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THE CONTACT PROCESS ON A FINITE SET. II

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In this paper we complete the work started in part I. We show that if σ_N is the time that the contact process on $\{1, \dots, N\}$ first hits the empty set, then for $\lambda > \lambda_c$ (the critical value for the process on Z) there is a positive constant $\gamma(\lambda)$ so that $(\log \sigma_N)/N \to \gamma(\lambda)$ in probability as $N \to \infty$. We also give a new simple proof that $\sigma_N / E\sigma_N$ converges to a mean one exponential. The keys to the proof of the first result are a "planar graph duality" for the contact process and an observation of J. Chayes and L. Chayes that exponential decay rates for connections in strips approach the decay rates in the plane as the width of the strip goes to ∞.

1. Introduction. The contact process is a Markov process with state space the subsets of Z and transition probabilities that satisfy

$$\begin{aligned} &P(x \notin \xi_t | \xi_0) \sim t, && \text{if } x \in \xi_0, \\ &P(x \in \xi_t | \xi_0) \sim \lambda t |\xi_0 \cap \{x-1, x+1\}|, && \text{if } x \notin \xi_0, \end{aligned}$$

as $t \to 0$, where $f(t) \sim g(t)$ means $f(t)/g(t) \to 1$. If we think of the sites in ξ_t as occupied by particles, then the dynamics can be described as: "Particles die at rate 1 and are born at vacant sites at rate λ times the number of occupied neighbors." It is by now well known that there is a unique Markov process with the above properties and there are several ways to construct it. See Liggett (1985), Chapter 6 [or Griffeath (1981)] for information on how to construct the process and the basic properties we use if no reference is given.

Let ζ_t^N denote the contact process on $\{1, \dots, N\}$ starting from all sites occupied and let $\sigma_N = \inf\{t: \zeta_t^N = \varnothing\}$. Since ζ_t^N is a Markov chain on a finite set $P(\sigma_N < \infty) = 1$ for all λ. Differences between the λ's appear when we let $N \to \infty$. To state our results, let ξ_t^0 be the contact process on Z starting from $\xi_0^0 = \{0\}$ and λ_c be its critical value

$$\lambda_c = \inf\{\lambda: P(\xi_t^0 \neq \varnothing \text{ for all } t) > 0\}.$$

In part I Durrett and Liu proved

THEOREM 1. *Let $\lambda < \lambda_c$ and*

$$\gamma_1(\lambda) = -\lim_{n\to\infty} \frac{1}{n} \log P(\xi_n^0 \neq \varnothing).$$

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Then as $N \to \infty$,

$$\frac{\sigma_N}{\log N} \to \frac{1}{\gamma_1(\lambda)}.$$

They proved some results about the behavior when $\lambda > \lambda_c$ but could not prove the following sharp result, which is the main result of this paper.

THEOREM 2. *Let* $\lambda > \lambda_c$. *There is a constant* $\gamma_2(\lambda) > 0$ *so that as* $N \to \infty$,

(a) $$\frac{1}{N}\log P(\tau^{\{1,\dots,N\}} < \infty) \to -\gamma_2(\lambda),$$

(b) $$\frac{1}{N}\log P(\hat{\tau}^{\{1,\dots,N\}} < \infty) \to -\gamma_2(\lambda),$$

(c) $$\frac{1}{N}\log P(\tau^{eq\cap\{1,\dots,N\}} < \infty) \to -\gamma_2(\lambda),$$

and

(d) $$\log\frac{\sigma_N}{N} \to \gamma_2(\lambda), \quad \textit{in probability}.$$

Here $\tau^A = \inf\{t\colon \xi_t^A = \varnothing\}$, where ξ_t^A is the contact process starting from $\xi_0^A = A$; $\hat{\tau}^A = \inf\{t\colon \hat{\xi}_t^A = \varnothing\}$, where $\hat{\xi}_t^A$ is the contact process on the half-line $\{1, 2, \dots\}$ starting from $\hat{\xi}_0^A = A$; and, finally, $\tau^{eq\cap\{1,\dots,N\}}$ is the extinction time for the contact process on Z starting from the upper invariant measure $(= \lim_{t\to\infty}\xi_t^Z)$ restricted to $\{1,\dots,N\}$.

The last process or more precisely (c) of Theorem 2 is the new ingredient here. It allows us to prove that the limit exists in (b) and is equal to the limit in (a), problems left unsolved in the first part. At first it may seem that (c) is not natural but the developments of Section 2 will show otherwise:

$$\begin{aligned} &P(\tau^{eq\cap\{1,\dots,N\}} < \infty) \\ &\qquad = P\big(\text{there is a "dual path" from } \big(N + \tfrac{1}{2}, 0\big) \text{ to } \big(\tfrac{1}{2}, 0\big)\big). \end{aligned}$$

Theorem 2 describes the asymptotic behavior of $\log \sigma_N$. The next result describes the behavior of σ_N itself.

THEOREM 3. *Let* $\beta_N = \inf\{t\colon P(\sigma_N > t) \le e^{-1}\}$. *As* $N \to \infty$,

(a) $$P(\sigma_N/\beta_N > x) \to e^{-x},$$

(b) $$E\sigma_N/\beta_N \to 1.$$

This result was first proved by Cassandro, Galves, Olivieri and Vares (1984) for large λ and later for all $\lambda > \lambda_c$ by Schonmann (1985). We present here a proof of (a) that is much shorter than the previous ones. We have not been able to improve the proof of (b) due to Galves and Vares, which is given in Schonmann (1985), so it is omitted.

