The Annals of Probability
1988, Vol. 16, No. 4, 1840–1853

CONDITIONING A LIFTED STOCHASTIC SYSTEM IN A PRODUCT SPACE

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We study the solution ξ_t to an elliptic stochastic equation and the solution (ξ_t, η_t) to a lift of this equation, under a condition on the behaviour of ξ_t. We obtain a "conditional" equation which gives the conditional behaviour, and we use it to deduce a conditioned version of the Stroock–Varadhan support theorem.

1. Introduction. In this paper we study the stochastic process $(\xi_t(\omega)x, \eta_t(\omega)y)$ in the product space $U \times \mathbb{R}^l$ (U an open domain in $\mathbb{R}^m$), which is determined by the following pair of (Itô) stochastic differential equations:

$$(1) \quad d(\xi_t(\omega)x) = X_N(\xi_t(\omega)x)\, dB_t(\omega) + X_D(\xi_t(\omega)x)\, dt,$$

$$(2) \quad d(\eta_t(\omega)y) = Y_N(\xi_t(\omega)x, \eta_t(\omega)y)\, dB_t(\omega) + Y_D(\xi_t(\omega)x, \eta_t(\omega)y)\, dt,$$

$$(\xi_0(\omega)x, \eta_0(\omega)y) = (x, y).$$

Here B_t is a Brownian motion on $\mathbb{R}^n$, X_D is a smooth map $U \to \mathbb{R}^m$ and Y_D is a smooth map $U \times \mathbb{R}^l \to \mathbb{R}^l$. Also X_N and Y_N are smooth maps $U \times \mathbb{R}^n \to \mathbb{R}^m$ and $U \times \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^l$, such that for each $(x, y) \in U \times \mathbb{R}^l$, the maps $X_N(x)$: $\mathbb{R}^n \to \mathbb{R}^m$ and $Y_N(x, y)$: $\mathbb{R}^n \to \mathbb{R}^l$ are linear. Also ω is an element of the probability space $(\Omega, \mathscr{F}, P)$ which corresponds to B_t.

One would perhaps more usually write $X_N(x)\, dB_t$ and $Y_N(x, y)\, dB_t$ as $\sum_{i=1}^n X_N^i(x)\, dB_t^i$ and $\sum_{i=1}^n Y_N^i(x, y)\, dB_t^i$, where $X_N^i(x) = X_N(x, e_i)$ and $Y_N^i(x, y) = Y_N(x, y, e_i)$, $\{e_1, \dots, e_n\}$ being the standard basis in $\mathbb{R}^n$ and $B_t^1, \dots, B_t^n$ being independent Brownian motions in $\mathbb{R}$. Thus X_n and Y_n are the collections of "noise" vector fields for (1) and (2), and X_D and Y_D are the "drift" vector fields. We will assume that the process $(\xi_t(x), \eta_t(y))$ is nonexplosive for all $(x, y) \in U \times \mathbb{R}^l$ and that (1) is elliptic, i.e., for each $x \in U$, the linear map $X_N(x)$: $\mathbb{R}^n \to \mathbb{R}^m$ is surjective (this implies $n \geq m$.)

Given the equation (1), the equation (2) might arise as a "lift" of (1); for example, we might take $Y_N(x, y) = DX_N(x)y$, $Y_D(x, y) = DX_D(x)y$, which will give $\eta_t(\omega)y = [D\xi_t(\omega)x]y$ (see [4] and the references therein). This is our motivation for studying (1) and (2). Notice that (1) is autonomous, i.e., it determines its solution by itself, but (2) is not.

Our aim is to study (for given $T > 0$, $x_0, x_T \in U$ and $y_0 \in \mathbb{R}^l$) the process $(\xi_t(\omega)x_0, \eta_t(\omega)y_0)$ when we impose the condition that $\xi_T(x_0) = x_T$. The behaviour of $\xi_T(\omega)x_0$ under the condition is well known and is studied by

Received November 1986; revised July 1987.
AMS 1980 *subject classification.* Primary 60H10.
Key words and phrases. Stochastic equation, conditional probability, lift, product space, Stroock–Varadhan support theorem, Doob h-transform.

considering the solution $\xi_t^{x_T}(\omega)x_0$ to the equation

$$(3)\qquad \begin{aligned} d\xi_t^{x_T}(\omega)x_0 = {} & X_N\big(\xi_t^{x_T}(\omega)x_0\big)\,dB_t(\omega) \\ & + X_D\big(\xi^{x_T}(\omega)x_0\big)\,dt + A_t\big(\xi_t^{x_T}(\omega)x_0\big)\,dt, \end{aligned}$$

where $A_t(x) = \nabla_x \log p_{T-t}(x, x_T)$ and $p_t(x, y)$ is the transition density associated with $\xi_t(x)$. In Section 2 (Theorem 2.1) we show that the conditioned behaviour of $(\xi_t(x_0), \eta_t(y_0))$ is given by the solution $(\xi^{x_T}(\omega)x_0, \eta^{x_T}(\omega)y_0)$ to the pair of equations (3) and the following:

$$(4)\qquad \begin{aligned} d\eta_t^{x_T}(\omega)y_0 = {} & Y_N\big(\xi_t^{x_t}(\omega)x_0, \eta_t^{x_T}(\omega)y_0\big)\,dB_t(\omega) \\ & + Y_D\big(\xi_t^{x_T}(\omega)x_0, \eta_t^{x_T}(\omega)y_0\big)\,dt \\ & + \tilde{A}_t\big(\xi_t^{x_T}(\omega)x_0, \eta_t^{x_T}(\omega)y_0\big)\,dt, \end{aligned}$$

where $\tilde{A}_t(x, y) = Y_N(x, y) \circ X_N^{-1}(x) \circ A_t(x)$. Here by $X_N^{-1}(x)$ (for $x \in U$) we mean the inverse of the linear map $X_N(x)$, chosen with range equal to the orthogonal complement of the kernel of $X_N(x)$. Using the fact that this complement is the image of the adjoint $X_N(x)^*$ and that this adjoint is injective [since $X_N(x)$ is surjective], we can easily deduce that our choice of $X_N^{-1}(x)$ is smooth. Equations (3) and (4) become singular as $t \to T$, and Theorem 2.1 only tells us about the conditional behaviour over a time interval $[0, T - \tau]$ (any $\tau > 0$). In Section 3 we show that the process $(\xi_t^{x_T}(x_0), \eta_t^{x_T}(y_0))$ is nonexplosive in spite of this singularity in (3) and (4), and we deduce (Theorem 3.3) that $(\xi_t^{x_T}(x_0), \eta_t^{x_T}(y_0))$ gives the conditional behaviour over the whole of $[0, T]$. Then we use (3) and (4) to show that given a local boundedness condition on Y_N, Y_D, the measure given by the conditional process is weakly continuous in x_T.

In Section 4 we use the conditional equation to deduce (Theorem 4) a conditional version of the Stroock–Varadhan support theorem (see [8] and [9]). This characterises the support of the conditional measure in a control-theoretic way, as in the usual support theorem, but with a corresponding condition on the controls.

