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A NOTE ON CAPACITARY MEASURES OF SEMIPOLAR SETS

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For a certain class of Markov processes, the λ-capacitary measure π_S^λ of a semipolar set S has the following property under a mild condition: A subset B of S is polar if and only if $\pi_S^\lambda(B) = 0$.

1. Introduction. Among results on semipolar sets, Dellacherie, Feyel and Mokobodzki's result (1982) is significant: A Borel set B is semipolar if and only if there exists a σ-finite measure m such that, for $A \subset B$, $m(A) = 0$ if and only if A is polar, under the hypothesis that B can contain none of its regular points. We call the measure m the DFM measure of B. In this note, without appealing to the DFM theory, we shall show that the λ-capacitary measure of a set S is a DFM measure of S if $S \subset \{x;\ P_x(T_S < \infty) < \varepsilon,\ \hat{P}_x(\hat{T}_S < \infty) < \varepsilon\}$ for some ε, $0 < \varepsilon < 1$. The processes discussed here are required to satisfy stronger hypotheses than those in DFM. In particular, the duality hypotheses are essential in our proof.

2. Capacitary measures of semipolar sets. The strong (classical) duality hypotheses in Chapter VI of Blumenthal and Getoor (1968) concern a pair of standard processes $X = (X_t)$ and $\hat{X} = (\hat{X}_t)$ with common state space $(E, \mathscr{E})$ and a σ-finite measure ξ. We assume that E is locally compact with a countable base. Writing $\xi(dx) = dx$, these duality hypotheses assert that the resolvents $(U^\lambda), (\hat{U}^\lambda)$ of X, $\hat{X}$, respectively, may be expressed in terms of potential kernel densities $u^\lambda(x, y)$ satisfying

(1) $U^\lambda(x, dy) = u^\lambda(x, y)\,dy$, $\hat{U}^\lambda(x, dy) = u^\lambda(y, x)\,dy$ for every $x \in E$;
(2) for every $y \in E$, $x \to u^\lambda(x, y)$ is λ-excessive relative to X and $x \to u^\lambda(y, x)$ is λ-excessive relative to $\hat{X}$;
(3) $(x, y) \to u^\lambda(x, y)$ is in $\mathscr{E}^* \times \mathscr{E}^*$

for each $\lambda > 0$. Here $\mathscr{E}^*$ is the σ-algebra of universally measurable subsets of E.

Recent capacity theory is far developed beyond the scope of the classical duality theory as is seen in Getoor (1984). In addition to the classical results, applying Getoor's modern theory to our case, our conditions (i) and (ii) enable us *to get a λ-capacitary measure for every Borel subset of E*. More precisely, for every Borel subset B and every $\lambda > 0$, there exist unique σ-finite measures π_B^λ, $\hat{\pi}_B^\lambda$ such that

$$E_x(\exp(-\lambda T_B)) = \int u^\lambda(x, y)\pi_B^\lambda(dy), \qquad \hat{E}_x(\exp(-\lambda \hat{T}_B)) = \int u^\lambda(y, x)\hat{\pi}_B^\lambda(dy).$$

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The measure π_B^λ is carried by $B \cup B^{c0-r}$ and $\hat{\pi}_B^\lambda$ is carried by $B \cup B^r$. Here T_B (resp. $\hat{T}_B$) is the hitting time of B defined by $T_B = \inf(t > 0,\ X_t \in B)$ [resp. $\hat{T}_B = \inf(t > 0,\ \hat{X}_t \in B)$]. The set B^r (resp. B^{c0-r}) denotes the set of regular points for B with respect to X (resp. $\hat{X}$). The total masses of π_B^λ and $\hat{\pi}_B^\lambda$ are identical, which we call *the λ-capacity of B, denoted by* $C^\lambda(B)$ *or* $C_X^\lambda(B)$. $C^\lambda(B)$ *may take the value* ∞. The λ-capacitary measure π_B^λ and the λ-capacity play an essential role in this paper. Our study is in the scope of the classical theory. A polar set is characterized as a set whose compact subsets have λ-capacity 0 for all $\lambda > 0$.

PROPOSITION. *Let* $X = (X_t)$ *and* $\hat{X} = (\hat{X}_t)$ *be standard processes on* $(E, \mathscr{E})$ *satisfying* (i) X *and* $\hat{X}$ *are in strong duality with respect to a σ-finite measure* ξ; (ii) $\lambda U^\lambda 1 = \lambda \hat{U}^\lambda 1 = 1$ *for all* $\lambda > 0$. *Consider a Borel set S such that, for some* ε, $1 \geq \varepsilon > 0$,

$$S \subset \left\{x;\ P_x(T_S < \infty) < \varepsilon,\ \hat{P}_x\left(\hat{T}_S < \infty\right) < \varepsilon\right\}.$$

Then there exists a σ-finite measure π *on S that satisfies*:

> *For every Borel subset B of S, B is polar, if and only if* $\pi(B) = 0$.