The paper is organized as follows. The "planar graph duality" is described in Section 2. Theorem 2 is proved in Section 3. Lemma 2 in that section is the result of J. Chayes and L. Chayes referred to in the abstract. Finally, the proof of Theorem 3 is given in Section 4. It is short and simple and almost independent of the developments that precede it. The only fact needed is that $\beta_N/N \to \infty$.

As usual the results in this paper and in part I have counterparts for oriented percolation (see Durrett (1984) for a description of this model) and the proofs are essentially the same but easier for that model because we can sum over all possible points rather than estimate the Lebesgue measure of the set of space–time points that can be reached (see the proofs of Lemmas 5 and 6 in Section 3).

What is more surprising is that there is also a close relation between our results and the work of Grimmett (1981) on sponge crossings in (unoriented) two-dimensional bond percolation. To describe the connection, consider bond percolation in $[1, N] \times [0, \infty)$ and let

$$\sigma_N = \sup\{l \geq 0\text{: there is a chain of occupied sites from } \{1,\dots,N\} \times \{0\} \text{ to } \{1,\dots,N\} \times \{l\}\}.$$

Grimmett (1981) showed that

$$\text{if } p < \tfrac{1}{2}, \text{ then } \sigma_N/\log N \to 1/\gamma(p), \quad \text{in probability},$$

$$\text{if } p > \tfrac{1}{2}, \text{ then } (\log \sigma_N)/N \to \gamma(p), \quad \text{in probability},$$

where $\gamma(p)$ is a positive constant ($=1/$the correlation length) with $\gamma(p) = \gamma(1-p)$.

Our proof of Theorem 2 can be adapted to give a proof of Grimmett's result for $p > \frac{1}{2}$ (which by the self-duality of bond percolation implies the result for $p < \frac{1}{2}$), and our proof of Theorem 3 can be modified to prove the new result that $\sigma_N/E\sigma_N$ converges in distribution to a mean one exponential. The modifications necessary to prove the second result are described in Section 4.

2. Planar graph duality. In this section we will describe a planar graph duality for the contact process. We begin by describing the analogous concept for oriented percolation, which has been described by Dhar, Barma and Phani (1981) and in somewhat greater generality by Redner (1981, 1982). Let $L = \{(m,n)\text{: } m,n \in Z \text{ and } m+n \text{ is even}\}$ and draw oriented bonds from (m,n) to $(m+1, n+1)$ and to $(m-1, n+1)$, which are open with probability p and closed with probability $1-p$. For reasons that will become clear, we will also draw oriented bonds from (m,n) to $(m+1, n-1)$ and to $(m-1, n-1)$, which are open with probability 0 and closed with probability 1.

To define the dual problem, let $L^* = \{(m,n)\text{: } m,n \in Z \text{ and } m+n \text{ is odd}\}$ and draw oriented bonds from (m,n) to $(m+1, n+1)$,$(m-1, n+1)$, $(m-1, n-1)$ and $(m+1, n-1)$. To make the link between the original process and the dual, we associate each dual bond with the bond on the original lattice, which is obtained by rotating the dual bond 90° clockwise around its midpoint, and we declare the dual bond to be closed (resp. open) if the associated

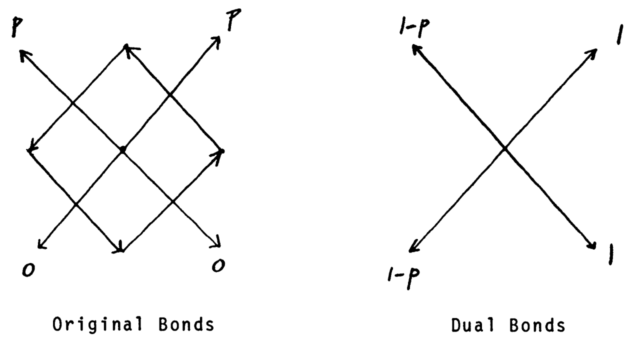

FIG. 1.

bond on the original lattice was open (resp. closed). See Figure 1 for help with the definitions and to observe that the dual model has bonds that are open with probability 1 if the x-coordinate increases in the direction of the orientation and are open with probability $1 - p$ if the x-coordinate decreases.

To explain the definitions and the connection between the two models, let n be an even integer and consider the trapezoid T with vertices $(0,0)$, (k,k), $(-n-k,k)$ and $(-n,0)$. Let S_1 be the segment from $(1,0)$ to $(k+1,k)$ and S_2 be the segment from $(-n-1,0)$ to $(-n-k-1,k)$.

PROPOSITION 1. *Either there is a path on the original lattice from the bottom of T to the top of T or a dual path from S_1 to S_2 that lies in $R \times [0,k]$ but not both.*

The last result can be obtained by combining the maxflow–min cut theorem with some facts about planar graphs (see Berge (1973), Chapter 5, especially page 85). However, a direct proof is not difficult and will show the reader that dual paths are just what previously have been called contours so we will give the details.

PROOF. First suppose there is a path from S_1 to S_2 and call it ω. By removing loops we can suppose without loss of generality that any point of L^* appears at most once in the path. With the loops removed an application of the Jordan curve theorem shows that ω divides the interior of T into two parts—one we call T_1, which lies below ω (its boundary contains the bottom of T), and the other T_2, which lies above ω. If we move along ω in the direction of the orientation, T_1 is always on our left and T_2 is on the right. From this we see that if there is a path of open bonds from the bottom to the top, then any time it crosses from T_1 to T_2 it does so along a bond that is a 90° clockwise rotation of a bond on ω but such a bond is closed by the definition of duality so no path exists.