The work of this paper is easily adapted to dealing with the solution to a stochastic dynamical system on a smooth finite-dimensional manifold and a lift to a smooth finite-dimensional fibre bundle. To do this, one can either embed the manifold in a Euclidean space or work in local charts (see [4]). [Note that for this formulation, the equations corresponding to (1) and (2) must be Stratonovitch.] In [3] we give an application of the conditioned support theorem in a manifold formulation.

2. The equation for the conditioned process. First we establish some notation. Throughout, we will restrict attention to the time interval $[0, T]$, and we will take the probability space $(\Omega, \mathscr{F}, P)$ to consist of (Borel) sets of Brownian paths in $\mathbb{R}^n$ which are measurable over this time interval. For $\tau \in [0, T]$ we will denote by $F_{T-\tau}$ the σ-algebra up to time $T - \tau$ and by $P_{T-\tau}$ the measure P restricted to $F_{T-\tau}$. (Thus $P_T \equiv P$.) We will abbreviate the solution $(\xi_t(\omega)x_0, \eta_t(\omega)y_0)$ of (1) and (2) to $G\xi_t(\omega)v_0$ [with $v_0 = (x_0, y_0)$], and the solution $(\xi_t^{x_T}(\omega)x_0, \eta_t^{x_T}(\omega)y_0)$ of (3) and (4) to $G\xi_t^{x_T}(\omega)v_0$. This notation emphasises that $G\xi_t^{x}(\omega)v_0$ is a lift of $\xi_t(x_0)$. Denote by $[\xi(P)x_0]$ and $[G\xi(P)v_0]$ the measures induced from $(\Omega, \mathscr{F}, P)$ on the spaces $C([0,T],U)$ and $C([0,T], U\times R^m)$ by the maps $\omega \to \{\text{Path } t \to \xi_t(\omega)x_0\}$ and $\omega \to \{\text{Path } t \to G\xi_t(\omega)v_0\}$ and for $\tau > 0$ take $[\xi^{x_T}(P_{T-\tau})x_0], [G\xi^{x_T}(P_{T-\tau})v_0]$ to denote the corresponding things. [Note that we must work to show that these are defined for $\tau = 0$ because (3) and (4) become singular as $t \to T$.] Finally, denote by $[\xi(P)x_0 | \xi_T(x_0) = x_T]$ and $[G\xi(P)v_0 | \xi_T(x_0) = x_T]$ the measures $[\xi(P)x_0]$ and $[G\xi(P)v_0]$ conditioned on the event $\xi_T(x_0) = x_T$. These conditional probabilities are defined up to a $p_T(x_0, -)$-null set of x_T's, or equivalently [since (1) is nondegenerate] up to a Lebesgue-null set of x_T's in U.

THEOREM 2.1. *Take $\tau \in (0, T]$. Take $x_0 \in U$, $y_0 \in \mathbb{R}^l$ and put $(x_0, y_0) = v_0$. Then up to a (Lebesgue) null set of x_T's in U, we have*

$$\text{(i)}\qquad [\xi(P_{T-\tau})x_0 | \xi_T(x_0) = x_T] = [\xi^{x_T}(P_{T-\tau})x_0],$$

$$\text{(ii)}\qquad [G\xi(P_{T-\tau})v_0 | \xi_T(x_0) = x_T] = [G\xi^{x_T}(P_{T-\tau})v_0].$$

PROOF. (i) This is well known. See [5].

(ii) Replace (1) and (2) by

$$d\xi_t(x_0) = X_N(\xi_t(x_0)) \circ \mathrm{Proj}^{\perp}(\xi_t(x_0))\, dB_t^{\perp} + X_D(\xi_t(x_0))\, dt, \tag{5}$$

$$\begin{aligned} d\eta_t(y_0) = {} & Y_N(\xi_t(x_0), \eta_t(y_0)) \\ & \circ\left[\mathrm{Proj}^{\parallel}(\xi_t(x_0))\, dB_t^{\parallel} + \mathrm{Proj}^{\perp}(\xi_t(x_0))\, dB_t^{\perp}\right] \\ & + Y_D(\xi_t(x_0), \eta_t(y_0))\, dt, \end{aligned} \tag{6}$$

where $\mathrm{Proj}^{\perp}(x)$ and $\mathrm{Proj}^{\parallel}(x)$ (for $x \in U$) denote orthogonal projection in $\mathbb{R}^n$ onto $\ker X_N(x)^{\perp}$ and $\ker X_N(x)$, respectively, and $B_t^{\perp}$, $B_t^{\parallel}$ are independent Brownian motions in $\mathbb{R}^n$. It is clear that the solutions of (5) and (6) have the same distributions as those of (1) and (2). Now (5) enables us to write $dB_t^{\perp}$ [or at least is component in $\ker X_N(\xi_t(x))^{\perp}$] in terms of $d\xi_t(x_0)$ and to eliminate $dB_t^{\perp}$ from (6) to obtain

$$\begin{aligned} d\eta_t(y_0) = {} & Y_N(\xi_t(x_0), \eta_t(y_0)) \\ & \circ\left[\mathrm{Proj}^{\parallel}(\xi_t(x_0))\, dB_t^{\parallel} + X_N^{-1}(\xi_t(x_0))\big(d\xi_t(x_0) - X_D(\xi_t(x_0))\, dt\big)\right] \\ & + Y_D(\xi_t(x_0), \eta_t(y_0))\, dt. \end{aligned} \tag{7}$$

Equation (7) shows how $\eta_t(y_0)$ is driven by $\xi_t(x_0)$ (as a process in $U \subset R^m$ with measure $[\xi(P)x_0]$) and the independent Brownian motion $B_t^{\parallel}$. The solution to (7) is a map ϕ: $C([0, T-\tau], \mathbb{R}^n) \times C([0, T-\tau], \mathbb{R}^m) \to C([0, T-\tau], \mathbb{R}^l)$, defined up to a $P_{T-\tau}^{\parallel} \otimes [\xi(P_{T-\tau}^{\perp})x_0]$-null set and which gives $\phi(P_{T-\tau}^{\parallel} \otimes [\xi(P_{T-\tau}^{\perp})x_0]) = [G\xi(P_{T-\tau})v_0]$. (Here $P^{\parallel}$ and $P^{\perp}$ denote the probabilities associated with $B_t^{\parallel}$ and $B_t^{\perp}$.)

