \geq \tfrac{1}{2}. \tag{3.8}$$

Therefore,

$$P\big(i_0 = \hat{i}_0 \neq \Delta | C_{t,r}\big) = \sum_{i=1}^{U(t)} P^2(i_0 = i | C_{t,r}) \geq \frac{1}{4U(t)}, \tag{3.9}$$

where the last inequality follows from (3.8) and the Cauchy–Schwarz inequality.

Since $\{\eta_{dt-1}^{(0,x_0)} \cap \hat{\eta}_{dt-1}^{(0,x_{2n})} \neq \varnothing\} \supset \{i_0 = \hat{i}_0 \neq \Delta\}$, it follows from (3.9) that (3.6) is bounded below by $\lim_t (t^k/4U(t))\alpha^4/4 > 0$. Since (3.6) is itself a lower bound of the left-hand side of (3.5), the proposition is proved. □

Proof of Theorem 1.2. Write $\mathbb{Z}^d = \mathbb{Z}^{2k} \times \mathbb{Z}^l$, where $l \geq k + 1$. By Lemma 3.2 and the Markov property, to prove (1.3) it suffices to show that for all

$u = (u_1, u_2)$, $v = (v_1, v_2)$,

$$\lim_t P\big(\xi_t^u \cap \hat{\xi}_t^v = \varnothing \,|\, (\pi\eta_s^u)(u_2) = (\pi\hat{\eta}_s^v)(v_2) = 1\,\forall\, s \geq 0\big) = 0.$$

By Proposition 2.3 this is implied by

$$\lim_t P\big(\xi_{(d+1)t}^u \cap \hat{\xi}_{(d+1)t}^v = \varnothing \,|\, |A_t^{u,v}| > Kt^l\big) = 0,$$

where $d = d(L)$ comes from Proposition 3.3 and L from (1.4) ($A_t^{u,v}$ and K are as in Proposition 2.3).

By (1.4) and (1.5) this is equivalent to

$$\lim_t P\big(\xi_{(d+1)t}^u \cap \hat{\xi}_{(d+1)t}^v = \varnothing \,|\, |A_t^{u,v}| > Kt^l \text{ and } \xi_t^u \cup \hat{\xi}_t^v \subset B_{Lt}\big) = 0.$$

Since ${}_t\xi_{(d+1)t} \subset \xi_{(d+1)t}$, it will suffice to show that

$$\lim_t P\big({}_t\xi_{(d+1)t}^u \cap {}_t\hat{\xi}_{(d+1)t}^v = \varnothing \,|\, |A_t^{u,v}| > Kt^l \text{ and } \xi_t^u \cup \hat{\xi}_t^v \subset B_{Lt}\big) = 0. \tag{3.10}$$

Given $|A_t^{u,v}| > Kt^l$, we have more than Kt^l elements x in $\mathbb{Z}^l$ such that $(\pi\xi_t^u)(x) = (\pi\hat{\xi}_t^v)(x) = 1$. Hence by Proposition 3.3 the left-hand side of (3.10) is bounded above $\lim_t(1 - M/t^k)^{Kt^l}$. Since $l \geq k + 1$, this is 0 and the theorem is proved. □

Remark. It is possible to use a simpler version of Proposition 2.3 (the inf in u and v can be deleted) and Griffeath's necessary and sufficient condition (1.3) to show that if (1.1) holds in dimension d for some $\lambda > \lambda_c^{(d)}$, then it holds in any dimension greater than d for that same λ. We do not write this proof because it will be similar to the proofs given in [11] and [12] although it avoids the use of linear structures with right angles. In fact this paper benefitted from talks with R. H. Schonmann concerning [11].

Note. Durrett and Schonmann [2] proved the following result for a discrete time version of the contact process in two dimensions:

Let $\lambda_c^{(2),k}$ be the critical value for the system on the strip $\mathbb{Z} \times \{-k, \ldots, k\}$ and observe that $\lambda_c^{(2),k}$ decreases as $k \to \infty$ to a limit λ^*. Then if $\lambda > \lambda^*$ the complete convergence theorem holds for the process on $\mathbb{Z}^2$.

Acknowledgment. I wish to thank the referee for his valuable comments and for pointing out a mistake in the original proof of Proposition 2.3.

REFERENCES

[1] Durrett, R. and Griffeath, D. (1982). Contact processes in several dimensions. *Z. Wahrsch. verw. Gebiete* **59** 535–552.

[2] Durrett, R. and Schonmann, R. H. (1987). Stochastic growth models. In *Proc. of the Workshop on Percolation Theory and Ergodic Theory of Infinite Particle Systems* 85–119. Springer, Berlin.

[3] Griffeath, R. (1978). Limit theorems for nonergodic set-valued Markov processes. *Ann. Probab.* **6** 379–387.

[4] GRIFFEATH, R. (1979). *Additive and Cancellative Interacting Particle Systems. Lecture Notes in Math.* **724**. Springer, Berlin.

[5] HARRIS, T. E. (1974). Contact interactions on a lattice. *Ann. Probab.* **2** 969–988.

[6] HARRIS, T. E. (1976). On a class of set-valued Markov processes. *Ann. Probab.* **4** 175–194.

[7] HARRIS, T. E. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378.

[8] HOLLEY, R. and LIGGETT, T. M. (1978). The survival of contact processes. *Ann. Probab.* **6** 198–206.

[9] LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

[10] RICHARDSON, D. (1973). Random growth in a tessellation. *Proc. Cambridge Philos. Soc.* **74** 515–528.

[11] SCHONMANN, R. H. (1987). A new proof of the complete convergence theorem for contact processes in several dimensions with large infection parameter. *Ann. Probab.* **15** 382–387.

[12] SCHONMANN, R. H. (1987). A new look at contact processes in several dimensions. In *Proc. of the Workshop on Percolation Theory and Ergodic Theory of Infinite Particle Systems* 245–250. Springer, Berlin.

[13] SCHONMANN, R. H. and VARES, M. E. (1986). The survival of the large dimensional basic contact process. *Probab. Theory Related Fields* **72** 387–393.

INSTITUTO DE MATEMÁTICA PURA E APLICADA
ESTRADA DONA CASTORINA 110
22460 RIO DE JANEIRO, RJ
BRAZIL