To complete the proof of Proposition 1, we have to show that if there is no path from the bottom to the top there is one on the dual from S_1 to S_2. The reader may find it helpful to consult Figure 2 while we do this. Let $A = \{(m,0)$:

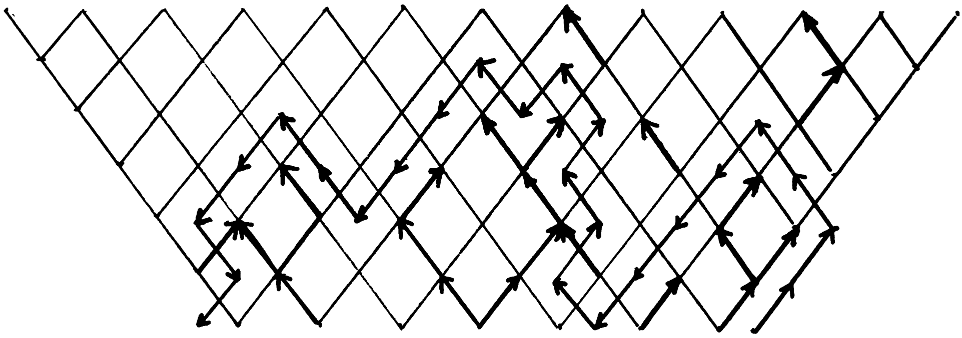

FIG. 2.

m is even and $-n \le m \le 0\}$. Let $C = \{x \in L$: x can be reached by an open path from some point in $A\}$. Let $D = \{(a, b) \in R^2$: $|a| + |b| \le 1\}$ and orient the boundary of D in a counterclockwise fashion. Finally, let $W = \bigcup_{z \in C}(z + D)$. C is for cluster, D is for diamond and W is for wet region. If we combine the boundaries of the $z + D$ with $z \in C$ and allow oppositely directed segments to cancel, then the boundaries that remain are closed paths on the dual and one of them Γ = the boundary of the unbounded component of $R \times (0, k) - W$ is the path we want. The reader should note that this path starts at $(1, 0)$ and ends at $(-n - 1, 0)$ but any path from S_1 to S_2 can be extended to such a path by adding a segment from $(1, 0)$ to the first point on the path and from the last point on the path to $(-n - 1, 0)$ since all the bonds added are open with probability 1. □

In the preceding argument we have used the notation of Section 10 of Durrett (1984) to show that the dual of oriented percolation is a process of "contours." This viewpoint clarifies a number of points in the contour argument described in Section 10 of Durrett (1984) but we will not go into this because our main aim is to define the corresponding dual for the contact process. One way to approach this is to write the contact process as a limit of oriented percolation process. Let $L_\varepsilon = \{(j, k\varepsilon/2)$, where $j, k \in Z\}$ and make bonds open independently with the following probabilities (here $m, n \in Z$):

$$\begin{aligned}
(m, n\varepsilon) &\to \left(m - 1, \left(n + \tfrac{1}{2}\right)\varepsilon\right), && \lambda\varepsilon,\\
(m, n\varepsilon) &\to \left(m, \left(n + \tfrac{1}{2}\right)\varepsilon\right), && 1 - (\varepsilon/2),\\
\left(m, \left(n + \tfrac{1}{2}\right)\varepsilon\right) &\to (m + 1, (n + 1)\varepsilon), && \lambda\varepsilon,\\
\left(m, \left(n + \tfrac{1}{2}\right)\varepsilon\right) &\to (m, (n + 1)\varepsilon), && 1 - (\varepsilon/2).
\end{aligned}$$

If we let $\varepsilon \to 0$ in the preceding construction, we get for each m a Poisson process of arrows to the right $(m, t_k) \to (m + 1, t_k)$, $k = 1, 2, \dots,$ at rate λ, a Poisson process of arrows to the left $(m, t'_k) \to (m - 1, t'_k)$, $k = 1, 2, \dots,$ at rate λ, and if we mark the closed bonds $(m, k\varepsilon/2) \to (m, (k + 1)\varepsilon/2)$ with δ's, we get a Poisson process of δ's at rate 1. The preceding limiting process is the graphical representation for the contact process introduced by Harris (1978) and developed by Griffeath (1979). In what follows we will assume that the reader is familiar with how the contact process is constructed from the graphical representation. See Liggett (1985) for a recent account.

With the duality for oriented percolation and the preceding limit in mind, we will now define the dual for the contact process. If we recall that the associated dual bond is a 90° counterclockwise rotation of the original bond and has the opposite state, then we are led to the following rules:

(i) $\downarrow$ is impossible so $\rightarrow$ is always allowed in the dual;
(ii) δ marks a hole so $\leftarrow$ is allowed only through δ's;
(iii) $\rightarrow$ open makes $\uparrow$ closed;
(iv) $\leftarrow$ open makes $\downarrow$ closed.

To explain the definitions and the connection between the two models, consider $T = R \times [0, t]$, let n be an integer, and let $A = \{0, -1, \dots, -n\}$.