Now if we fix $\gamma \in C([0, T-\tau], \mathbb{R}^m)$, then from the preceding we see that the map $\phi_\gamma \equiv \phi(\cdot, \gamma)$: $C([0, T-\tau], \mathbb{R}^n) \to C([0, T-\tau], \mathbb{R}^l)$ sends $P^{\parallel}_{T-\tau}$ to the conditional measure $[G\xi(P^{\perp}_{T-\tau})v_0 | \xi_1(x_0) = \gamma_t$ for $t \in [0, T-\tau]]$ (This is true for $[\xi(P^{\perp}_{T-\tau})x_0]$ a.e. τ.) Note that by the substitution result of [6], Chapter 3, Proposition 4B, we can recover (6) by substituting for $d\xi_t(x_0)$ in (7), using (5). Now suppose we change the measure $[\xi(P^{\perp}_{T-\tau})x_0]$ to the equivalent measure $[\xi^{x_T}(P^{\perp}_{T-\tau})x_0] \equiv [\xi(P^{\perp}_{T-\tau})x_0 | \xi_T(x_0) = x_T]$ [see part (i) of the result.] Then $[G\xi(P^{\perp}_{T-\tau})v_0 | \xi_t(x_0) = \gamma_t$ for $t \in [0, T-\tau]]$ is still given by (7), but to get $[G\xi(P^{\perp}_{T-\tau})v_0 | \xi_T(x_0) = x_T]$, we must now substitute $d\xi^{x_T}_t(x_0)$ instead of $d\xi_t(x_0)$ in (7), where $d\xi^{x_T}_t(x_0)$ is given by the following, which is equivalent to (3):

$$
\begin{aligned}
d\xi^{x_T}_t(x_0) &= X_N(\xi^{x_T}_t(x_0)) \circ \mathrm{Proj}^{\perp}(\xi^{x_T}_T(x_0))\, dB^{\perp}_t \\
&\quad + X_D(\xi^{x_T}_t(x_0))\, dt + A_t(\xi^{x_T}_t(x_0))\, dt.
\end{aligned}
\tag{8}
$$

Making this substitution yields an equation whose solution has the same distributions as (4), and the result follows. □

COROLLARY 2.2 (Girsanov transformation to get conditioned probability). *Take* $\tau \in (0, T]$. *Take* $x_0, x_T \in U$, $y_0 \in \mathbb{R}^l$ *and put* $(x_0, y_0) = v_0$. *Define the probability* $P^{x_T}_{T-\tau}$ *on the measurable space* $(\Omega, \mathscr{F}_{T-\tau})$ *by*

$$
\begin{aligned}
\frac{dP^{x_T}_{T-\tau}}{dP_{T-\tau}}(\omega) = \exp\Bigg\{ &\int_0^{T-\tau} \langle X_N^{-1}(\xi_t(\omega)x_0) \circ A_t(\xi_t(\omega)x_0), dB_t(\omega)\rangle \\
&- \frac{1}{2}\int_0^{T-\tau} \| X_N^{-1}(\xi_t(\omega)x_0) \circ A_t(\xi_t(\omega)x_0\|^2\, dt \Bigg\},
\end{aligned}
$$

where X_N^{-1} *is chosen as in* (4). *Then*

(i) $$[\xi(P^{x_T}_{T-\tau})x_0] = [\xi^{x_T}(P_{T-\tau})x_0],$$

(ii) $$[G\xi(P^{x_T}_{T-\tau})v_0] = [G\xi^{x_T}(P_{T-\tau})v_0].$$

Note. Part (i) of Corollary 2.2 is well known and any (measurable) choice of X_N^{-1} will do for this. But to get part (ii) we must use the previous choice of X_N^{-1}.

3. The conditioned measure over the time interval $[0, T]$. In this section we will show that under the condition that $|Y_N(x, y)|, |Y_D(x, y)|$ are bounded for $(x, y) \in V \times \mathbb{R}^l$, where V is compact in U, the solution $(\xi^{x_T}_t(x_0), \eta^{x_T}_t(y_0))$ to (3) and (4) can be defined over the time interval $[0, T]$ in spite of the fact that they become singular as $t \to T$, and that they give the conditional probabilities as in Theorem 2.1, but over $[0, T]$. Our technique is first to give an estimate (Lemma 3.2) concerning $|\xi_t(x_0) - x_T|$ given $\xi_T(x_0) = x_T$, which allows us to show that $\xi^{x_T}_t(x_0) \to x_T$ in a nice way, and then that (4) is nonexplosive. Note that our work applies to $(\xi^{x_T}_t(x_0), \eta^{x_T}_t(y_0))$ for every $x_T \in U$. With this definition, we show that the conditional probability is weakly continuous in x_T.

Note that we can define $[\xi(P)x_0|\xi_T(x_0) = x_T]$ for each $x_T \in U$ as giving the probability

$$\frac{1}{p_T(x_0, x_T)} \int_{x_q \in B_q} \cdots \int_{x_1 \in B_1} p_{t_1}(x_0, x_1) p_{t_2 - t_1}(x_1, x_2) \times \cdots \times p_{T - t_q}(x_q, x_T)\, dx_1 \cdots dx_q$$

to the cylinder set

$$\{\gamma \in C([0, T]U) \colon \gamma_{t_i} \in B_i \text{ for } 0 \le t_1 < t_2 < \cdots < t_q \le T\}.$$

Also it follows from [2], Proposition 2.5, page 69, that with this definition we have $[\xi(P)x_0|\xi_T(x_0) = x_T] = [\xi^{x_T}(P)x_0]$, although we will obtain this independently.

We will make much use of the following estimates.

Suppose we have a (nondegenerate) stochastic equation

$$d\zeta_t(x) = Z_N(\zeta_t(x))\, dB_t + Z_D(\zeta_t(x))\, dt, \qquad \zeta_0(x) = x, \tag{9}$$

in an open domain D in $\mathbb{R}^m$, with corresponding transition densities $p_t^Z(x, y)$. Suppose there exists N_1 such that the system is uniformly elliptic in D with bound N_1, i.e.,

$$\sup_{x \in D} \{\|[Z_N(x) Z_N(x)^*]^{-1}\| \le N_1, \|Z_N\|_{C^2} \le N_1, \|Z_D\|_{C^2} \le N_1\}.$$

Then given a closed domain V in D, there exist $c_1, \ldots, c_6 > 0$ such that for all $x, y \in V$, we have

$$c_1 t^{-m/2} \exp(-c_2|x - y|^2/t) \le p_t^Z(x, y) \le c_3 t^{-m/2} \exp(-c_4|x - y|^2/t), \tag{10}$$

$$|\nabla_x p_t^Z(x, y)| \le c_5 t^{-m/2 - 1/2} \exp(-c_6|x - y|^2/t). \tag{11}$$

In fact, we can choose $c_1, \ldots, c_6$ uniformly given D, V and N_1. [For (10) see [1]; for (11) see [7].]

LEMMA 3.1. *Take a strictly increasing continuous function F: $[0, 1] \to R^{\ge 0}$, which is bounded by* 1, *and is such that $F(t) = (t \log\log t^{-1})^{1/2}$ for say $t \le 1/e^e$. Take $N_1 > 0$, $\delta > 0$, $R > 0$. Then there exists $N_2 > 0$ for which we have the following*:

If we have a stochastic equation that is uniformly elliptic in $\mathbb{R}^m$ with bound N_1 and solution (starting from x) denoted by $\zeta_1(x)$, then, given z_0, z_1 in $\mathbb{R}^m$ with $|z_1 - z_0| \le R$, we have

$$P\{|\zeta_t(z_0) - z_1| \le N_2 F(1 - t) \text{ for all } t \in [0, 1] | \zeta_1(z_0) = z_1\} \ge 1 - \delta.$$

PROOF. (1) Note first that we have $\rho > 0$, depending only on N_1, such that

$$P\{|\zeta_t(x) - x| \le 2F(t) \text{ for } t \in [0, \rho]\} \ge (1 - \delta).$$

This is an easy consequence of the law of the iterated logarithm.