PROPOSITION 2. *Either there is a path from $A \times \{0\}$ to the top of T in the contact process or a dual path from $(\frac{1}{2}, 0)$ to $(-n - \frac{1}{2}, 0)$ that lies in T but not both.*

PROOF. The first part of the argument (if there is a dual path there is no path in the contact process) is almost exactly as before so we proceed to the second part. This time the reader should follow along on Figure 3 and look at Griffeath (1981), page 160, or Gray and Griffeath (1982) when we are done.

Let $C = \bigcup_s(\xi_s^A \times \{s\})$. Let $D = \{(a, 0) \in R^2\colon |a| \le \frac{1}{2}\}$ with the left and right endpoints considered as infinitesimal arrows pointing down and up, respectively. Suppose C does not intersect the top of T. If we let $W = \bigcup_{z \in C}(z + D)$ and let Γ be the boundary of the unbounded component of $R \times (0, \infty) - W$ with Γ oriented in the way dictated by the infinitesimal arrows, then Γ is a path on the dual and the "contours" used by Gray and Griffeath. □

In Section 3 we will need several variations of the last result. We will prove the one that was mentioned in the Introduction now. It is the most difficult. After seeing its proof and the previous argument, we think the reader can supply the details in the remaining cases.

PROPOSITION 3.

$$P\left(\tau^{eq \cap \{1, \dots, n\}} < \infty\right)$$
$$= P\left(\textit{there is a dual path from } \left(n + \tfrac{1}{2}, 0\right) \textit{ to } \left(\tfrac{1}{2}, 0\right)\right).$$

PROOF. Let $T = R \times [-t, t]$ and let $A = \{1, \dots, n\}$. We claim that either there is a path from the bottom to the top of T that passes through $A \times \{0\}$ or there is a dual path from $(n + \frac{1}{2}, 0)$ to $(\frac{1}{2}, 0)$ in T but not both. As in the two previous proofs the existence of a dual path prevents the occurrence of a path in the contact process so we will suppose there is no contact path and construct a dual path.

First modify the graphical representation so that δ's occur at all points on $A^c \times \{0\}$. Second truncate the picture by picking $l \le 0$ $(r \ge n + 1)$ so that there

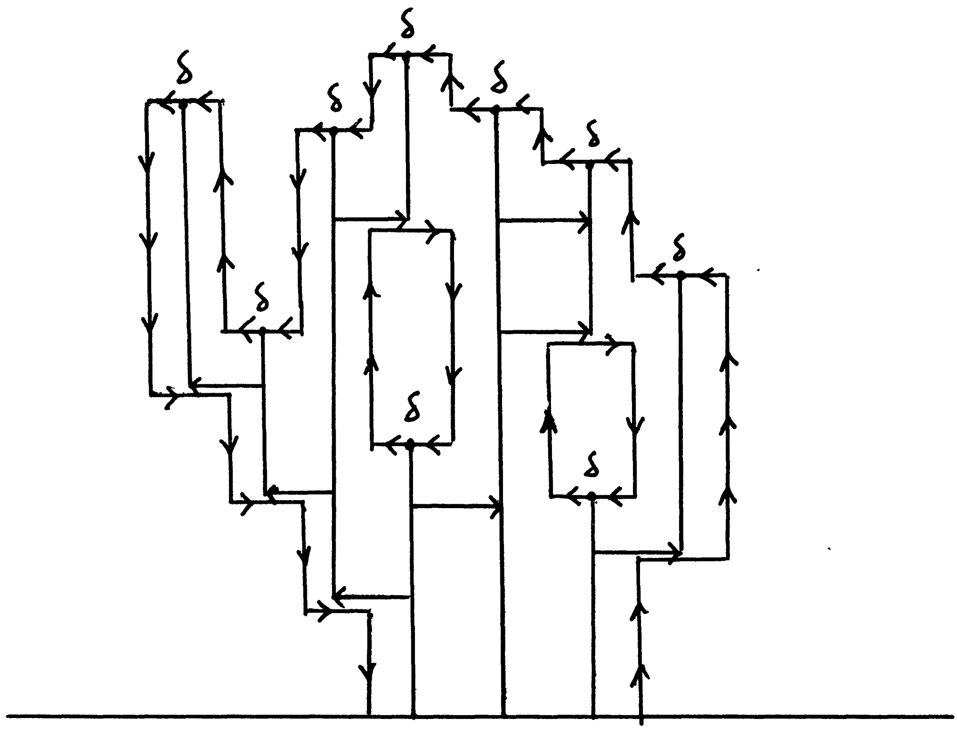

FIG. 3.

are no δ's at l (resp. r) at times $s \in [-t, 0]$ and there are no arrows from l to $l-1$ (resp. r to $r+1$) at times $s \in [0, t]$. Let C be the set of space–time points that can be reached starting from $\{l, l+1, \dots, r-1, r\}$ occupied at time $-t$ in the modified graphical representation. Let $D = \{(a, 0) \in R^2\colon |a| \le \frac{1}{2}\}$ with the left and right endpoints considered as infinitesimal arrows pointing down and up, respectively.

Suppose C does not intersect the top of T, let Γ be the boundary of the component of $((l-\frac{1}{2}, r+\frac{1}{2}) \times (-t, t)) - W$ that touches $(l-\frac{1}{2}, r+\frac{1}{2}) \times \{t\}$ and orient Γ in the way dictated by the infinitesimal arrows. Γ is a path on the dual that starts at $(r+\frac{1}{2}, 0)$ and ends at $(l-\frac{1}{2}, 0)$. (See Figure 4.) Let p be the last point of Γ on $[n+\frac{1}{2}, \infty) \times \{0\}$ and let q be the first point of Γ on $(-\infty, \frac{1}{2}]$. The path we want is obtained by moving from $(n+\frac{1}{2}, 0)$ to p along the x-axis, going from p to q along Γ and then from q to $(\frac{1}{2}, 0)$ along the x-axis. The bonds we have added are all from left to right (which is always allowed) so the result is a dual path.

Let $B(t)$ be the set of x such that $(x, 0)$ can be reached by a path starting from some $(y, -t)$. At this point we have shown

$$P\left(\xi_t^{B(t) \cap \{1, \dots, n\}} = \varnothing\right)$$
$$= P\big(\text{there is a dual path from } \left(n+\tfrac{1}{2}, 0\right) \text{ to } \left(\tfrac{1}{2}, 0\right) \text{ in } [-t, t] \times R.$$

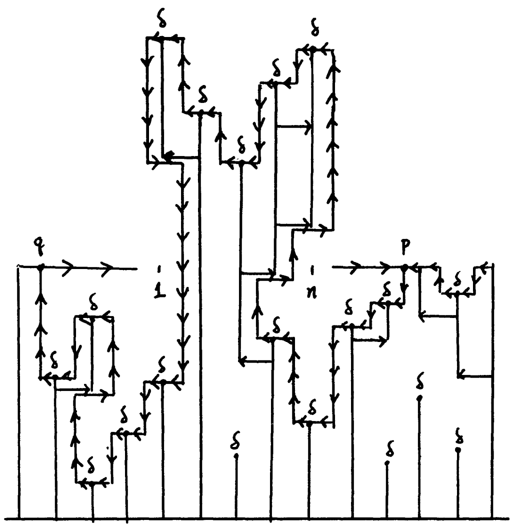

FIG. 4.

As $t \uparrow \infty$, $B(t)$ decreases to a limit $B(\infty)$. For each t, $B(t)$ has the same distribution as ξ_t^Z so the limit $B(\infty)$ has the equilibrium distribution. Now as $t \uparrow \infty$ the left-hand side increases to

$$P\big(\xi_t^{B(\infty) \cap \{1, \dots, n\}} = \varnothing \text{ for some } t\big)$$

and the right-hand side increases to

$$P\big(\text{there is a dual path from } \big(n + \tfrac{1}{2}, 0\big) \text{ to } \big(\tfrac{1}{2}, 0\big)\big)$$

and we have proved Proposition 3. □

3. Proof of Theorem 2. In this section we will prove Theorem 2. Let $F_n = \{(-\frac{1}{2}, 0) \to (-n - \frac{1}{2}, 0)\}$, where $(-\frac{1}{2}, 0) \to (-n - \frac{1}{2}, 0)$ is short for "there is a dual path from" $(-\frac{1}{2}, 0)$ to $(-n - \frac{1}{2}, 0)$. Let $W_k = \{(x, y)\colon -k \le y \le k\}$ and let $F_n^k = \{(-\frac{1}{2}, 0) \to (-n - \frac{1}{2}, 0) \text{ in } W_k\}$.

LEMMA 1. *There are constants δ and δ_k so that as $n \to \infty$,*

$$\frac{1}{n} \log P(F_n) \to -\delta, \qquad \frac{1}{n} \log P\big(F_n^k\big) \to -\delta_k$$

and for any k,

$$P\big(F_n^k\big) \le \exp(-\delta_k n).$$

PROOF. For $1 \le k \le \infty$ let $F_{m,n}^k = \{(-m - \frac{1}{2}, 0) \to (-n - \frac{1}{2}, 0) \text{ in } W_k\}$. $F_{0,n}^k \supset F_{0,m}^k \cap F_{m,n}^k$ and the $F_{m,n}^k$ are decreasing events so Harris' inequality implies

$$P\big(F_{0,n}^k\big) \ge P\big(F_{0,m}^k \cap F_{m,n}^k\big) \ge P\big(F_{0,m}^k\big) P\big(F_{m,n}^k\big).$$

If we let $a_n = \log P(F^k_{0,n})$ the last inequality can be rewritten as

$$a_n \geq a_m + a_{n-m}.$$

It is well known and easy to prove (see Durrett (1984), page 1017) that this implies

$$\frac{1}{n}a_n \to \sup_m \frac{1}{m}a_m,$$

so if we let $-\delta_k$ denote the right-hand side and write δ for δ_∞ we have proved the desired results. □

The next result is an observation due to J. Chayes and L. Chayes. The only difficult thing about the proof is to believe that all the inequalities go in the right direction.

LEMMA 2. $\lim_{k\to\infty}\delta_k = \delta$.

PROOF. First observe that δ_k decreases as k increases so the limit exists and is $\geq \delta$. Let $\varepsilon > 0$ and pick N so that

$$P(F_N) \geq \exp(-(\delta+\varepsilon)N).$$

As k increases to ∞, $F^k_N \uparrow F_N$ so $P(F^k_N) \uparrow P(F_N)$. From Lemma 1 we have $P(F^k_N) \leq \exp(-\delta_k N)$. Putting the last three results together gives

$$\begin{aligned}\exp(-(\delta+\varepsilon)N) &\leq P(F_N)\\ &= \lim_{k\to\infty} P\big(F^k_N\big)\\ &\leq \lim_{k\to\infty}\exp(-\delta_k N)\\ &= \exp\Big(-\Big(\lim_{k\to\infty}\delta_k\Big)N\Big).\end{aligned}$$

The last result implies

$$-\Big(\lim_{k\to\infty}\delta_k\Big) \geq -(\delta+\varepsilon)$$

and since ε is arbitrary

$$\lim_{k\to\infty}\delta_k \leq \delta. \qquad \square$$

LEMMA 3. *Let*

$$\lim_{n\to\infty}\frac{1}{n}\log P(\tau^{\{1,\dots,n\}} < \infty) = -\gamma.$$

Then $\gamma = \delta$.