(2) From [2], Section 2, we see that the distribution of $\zeta_t(z_0)$ given $\zeta_1(z_0) = z_1$ is the same, with time reversal $t \to 1 - t$, as the distribution of the adjoint

process $\hat{\zeta}_t(z_1)$ given $\hat{\zeta}_1(z_1) = z_0$. If $\zeta_t(x)$ is given by (9), then $\hat{\zeta}_t(x)$ is given by

$$d\hat{\zeta}_t(x) = -Z_N\big(\hat{\zeta}_t(x)\big)\, d\hat{B}_t - Z_D\big(\hat{\zeta}_t(x)\big)\, dt$$

($\hat{B}_t$—Brownian motion, independent of B_t), and the conditioned distribution is given by $\hat{\zeta}_t^{z_0}(z_1)$, where

$$d\hat{\zeta}_t^{z_0}(z_1) = -Z_N\big(\hat{\zeta}_t^{z_0}(z_1)\big)\, d\hat{B}_t - Z_D\big(\hat{\zeta}_t^{z_0}(z_1)\big)\, dt + \hat{A}_t^{z_0}\big(\zeta^{z_0}(z_1)\big)\, dt,$$

where $\hat{p}^z(x, y)$ is the corresponding transition density to $\hat{\zeta}_t(x)$, and $\hat{A}_t^{z_0}(x) = \nabla_x \log \hat{p}_{1-t}^z(x, z_0)$.

Now (10) and (11) yield that there exists $N_3 > 0$ such that for $|z_0 - z_1| \le R$, $|x - z_1| \le 1$, $t \in [0, \frac{1}{2}]$, we have $|\hat{A}_t^{z_0}(x)| \le N_3$. Therefore we can apply part (1) of this proof (with N_1 replaced by $N_1 \vee N_3$) to yield that there is $\rho > 0$ such that for $|z_0 - z_1| \le R$, we have $2F(\rho) \le 1$ and

$$P\{|\zeta_t(z_0) - z_1| \le 2F(1-t) \text{ for } t \in [1-\rho, 1] | \zeta_1(z_0) = z_1\} \ge (1-\delta).$$

(3) Given $N > 0$, denote by $p_t^N(x, y)$ the density of the transition probability given by

$$p_t^N(x, dy) = P\{\zeta_t(x) \in dy \text{ and } |\zeta_s(x) - z_1| \le N \text{ for all } s \in [0, t]\}.$$

[Thus $p_t^N(x, y)$ corresponds to killing the process when it exits from the ball $B_{z_1}(N)$ at z_1 of radius N.] Using (10) and the fact that

$$p_t^N(x, y) = p_t(x, y) - \int_{(s, z) \in [0, t] \times \partial B_{z_1}(N)} p_s(z, y)\, d\nu(s, z)$$

[where the measure ν gives the exit time and position of $\zeta_t(x)$ from $B_{z_1}(N)$], we see that there exists N_4 such that

$$p_{1-\rho}^{N_4}(z_0, y) \ge (1-\delta) p_{1-\rho}(z_0, y) \quad \text{if } |y - z_1| \le 1 \text{ and } |z_0 - z_1| \le R.$$

Let us make the following abbreviations:

$$A_\rho^N\colon \{|\zeta_t(z_0) - z_1| \le N \text{ for } t \in [0, 1-\rho]\},$$
$$B_\rho\colon \{|\zeta_t(z_0) - z_1| \le 2F(1-t) \text{ for } t \in [1-\rho, 1]\},$$
$$D_\rho^y\colon \{\zeta_{1-\rho}(z_0) \in dy\},$$
$$C\colon \{\zeta_1(z_0) = z_1\}.$$

Then we have

$$\begin{aligned} P\big(A_\rho^{N_4} \cap B_\rho | C\big) &= \int_{y \in B_{z_1}(1)} P\big(A_\rho^{N_4} \cap D_\rho^y \cap B_\rho | C\big) \\ &= \int_{y \in B_{z_1}(1)} P\big(A_\rho^{N_4} \cap D_\rho^y | C\big) P\big(B_\rho | A_\rho^{N_4} \cap D_y \cap C\big) \\ &= \int_{y \in B_{z_1}(1)} p_{1-\rho}^{N_4}(z_0, y) P\big(D_\rho^y | C\big) P\big(B_\rho | D_\rho^y \cap C\big) \end{aligned}$$

[Note $P(B_\rho | A_\rho^{N_4} \cap D_\rho^y \cap C) = P(B_\rho | D_\rho^y \cap C)$ because $A_\rho^{N_4}$ and B_ρ are independent, given D_ρ^y.]

$$\begin{aligned}&\geq (1-\rho)\int_{y\in B_{z_1}(1)} p_{1-\rho}(z_0, y) P(D_\rho^y|C) P(B_\rho|D_\rho^y \cap C)\\&= (1-\delta)P(B_\rho|C)\\&\geq (1-\delta)^2,\end{aligned}$$

by part (2).

The result follows [with $(1-\delta)$ replaced by $(1-\delta)^2$] taking $N_2 \geq 2$ and such that $N_2 F(1-\rho) \geq N_4$. □

LEMMA 3.2. *Take $\delta > 0$, $R > 0$, $x_T \in U$. Then there exist $N_2 > 0$, $\tau_1 > 0$ such that for any $\tau \leq \tau_1$ we have the following*:

If $|x_{T-\tau} - x_T| \leq \tau^{1/2}R$, then

$$P\left\{|\xi_{T-\tau,t}(x_{T-\tau}) - x_T| \leq \tau^{1/2} N_2 F\left(\frac{T-t}{\tau}\right)\right.$$
$$\left. \textit{for } t \in [T-\tau, T] | \xi_{T-\tau,T}(x_{T-\tau}) = x_T\right\} \geq 1-\delta.$$

[*Here by $\xi_{T-\tau,t}(x_{T-\tau})$ we mean the solution to* (1) *starting from $x_{T-\tau}$ at time $T-\tau$*.]

PROOF. (1) Take a closed ball in U, centered at x_T, of radius say $r > 0$. Extend the coefficients X_N, X_D of (1) from this ball [denoted $\overline{B}_r(x_T)$] to all of $\mathbb{R}^m$ [and denote them again by X_N, X_D and the associated solutions by $\xi_t(x)$] so that they make up a system which is uniformly elliptic with bound say N_1. Now take any $t \in (0, T \wedge 1]$ and consider the scaling transformation

$$\Theta^\tau : \mathbb{R}^m \times [T-\tau, T] \to \mathbb{R}^m \times [0,1], \qquad (x,t) \to \left(\tau^{-1/2}x, \tau^{-1}(t-T)+1\right).$$

Take $x \in \mathbb{R}^m$ and put $y = \tau^{-1/2}x$. Then the process $\xi_t^\tau(y)$, which is defined as $\xi_{T-\tau,t}(x)$ transformed via Θ^τ, satisfies the equation

$$d(\xi_t^\tau(y)) = X_N^\tau(\xi_t^\tau(y))\, dB_t^\tau + X_D^\tau(\xi_t^\tau(y))\, dt,$$

where $X_N^\tau(z) = X_N(\tau^{1/2}z)$, $X_D^\tau(z) = \tau^{1/2}X_D(\tau^{1/2}z)$ and $B_t^\tau(\omega) = \tau^{-1/2}B_{(T-\tau)+\tau t}(\omega)$, so that (at least as far as its increments are concerned) B_t^τ is again a Brownian motion. Thus $\xi_t^\tau(y)$ is the solution of a stochastic equation which is again uniformly elliptic with bound N_1, and so by Lemma 3.1 there exists $N_2 > 0$ such that if $|y_0 - y_1| \leq R$, then

$$P\{|\xi_t^\tau(y_0) - y_1| \leq N_2 F(1-t) \text{ for } t \in [0,1] | \xi_1^\tau(y_0) = y_1\} \geq 1-\delta.$$

But this probability must be the same as that of the statement, by the correspondence induced by the scaling Θ^τ.

(2) Using the technique of the proof of Lemma 3.1, we can show that independently of the extension of X_N, X_D beyond $\overline{B}_r(X_T)$, we have

$$\sup_{\{|x_{T-\tau}-x_T|\le \tau^{1/2}R\}} P\{|\xi_{T-\tau,t}(x_{T-\tau}) - x_T| \le r/2 \text{ for } t \in [T-\tau, T] | \xi_{T-\tau,T}(x_{T-\tau}) = x_T\} \to 1 \quad \text{as } \tau \to 0.$$

Let us make the following abbreviations:

$$\{\xi^i_{T-\tau,t}(x_{T-\tau}) - x_T| \le r/2 \text{ for } t \in [T-\tau, T]\} = R^\tau_i,$$

$$\{\xi^i_{T-\tau,T}(x_{T-\tau}) = x_T\} = C^\tau_i,$$

$$\left\{|\xi^i_{T-\tau,t}(x_{T-\tau}) - x_T| \le \tau^{1/2}N_2F\left(\frac{T-t}{\tau}\right)\right\} = B^\tau_i.$$

Here ξ^1, ξ^2 correspond to two extensions of X_N, X_D beyond $B_\tau(x_T)$. Note that if $\tau \le 1/N_2^2$, then $B^\tau_i \subset R^\tau_i$ and that $P(B^\tau_i|C^\tau_i \cap R^\tau_i)$ is independent of i. Also

$$P(B^\tau_i|C^\tau_i) = P(B^\tau_i|C^\tau_i \cap R^\tau_i)P(R^\tau_i|C^\tau_i).$$

From the preceding we see that we can find $\tau_1 > 0$ (with $\tau_1 \le 1 \wedge T \wedge 1/N_2^2$) such that for $\tau < \tau_1$, we have $P(R^\tau_t|C^\tau_i) \ge 1 - \delta$. Thus

$$P(B^\tau_i|C^\tau_i) \le P(B^\tau_i|C^\tau_i \cap R^\tau_i) \le (1-\delta)^{-1}P(B^\tau_i|C^\tau_i),$$

and it follows that the estimate of part (1) depends only on the uniform ellipticity over $B_r(x_T)$, if $\tau \le \tau_1$ and we throw in an extra factor $(1-\delta)^2$. □

THEOREM 3.3. *Assume that the coefficients $Y_N(x, y), Y_D(x, y)$ of* (2) *are bounded over sets $V \times \mathbb{R}^l$, where V is a compact subset of U. Take $x_0, x_T \in U$, $y_0 \in \mathbb{R}^l$ and put $v_0 = (x_0, y_0)$.*

(i) *Then $G\xi^{x_T}_t(v_0)$ [$\equiv (\xi^{x_T}_t(x_0), \eta^{x_T}_t(y_0))$—the solution of* (3) *and* (4)] *tends a.s. to a limit as $t \to T$, which lies in $\{x_T\} \times \mathbb{R}^l$ and which we will denote by $G\xi^{x_T}_T(v_0)$ [$\equiv (x_T, \eta^{x_T}_T(y_0))$]. Thus we can define the measure $[G\xi^{x_T}(P)v_0]$ as that induced from $(\Omega, \mathscr{F}, P)$ via equations* (3) *and* (4).

(ii) *We have*

$$[G\xi^{x_T}(P)v_0] = [G\xi(P)v_0|\xi_T(x_0) = x_T],$$

whenever this RHS is defined, i.e., up to a null set of x_T's.

PROOF. (i) (1) Take $\delta > 0$ and $R > 0$ sufficiently large so that for $\tau > 0$ sufficiently small, say $\tau \le \tau_0$, we have

$$P\{|\xi_{T-\tau}(x_0) - x_T| \le \tau^{1/2}R|\xi_T(x_0) = x_T\} \ge 1 - \delta.$$

Take N_2, τ_1 as in Lemma 3.2. We will show that, assuming the behaviour of Lemma 3.2 and taking $\tau \le \tau_0 \wedge \tau_1 \wedge \tau_2$, where τ_2 is such that $\overline{B}_r(x_T) \subset U$ for $r = \tau_2^{1/2}N_2$, then $G\xi^{x_T}_{T-\tau,t}(v_{T-\tau})$ is a Cauchy sequence for $v_{T-\tau} \equiv (x_{T-\tau}, y_{T-\tau})$,

$|x_{T-\tau} - x_T| \le \tau^{1/2}R$ and any $y_{T-r} \in \mathbb{R}^l$. It will then follow that $G\xi_t^{x_T}(v_0)$ is Cauchy with probability at least $(1-\delta)^2$. [Here $G\xi^{x_T}_{T-\tau,t}(v_{T-\tau}) \equiv (\xi^{x_T}_{T-\tau,t}(x_{T-\tau}), \eta^{x_T}_{T-\tau,t}(y_{T-\tau}))$ is the solution to (3) and (4) starting at v_{T-r} at time $T-\tau$.]

(2) Taking $t \in (0, \tau_0 \wedge \tau_1]$ and any τ' with $0 < \tau' < \tau$, we see by Theorem 2.1 (with $x_0, [0,T], \tau$ replaced by $x_{T-\tau}, [T-\tau, T], \tau'$) and Lemma 3.2 that

$$P\{\Omega_{T-\tau,T-\tau'}\} \ge 1-\delta,$$

where

$$\Omega_{T-\tau,T-\tau'} = \left\{|\xi^{x_T}_{T-\tau,t}(x_{T-\tau}) - x_T| \le \tau^{1/2}N_2F\left(\frac{T-t}{\tau}\right) \text{ for } t \in [T-\tau, T-\tau']\right\}.$$

Since $\Omega_{T-\tau,T-\tau'}$ decreases as τ' increases, we can deduce from this that

$$|\xi^{x_T}_{T-\tau,t}(x_{T-\tau}) - x_T| \le \tau^{1/2}N_2F\left(\frac{T-t}{\tau}\right) \quad \text{for } t \in [T-\tau, T), \tag{12}$$

with probability at least $1-\delta$, and thus $\xi^{x_T}_{T-\tau,t}(x_{T-\tau}) \to x_T$ as $t \to T$, with probability at least $1-\delta$.