PROOF. The limit exists by supermultiplicativity and is < 0 (see Durrett (1984), Section 10). In Section 2 we showed $P(F_n) = P(\tau^{eq\cap\{1,\dots,n\}} < \infty)$ so $P(F_n) \geq P(\tau^{\{1,\dots,n\}} < \infty)$ and $-\delta \geq -\gamma$. To prove the other direction, we observe that if $W_0^{2k} = \{(x, y): 0 \le y \le 2k\}$, then $\{\tau^{\{1,\dots,n\}} \le 2k\}$ occurs provided that the following three events occur: $\{(n + \frac{1}{2}, k) \to (\frac{1}{2}, k) \text{ in } W_0^{2k}\}$, {there is no arrow from a point (n, s) to $(n + 1, s)$ for $0 \le s \le k$} and {there is no arrow from a point $(1, s)$ to $(0, s)$ for $0 \le s \le k$}. From Harris' inequality it follows that these three events are positively correlated, hence

$$P(\tau^{\{1,\dots,n\}} \le 2k) \ge P\big((n + \tfrac{1}{2}, k) \to (\tfrac{1}{2}, k) \text{ in } W_0^{2k}\big) \cdot e^{-2\lambda k},$$

and taking logs, dividing by n and letting $n \to \infty$ now shows $-\gamma \ge -\delta_k$. Since k is arbitrary the desired result follows from Lemma 2. □

LEMMA 4. *Let G_n = {there is a y so that $(n + \frac{1}{2}, 0) \to (\frac{1}{2}, y)$}. As $n \to \infty$,*

$$\frac{1}{n}\log P(G_n) \to -\gamma,$$

where γ is the constant of Lemma 3.

PROOF. $P(G_n) \ge P(F_n)$ so the $\liminf \ge -\gamma$. To prove the $\limsup \le -\gamma$, observe that Durrett and Griffeath (1983) have shown that there are constants $C, \varepsilon \in (0, \infty)$ independent of A so that

$$P(t \le \hat{\tau}^A < \infty) \le Ce^{-\varepsilon t}.$$

From the last fact it follows that if we pick $\theta > (2\gamma + \log 2)/\varepsilon$ and let G_n^θ = {there is a y with $|y| \le n\theta$ so that $(n + \frac{1}{2}, 0) \to (\frac{1}{2}, y)$}, then

$$\limsup_{n\to\infty} \frac{1}{n}\log P(G_n^\theta) = \limsup_{n\to\infty} \frac{1}{n}\log P(G_n).$$

To see this, let G_n^+ = {there is a $y > 0$ so that $(n + \frac{1}{2}, 0) \to (\frac{1}{2}, y)$}, observe that $G_n^+ - G_n^\theta \subset$ {there is an $A \subset \{1,\dots,n\}$ so that $\theta n \le \tau^A < \infty$} and that the probability of the last event is smaller than $2^n \cdot C\exp(-\theta\varepsilon n)$.

Let $F_n(x, y)$ = {there is a dual path from $(n + \frac{1}{2}, x)$ to $(\frac{1}{2}, y)$}. Repeating the first argument in the proof of Lemma 1 and using the $(x, y) \to (x, -y)$ reflection symmetry of the dual gives

$$\begin{aligned} P(F_{2n}(0,0)) &\ge P(F_n(0, y))P(F_n(y, 0)) \\ &= P(F_n(0, y))^2. \end{aligned}$$

Taking logs of both sides, dividing by $2n$ and letting $n \to \infty$ gives

$$\limsup_{n\to\infty} \frac{1}{n} \sup_y \log P(F_n(0, y)) \le -\gamma$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\log\left(\int_{-\theta n}^{\theta n+1} P(F_n(0, y))\,dy\right) \le -\gamma.$$

Now, Fubini's theorem implies

$$\int_{-\theta n}^{\theta n+1} P(F_n(0, y))\, dy = E\left|\left\{y \in [-\theta n, \theta n + 1] \colon \left(n + \tfrac{1}{2}, 0\right) \to \left(\tfrac{1}{2}, y\right)\right\}\right|,$$

where $|\{\cdots\}|$ indicates the Lebesgue measure of the indicated set, so the proof will be complete once we show

$$(*) \qquad E\left|\left\{y \in [-\theta n, \theta n + 1] \colon \left(n + \tfrac{1}{2}, 0\right) \to \left(\tfrac{1}{2}, y\right)\right\}\right| \geq e^{-\lambda} P(G_n^\theta).$$

To see the last claim, pick some path from $(n + \frac{1}{2}, 0)$ to $(\frac{1}{2}, y)$, which stays in $H(\frac{1}{2}) = \{(x, y)\colon x > \frac{1}{2}\}$ and ends at a point $y_0 \in [-\theta n, \theta n]$. The existence of such a path is measurable with respect to the graphical representation restricted to $H(\frac{1}{2})$ and if there are no arrows from 0 to 1 in $[y_0, y_0 + 1]$ we can reach all the points $(\frac{1}{2}, y)$ with $y \in [y_0, y_0 + 1]$. The probability of the last event is $e^{-\lambda}$ so we have shown $(*)$ and the result follows. □

From Lemma 4 it follows immediately that we have

COROLLARY 5. *As $n \to \infty$,*

$$\frac{1}{n} \log P(\hat{\tau}^{\{1, \dots, n\}} < \infty) \to -\gamma.$$

PROOF.

$$G_n \supset \{\hat{\tau}^{\{1, \dots, n\}} < \infty\} \supset \{\tau^{\{1, \dots, n\}} < \infty\},$$

so the result follows from Lemmas 3 and 4. □

Combining the last Corollary with Lemma 3 shows that the limits in (a), (b) and (c) of Theorem 2 exist and are equal so it only remains to prove part (d).