(3) We show that taking $\tau \le \tau_0 \wedge \tau_1 \wedge \tau_2$ and assuming (12), $\eta^{x_T}_{T-\tau,t}(y_{T-\tau})$ is a Cauchy sequence a.s. From (4) we have

$$\begin{aligned}\eta^{x_T}_{T-\tau,t}(y_{T-\tau}) = y_{T-\tau} &+ \int_{T-\tau}^t Y_N\left(\xi^{x_T}_{T-\tau,s}(x_{T-\tau}), \eta^{x_T}_{T-\tau,s}(y_{T-\tau})\right) dB_s \\ &+ \int_{T-\tau}^t Y_D\left(\xi^{x_T}_{T-\tau,s}(x_{T-\tau}), \eta^{x_T}_{T-\tau,s}(y_{T-\tau})\right) ds \\ &+ \int_{T-\tau}^t \tilde{A}_s\left(\xi^{x_T}_{T-\tau,s}(x_{T-\tau}), \eta^{x_T}_{T-\tau,s}(y_{T-\tau})\right) ds,\end{aligned} \tag{13}$$

where $\tilde{A}_s(x,y) = Y_N(x,y) \circ X_N^{-1}(x) \circ A_s(x)$, with $A_s(x)$ as in (3).

Put

$$\sup_{\substack{|x-x_T| \le \tau_2^{1/2}N_2 \\ y \in \mathbb{R}^l}} \{|Y_N(x,y)|, |Y_D(x,y)|, |Y_N(x,y) \circ X_N^{-1}(x)|\} = N_1$$

and denote the last three terms in (13) by $a_t^{(1)}, a_t^{(2)}, a_t^{(3)}$. Then since $\xi^{x_T}_{T-\tau,s}(x_{T-\tau})$ does not escape from $B_r(x_T)$, assuming (12), we see that $a_t^{(1)}, a_t^{(2)}$ are Cauchy.

To deal with $a_t^{(3)}$, take N_3, N_4 such that for $|x - x_T| \le r$ we have $|A_s(x)| \le N_3(T-s)^{-1/2}\exp\{N_4|x-x_T|^2/(T-s)\}$. [We see from (10) and (11) that such N_3, N_4 exist.] Then, assuming (12), we have for $T - \tau/e^e \le t < r < T$ that

$$\begin{aligned}|a_r^{(3)} - a_t^{(3)}| &\le N_1N_3\int_t^r (T-s)^{-1/2} \\ &\quad \times \exp\left\{N_4\tau N_2^2\left(\frac{T-s}{\tau}\right)\left\{\log\log\left(\frac{\tau}{T-s}\right)\right\}\Big/(T-s)\right\} ds \\ &= N_1N_3\int_t^r (T-s)^{-1/2}\left\{\log\left(\frac{\tau}{T-s}\right)\right\}^{N_2^2N_4} ds.\end{aligned} \tag{14}$$

Now take $n > 2N_2^2N_4$ and take N_5 such that $\log t^{-1} \le N_5 t^{-1/n}$ for $t \le 1/e^e$. Also put $\alpha = \frac{1}{2} - N_2^2N_4/n$ and note that $\alpha \in (0, \frac{1}{2})$. Then

$$(14) \le N_1N_3N_5\int_t^r (T-s)^{\alpha-1}\,ds = -N_1N_3N_5\big[(T-s)^\alpha/\alpha\big]_{s=t}^{s=r},$$

and we see that assuming (12), $a_t^{(3)}$ is Cauchy.

(ii) This follows immediately from part (i) and Theorem 2.1. □

LEMMA 3.4. *Take $\varepsilon > 0$, $\delta > 0$, $R > 0$, $x_T \in U$. Assume Y_N, Y_D are locally bounded as in Theorem 3.3. Then there exists $\tau_3 > 0$ such that if $\tau \le \tau_3$, then for any $(x_{T-\tau}, y_{T-\tau})$ ($\equiv v_{T-\tau}$ say) in $U \times \mathbb{R}^m$ with $|x_{T-\tau} - x_T| \le \tau^{1/2}R$, we have*

$$P\Big\{\sup_{t\in[T-\tau,T]} |G\xi_{T-\tau,t}^{x_T}(v_{T-\tau}) - v_{T-\tau}| \le \varepsilon\Big\} \ge 1-\delta.$$

[Here $G\xi_{T-\tau,t}^{x_T}(v_{T-\tau}) \equiv (\xi_{T-\tau,t}^{x_T}(x_{T-\tau}), \eta_{T-\tau,t}^{x_T}(y_{T-r}))$ is the solution to (3) *and* (4) *starting from time $T-\tau$.]*

PROOF. Take $\tau_0, \tau_1, \tau_2, N_1, \dots, N_5, \alpha$ as in the proof of Theorem 3.3(i). We will show that it suffices to take τ_3 such that

(i) $\tau_3 \le \varepsilon^2\delta/N_1^2$,
(ii) $N_1\tau_3 \le \varepsilon$,
(iii) $\frac{1}{2}\tau_3^{1/2}N_1N_3\exp\{N_2^2N_4e^e\} + (1/\alpha)N_5(\tau_3/e^e)^\alpha \le \varepsilon$,
(iv) $N_2\tau_3^{1/2} \le \varepsilon$, $\tau_3^{1/2}R \le \varepsilon$.

So take $\tau \le \tau_3$ and assume (12). Then from condition (iv) on τ_3 we see that $\sup_{t\in[T-\tau,T]}|\xi_{T-\tau,t}^{x_T}(x_{T-\tau}) - x_{T-\tau}| \le 2\varepsilon$. Take $a_t^{(1)}, a_t^{(2)}, a_t^{(3)}$ as in the proof of Theorem 3.3(i). Then condition (ii) yields easily that $\sup_{t\in[T-\tau,T]}|a_t^{(2)}| \le \varepsilon$.

Also, using condition (iii), we have

$$\begin{aligned} |a_t^{(3)}| &\le N_1N_3\int_{T-\tau}^t (T-s)^{-1/2}\exp\{N_4\tau N_2^2/(T-s)\}\,ds \\ &\le \exp\{N_2^2N_4e^e\}\int_{T-\tau}^T (T-s)^{-1/2}\,ds \le \varepsilon, \end{aligned}$$

if $t \in [T-\tau, T-\tau/e^e]$, and

$$\begin{aligned} |a_t^{(3)}| &\le N_1N_3\int_{T-\tau}^{T-\tau/e^e} (T-s)^{-1/2}\exp\{N_4\tau N_2^2/(T-s)\}\,ds \\ &\quad + N_1N_3\int_{T-\tau/e^e}^T (T-s)^{-1/2}\Big\{\log\Big(\frac{\tau}{1-s}\Big)\Big\}^{N_2^2N_4}\,ds \\ &\le \frac{1}{2}\tau^{1/2}\exp\{N_2^2N_4e^e\} + \frac{1}{\alpha}N_5\Big(\frac{\tau}{e^e}\Big)^\alpha \\ &\le \varepsilon, \end{aligned}$$

if $t \in [T-\tau/e^e, T]$.