LEMMA 6. *As $N \to \infty$,*

$$(\log \sigma_N)/N \to \gamma, \quad \textit{in probability.}$$

PROOF. In Durrett and Liu (1988) it was shown that for any $\varepsilon > 0$,

$$P\left((\log \sigma_N)/N > \gamma + \varepsilon\right) \to 0,$$

as $N \to \infty$, so all we have to do is prove the other bound. Now

$$P(\sigma_N \leq t) \leq P\Big(\left(N + \tfrac{1}{2}, x\right) \to \left(\tfrac{1}{2}, y\right) \text{ in } \left[\tfrac{1}{2}, N + \tfrac{1}{2}\right] \times [0, \infty) \quad \text{for some } x, y \leq t\Big)$$

and Lemma 4 implies that if N is large

$$\int_0^{t+1} dx\, P\Big(\text{for some } y \left(N + \tfrac{1}{2}, x\right) \to \left(\tfrac{1}{2}, y\right) \text{ in } \left[\tfrac{1}{2}, N + \tfrac{1}{2}\right] \times R\Big) \leq (t + 1)\exp(-\gamma(1 - \varepsilon)N).$$

The preceding integral is

$$E\left|\left\{x\colon \text{there is a } y \text{ with } \left(N + \tfrac{1}{2}, x\right) \to \left(\tfrac{1}{2}, y\right) \text{ in } \left[\tfrac{1}{2}, N + \tfrac{1}{2}\right] \times R\right\}\right|,$$

where again $|\{\cdots\}|$ denotes the Lebesgue measure of the indicated set. To convert the last estimate into a bound on $P(\sigma_N \le t)$, we observe that the existence of a path from $\{N+\frac{1}{2}\} \times [0, t]$ to $\{\frac{1}{2}\} \times [0, t]$ in $[\frac{1}{2}, N+\frac{1}{2}] \times [0, \infty)$ is measurable with respect to the graphical representation restricted to $[\frac{1}{2}, N+\frac{1}{2}] \times [0, \infty)$, and if there is no arrow from $N+1$ to N in $[x_0, x_0+1]$, then all the points $(N+\frac{1}{2}, x)$ with $x_0 \le x \le x_0 + 1$ are the starting points of paths to $(\frac{1}{2}, \cdot)$ so

$$\begin{aligned} E\Big|\Big\{x \in [0, t+1]\colon &\text{there is a } y \text{ with } \left(N+\tfrac{1}{2}, x\right) \to \left(\tfrac{1}{2}, y\right) \\ &\qquad\qquad \text{in } \left[\tfrac{1}{2}, N+\tfrac{1}{2}\right] \times [0, \infty)\Big\}\Big| \\ \ge e^{-\lambda} P\Big\{&\text{there are } x, y \le t \text{ with } \left(N+\tfrac{1}{2}, x\right) \to \left(\tfrac{1}{2}, y\right) \\ &\qquad\qquad \text{in } \left[\tfrac{1}{2}, N+\tfrac{1}{2}\right] \times [0, \infty)\Big\}. \qquad \square \end{aligned}$$

4. Proof of part (a) of Theorem 3. Let $V_y = \{(x, y)\colon 1 \le x \le N\}$, let $\sigma_N = \sup\{y\colon$ there is a path from V_0 to V_y in $[1, N] \times [0, \infty)\}$, and let $\beta_N = \inf\{t\colon P(\sigma_N > t) \le e^{-1}\}$. To prove

(a) $$P(\sigma_N/\beta_N \ge x) \to e^{-x},$$

it is enough to show

(â) for any $x, t > 0$,

$$|P(\sigma_N > \beta_N(t+s)) - P(\sigma_N > \beta_N s) P(\sigma_N > \beta_N t)| \to 0, \quad \text{as } N \to \infty.$$

To do this, we use a little bit of the renormalized bond construction of Durrett (1984) and Durrett and Schonmann (1987). First recall the definition of the edge speed $\alpha(\lambda)$. Let $r_t = \sup \xi_t^{(-\infty, 0]}$, i.e., the rightmost particle at time t when the contact process starts with all nonpositive integers occupied. Durrett (1980) showed that

$$\text{as } t \to \infty, \quad r_t/t \to \alpha(\lambda) \quad \text{almost surely}$$