Thus, assuming (12) and using (13), we have

$$P\Big\{\sup_{t\in[T-\tau,T]}|\eta^{x_T}_{T-\tau,t}(y_{T-\tau})-y_{T-\tau}|\le 4\varepsilon\Big\}$$
$$\ge P\bigg\{\sup_{t\in[T-\tau,T]}\bigg|\int_{T-\tau}^t Y_N\big(\xi^{x_T}_{T-\tau,s}(x_{T-\tau}),\eta^{x_T}_{T-\tau,s}(y_{T-\tau})\big)\,dB_s\bigg|\le\varepsilon\bigg\}.$$

Assuming (12), we have $|\xi^{x_T}_{T-\tau,s}(x_{T-\tau})-x_T|\le\tau_2^{1/2}N_2$ for $s\in[T-\tau,T]$, and this last probability is unaltered if we set $Y_N(x,y)$ equal to 0 for $|x-x_T|>\tau_2^{1/2}N_2$. Altering Y_N in this way, we have $|Y_N|\le N_1$ and hence, using the martingale inequality, we deduce that this last probability [not assuming (12)] is at least $1-\delta$. Thus the probability that (12) is true (and hence this inequality), and that the event of this last probability is true, is at least $1-2\delta$. This gives the result, with ε and δ replaced by 4ε and 2δ. □

THEOREM 3.5. *Assume Y_N, Y_D are locally bounded as in Theorem* 3.3. *Take $x_0, x_T\in U$, $y_0\in\mathbb{R}^l$ and put $(x_0,y_0)=v_0$. Take $e>0$, $\delta>0$. Then there exists $\rho>0$ such that if $|x_T-x_T'|\le\rho$,*

$$P\Big\{\omega:\sup_{t\in[0,T]}|G\xi^{x_T}_t(\omega)v_0-G\xi^{x_T'}_t(\omega)v_0|\le\varepsilon\Big\}\ge 1-\delta.$$

COROLLARY 3.6. *If Y_N, Y_D are locally bounded as in Theorem* 3.3, *then for each $v_0\in U\times\mathbb{R}^l$, the measure $[G\xi^{x_T}(P)v_0]$ is weakly continuous in x_T.*

PROOF OF THEOREM 3.5. Take $R>0$, $\tau_0>0$ such that if $\tau\le\tau_0$, then

$$P\big\{|\xi^{x_T}_{T-\tau}(x_0)-x_T|\le\tau^{1/2}R\big\}\ge 1-\delta.$$

Also take τ_3 as in Lemma 3.4. Then for $\tau\le\tau_0\wedge\tau_3$ we have

$$P\Big\{\omega:\sup_{t\in[T-\tau,T]}|G\xi^{x_T}_t(\omega)v_0-G\xi^{x_T}_{T-\tau}(\omega)v_0|\le\varepsilon\Big\}\ge 1-2\delta.$$

Now note that we can make all our estimates locally uniform, and hence we can choose $\tau_4>0$ and a neighbourhood W of x_T such that for $\tau\le\tau_4$, $x_T'\in W$, we have

$$P\Big\{\omega:\sup_{t\in[T-\tau,T]}|G\xi^{x_T'}_t(\omega)v_0-G\xi^{X_T'}_{T-\tau}(\omega)v_0|\le\varepsilon\Big\}\ge 1-2\delta.$$

The result follows (with ε,δ replaced by $3\varepsilon, 5\delta$) from this and Theorem 3.5 itself, but with "$\sup_{t\in[0,T]}$" replaced by "$\sup_{t\in[0,T-\tau_4]}$"—this is a standard result because the coefficients of (3) and (4) are continuous in x_T for $t\in[0,T-\tau_4]$. □

Note (Removing the local boundedness condition of Theorem 3.3 on Y_N, Y_D). Suppose that instead of this condition, we merely assume that $G\xi_t(v_0)$ is nonexplosive. This means that $P\{\sup_{t\in[0,T]}|G\xi_t(v_0)|<\infty\}=1$ and implies that for a.e. x_T, we have

$$P\Big\{\sup_{t\in[0,T]}|G\xi_t(v_0)|<\infty\,|\,\xi_T(x_0)=x_T\Big\}=1. \tag{15}$$

Now choose $M > 0$ and truncate Y_N, Y_D off the set $\{(x, y): |y| > M\}$, i.e., alter them so that they are locally bounded. Then the preceding results do hold for the truncated systems, and moreover the alteration does not affect the solutions to (1) and (2) and (3) and (4) if we kill them on exiting from $\{|y| \le M\}$. Letting $M \to \infty$, we deduce that for x_T such that (15) holds, (3) and (4) do not explode as $t \to T$. The truncation technique also enables us to prove Theorem 3.3(ii) for x_T such that (15) holds, merely assuming that $G\xi(v_0)$ is nonexplosive.

4. A conditional version of the Stroock–Varadhan support theorem. This theorem applies to Stratonovitch equations, therefore we must convert (1) and (2) to Stratonovitch form when giving the action of the controls. So do this and denote the new drifts by $\tilde{X}_D$ and $\tilde{Y}_D$. Also denote by C the space of piecewise smooth maps ("controls") $[0, T] \to \mathbb{R}^n$. For $c \in C$ and $(x, y) \in U \times \mathbb{R}^l$ denote by $(\xi_t(c)x, \eta_t(c)y)$ the solution to the pair of time dependent, ordinary differential equations

$$d\xi_t(c)x = X_N(\xi_t(c)x)\dot{c}_t\,dt + \tilde{X}_D(\xi_t(c)x)\,dt,$$

$$d\eta_t(c)y = Y_N(\xi_t(c)x, \eta_t(c)y)\dot{c}_t\,dt + \tilde{Y}_D(\xi_t(c)x, \eta_t(c)y)\,dt.$$

The control c here stands in place of the noise in (the Stratonovitch form of) (1) and (2): One thinks of choosing a control c to steer the solution along a desired path in $U \times \mathbb{R}^l$.

THEOREM 4.1 (Conditioned Stroock–Varadhan support theorem). *Take* $(x_0, y_0) \equiv v_0 \in U \times \mathbb{R}^l$. *Then for all but a null set of x_T's in U, we have*

$$\begin{aligned} \textit{Support}\big[G\xi(P)v_0 | \xi_T(x_0) = x_T\big] \\ = \textit{Closure}\big[G\xi(c)v_0\colon c \in C,\ \xi_t(c)x_0 = x_T\big]. \end{aligned} \tag{16}$$

PROOF. (1) First assume Y_N and Y_D are locally bounded as in Theorem 3.3. Take $x_T \in U$. We will prove the result with $[G\xi(P)v_0|\xi_T(x_0) = x_T]$ replaced by $[G\xi^{x_T}(P)v_0]$.

(2) Here we prove "$\subset$." So take a path γ in the LHS of (16). Take $\varepsilon > 0$. We will find a control c such that $\xi_T(c)x_0 = x_T$ and $\sup_{t\in[0,T]}\{|\gamma_t - G\xi_t(c)v_0|\} \le \varepsilon$. The idea is to find c over the time interval $[0, T - \tau]$ for suitable $\tau > 0$ using the usual support theorem and then to steer the solution to x_T over the time interval $[T - \tau, T]$.

Take $r \le \varepsilon$ such that the closed ball $\overline{B}_r(x_T)$ lies in U and put

$$N_1 = \sup_{\substack{|x - x_T| \le r \\ y \in \mathbb{R}^l}} \big\{\|Y_N(x, y) \circ X_N^{-1}(x)\|, \|\tilde{Y}_D\|, \|\tilde{X}_D\|, 1\big\}.$$

Take $\tau > 0$ such that $\tau \le \varepsilon/6(N_1 + N_1^2)$ and $\sup_{t\in[T-\tau,T]}\{|\gamma_t - \gamma_{T-\tau}|\} \le \varepsilon/6N_1$.

Take a control $c^\tau\colon [0, T - \tau] \to \mathbb{R}^n$ such that

$$\sup_{t\in[0,T-\tau]} |G\xi_t(c^\tau)v_0 - \gamma_t| \le \varepsilon/6N_1.$$

Then

$$|\xi_{T-\tau}(c^\tau)x_0 - x_T| (\equiv \rho \text{ say}) \le |\xi_{T-\tau}(c^\tau)x_0 - \pi(\gamma_{T-\tau})| + |\pi(\gamma_{T-\tau}) - x_T| \\ \le \varepsilon/3N_1.$$

We will extend c^τ to $c\colon [0,T] \to \mathbb{R}^n$ so as to steer $\xi_t(c)x_0$ along the straight line $\alpha\colon [T-\tau, T] \to U$ (with constant speed ρ/τ), such that $\alpha_{T-t} = \xi_{T-\tau}(c^\tau)x_0$ and $\alpha_T = x_T$. To do this, simply put $c_T = c_t^\tau$ for $t \in [0, T-\tau]$, and $\dot{c}_t = X_N^{-1}(\alpha_t)(\dot{\alpha}_t - \tilde{X}_D(\alpha_t))$ for $t \in (T-\tau, T]$.

With this choice of c_t,

$$\frac{d\eta_t(c)y_0}{dt} = Y_N(\eta_t(c)y_0)\circ\left[X_N^{-1}(\alpha_t)(\dot{\alpha}_t - \tilde{X}_D(\alpha_t))\right] + \tilde{Y}_D(\eta_t(c)y_0),$$

for $t \in [T-\tau, T]$, and the speed of $\eta_t(c)y_0$ is bounded by $(\rho/\tau)N_1 + N_1^2 + N_1$. Thus

$$\sup_{t\in[T-\tau,T]}\{|G\xi_t(c)v_0 - G\xi_{T-\tau}(c)v_0|\} \le \rho N_1 + \tau(N_1^2 + N_1) \\ \le \varepsilon/3 + \varepsilon/6 = \varepsilon/2,$$

and we have for $t \in [T-\tau, T]$ that

$$|G\xi_t(c)v_0 - \gamma_t| \le |G\xi_t(c)v_0 - G\xi_{T-\tau}(c)v_0| \\ + |G\xi_{T-\tau}(c)v_0 - \gamma_{T-\tau}| + |\gamma_{T-\tau} - \gamma_t| \\ \le 5\varepsilon/6.$$

(3) Here we prove "$\supset$." So take a piecewise smooth control c such that $\xi_T(c)x_0 = x_T$. Take $\varepsilon > 0$. We will show that

$$P\left\{\sup_{t\in[0,T]}|G\xi_t^{x_T}(v_0) - G\xi_t(c)v_0| \le 3\varepsilon\right\} > 0. \tag{17}$$

Take $R > 0$, $\delta > 0$ and then take τ_3 as in Lemma 3.4. Take K such that

$$|G\xi_t(c)v_0 - G\xi_s(c)v_0| \le K|t-s| \quad \text{for all } s,t \in [0,T].$$

Finally, take $\tau > 0$ such that $\tau \le \tau_3$, $2K\tau \le \tau^{1/2}R$, $K\tau \le \varepsilon$. Now if $v_{T-\tau}$ satisfies $|v_{T-\tau} - G\xi_{T-\tau}(c)v_0| < K\tau$, then putting $v_{T-\tau} \equiv (x_{T-\tau}, y_{T-\tau})$, we have

$$|x_{T-\tau} - x_T| \le |x_{T-\tau} - \xi_{T-\tau}(c)x_0| + |\xi_{T-\tau}(c)x_0 - x_T| \le 2K\tau \le \tau^{1/2}R,$$

and by Lemma 3.4 we have

$$P\left\{\omega\colon \sup_{t\in[T-\tau,T]}|G_{T-\tau,t}(\omega)v_{T-\tau} - v_{T-\tau}| \le \varepsilon\right\} \ge 1-\delta.$$

Therefore, with probability at least $1-\delta$, and uniformly over $t \in [T-\tau, T]$, we have

$$|G\xi_t(c)v_0 - G\xi_{T-\tau,t}(\omega)v_{T-\tau}| \\ \le |G\xi_t(c)v_0 - G\xi_{T-\tau}(c)v_0| + |G\xi_{T-\tau}(c)v_0 - v_{T-\tau}| \\ + |v_{T-\tau} - G\xi_{T-\tau,t}(\omega)v_{T-\tau}| \\ \le 3\varepsilon.$$

The estimate (17) now follows taking $v_{T-\tau} = G\xi_{T-\tau}^{x_T}(\omega)v_0$ in the preceding and considering the following, which is the usual support theorem:

$$P\left\{\omega:\ \sup_{t\in[0,T-r]} |G\xi_t^{x_T}(\omega)v_0 - G\xi_t(c)v_0| \le K\tau\right\} > 0.$$

(4) To remove the local boundedness condition on Y_N, Y_D, apply a truncation argument, as in the note at the end of Section 3. □

Note. A control system is naturally associated with a Stratonovitch stochastic equation and the usual Stroock–Varadhan support theorem is most conveniently formulated for Stratonovitch equations. However, it seems that we must have an Itô system in order to do the analysis of Section 3. Also it does not seem sensible to study the control system associated with equations (3) and (4) even in their Stratonovitch form.

Acknowledgments. The work presented here was begun while I was a SERC Research Fellow at Warwick University (England) and completed while I was visiting the Institute for Mathematics and its Applications at Minnesota. I would like to take this opportunity to thank the organisers of the IMA for inviting me to visit and for their generous hospitality. Also I would like to thank those who have helped me with the work presented here: D. Elworthy and K. Watling at Warwick; also M. Cranston, E. Fabes and especially D. Stroock, who showed me how to deal with the technical problems of Section 3.

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