and

$$\lambda_c = \inf\{\lambda\colon \alpha(\lambda) > 0\}.$$

Let $s > 0$, write α for $\alpha(\lambda)$ and suppose N is large enough so that $\beta_N s > 2N/\alpha$ (it follows from Theorem 2 that $(\log \beta_N)/N \to \gamma > 0$). Let $L = N/0.9\alpha$. Let R_2 be a parallelogram with vertices $(-0.1\alpha L, \beta_N s - 0.5L)$, $(0, \beta_N s - 0.5L)$, $(N + 0.1\alpha L, \beta_N s + 0.5L)$ and $(N, \beta_N s + 0.5L)$. Let R_1 be a parallelogram with vertices $(N, \beta_N s + 0.3L)$, $(N + 0.1\alpha L, \beta_N s + 0.3L)$, $(0, \beta_N s + 1.3L)$ and $(-0.1\alpha L, \beta_N s + 1.3L)$. Let R_3 be a parallelogram with vertices $(-0.1\alpha L, \beta_N s - 0.3L)$, $(0, \beta_N s - 0.3L)$, $(N + 0.1\alpha L, \beta_N s - 1.3L)$ and $(N, \beta_N s - 1.3L, N)$. See Figure 5 for a picture. Let $A_{N,s}$ be the event that there is a path from the bottom to the top of all the parallelograms. When $A_{N,s}$ occurs a path from V_0 to $V_{\beta_N s}$ and a path from $V_{\beta_N s}$ to $V_{\beta_N(s+t)}$ can be connected using

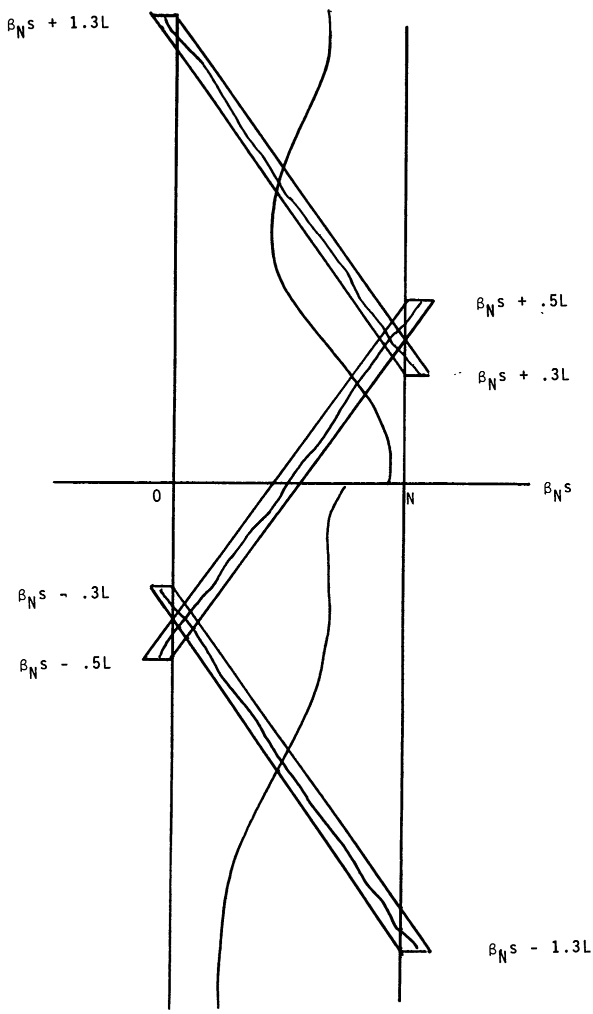

FIG. 5.

paths in the tubes so

$$\begin{aligned} P\big(V_0 \to V_{\beta_N s}\big)P\big(V_{\beta_N s} \to V_{\beta_N(s+t)}\big) &\ge P\big(V_0 \to V_{\beta_N(s+t)}\big) \\ &\ge P\big(V_0 \to V_{\beta_N(s+t)}, A_{N,s}\big) \\ &= P\big(V_0 \to V_{\beta_N s}, V_{\beta_N s} \to V_{\beta_N(s+t)}, A_{N,s}\big) \\ &\ge P\big(V_0 \to V_{\beta_N s}\big)P\big(V_{\beta_N s} \to V_{\beta_N(s+t)}\big)P\big(A_{N,s}\big), \end{aligned}$$

where the last line follows from Harris' inequality. In other words,

$$\begin{aligned} P(\sigma_N > \beta_N s)P(\sigma_N > \beta_N t) &\ge P(\sigma_N > \beta_N(s+t)) \\ &\ge P(\sigma_N > \beta_N s)P(\sigma_N > \beta_N t)P(A_{N,s}), \end{aligned}$$

or rearranging the last line

$$0 \le P(\sigma_N > \beta_N(s+t)) - P(\sigma_N > \beta_N s)P(\sigma_N > \beta_N t) \le P(A^c_{N,s}).$$

As $N \to \infty$, $P(A^c_{N,s}) \to 0$ (see Durrett (1984), Section 9, or Durrett and Schonmann (1987)) so we have proved (a).

To extend the argument above to (unoriented) bond percolation, use the same approach but define $A_{N,s}$ to be the intersection of the events:

$$\{\text{there is a left to right crossing of } [1, N] \times [\beta_N s - N, \beta_N s]\},$$

$$\{\text{there is a left to right crossing of } [1, N] \times [\beta_N s, \beta_N s + N]\},$$

$$\{\text{there is a top to bottom crossing of } [1, N] \times [\beta_N s - N, \beta_N s + N]\}.$$

When $p > \frac{1}{2}$ the last three events have probabilities that approach 1 as $N \to \infty$ and when they occur a path from V_0 to $V_{\beta_N s}$ and a path from $V_{\beta_N s}$ to $V_{\beta_N(s+t)}$ can be connected, so repeating the preceding arguments proves (â) and (a) follows. □

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