If $\varepsilon < 1$, *then the λ-capacitary measure* π_S^λ, $\lambda > 0$, *may be chosen as the measure* π.

We shall divide the proof into several steps.

Step 1. Consider the case $0 < \varepsilon < 1$. We shall show:

(2.1) For every bounded Borel subset K of S, $\pi_S^\lambda(K)$ is bounded in λ.

More strictly, we shall prove:

(2.2) For every bounded Borel subset K of S, $C^\lambda(K)$ is bounded in λ.

In the proof the following identity plays the key role:

$$\pi_B^\lambda(dy) = \pi_B^\mu(dy) + (\lambda - \mu)\int E_x(\exp(-\mu T_B))\, dx\, \hat{E}_x\left(\exp\left(-\lambda \hat{T}_B\right),\ \hat{X}_{\hat{T}_B} \in dy\right), \tag{2.3}$$

for $\lambda > \mu > 0$ and for every Borel subset B of E.

The identity was stated as exercise 4.15 in Chapter VI of Blumenthal and Getoor (1968) under certain regularity conditions on the resolvent. A modern version of (2.3) is shown in Getoor (1984). Define a set G by

$$G = \left\{x;\ P_x(T_K < \infty) > \delta\right\},$$

for a constant δ with $\varepsilon < \delta < 1$. Then it follows from (2.3) that

$$
\begin{aligned}
C^{\lambda}(K) &\le C^{\mu}(K) + (\lambda-\mu)\int_{G^c} E_x(\exp(-\mu T_K))\,dx\,\hat{E}_x\big(\exp(-\lambda \hat{T}_K)\big) \\
&\quad + (\lambda-\mu)\int_G \hat{E}_x\big(\exp(-\lambda\hat{T}_K)\big)\,dx \qquad (2.4)\\
&\le C^{\mu}(K) + \delta\lambda\int_{G^c}\hat{E}_x\big(\exp(-\lambda\hat{T}_K)\big)\,dx + \lambda\int_G \hat{E}_x\big(\exp(-\lambda\hat{T}_K)\big)\,dx.
\end{aligned}
$$

Using condition (ii), we see

$$C^{\lambda}(K) = \lambda\int_E\int u^{\lambda}(x,y)\hat{\pi}_K^{\lambda}(dy)\,dx = \lambda\int_E \hat{E}_x\big(\exp(-\lambda\hat{T}_K)\big)\,dx.$$

Hence it follows from (2.4) that

$$(1-\delta)\lambda\int_E \hat{\pi}_K^{\lambda}(dy)\left\{\int_{G^c} u^{\lambda}(y,x)\,dx\right\} \le C^{\mu}(K), \quad \text{for } \lambda > \mu.$$

Noting that $\hat{\pi}_K^{\lambda}(dy) \ge \hat{\pi}_K^{\nu}(dy)$ for $\lambda > \nu$, we have

$$(1-\delta)\lambda\int_E \hat{\pi}_K^{\nu}(dy)\left\{\int_{G^c} u^{\lambda}(y,x)\,dx\right\} \le C^{\mu}(K), \quad \text{for } \lambda > \nu > \mu. \tag{2.5}$$

On the other hand, we have

$$\lim_{\lambda\to\infty}\lambda U^{\lambda}1_G(x) = 0, \quad \text{on } K. \tag{2.6}$$

Indeed, for $\lambda > \nu$, it follows from the resolvent equation that

$$
\begin{aligned}
E_x(\exp(-\nu T_G)) &= U^{\lambda}\pi_G^{\nu}(x) + (\lambda-\nu)\int u^{\lambda}(x,z)\int u^{\nu}(z,y)\pi_G^{\nu}(dy)\,dz \\
&\ge U^{\lambda}\pi_G^{\nu}(x) + (\lambda-\nu)\int_G u^{\lambda}(x,z)E_z(\exp(-\nu T_G))\,dz \\
&= U^{\lambda}\pi_G^{\nu}(x) + (\lambda-\nu)U^{\lambda}1_G(x).
\end{aligned}
$$

Here we have used $\xi(G\cap G^{ir}) = 0$. Hence we have

$$E_x(\exp(-\nu T_G)) \ge \limsup_{\lambda\to\infty}\lambda U^{\lambda}1_G(x).$$

But $K \subset G^{ir}$, because $P_{X(t)}$ $(T_K < \infty)$ takes positive time to get from ε to δ. Hence $\lim_{\nu\to\infty}E_x(\exp(-\nu T_G)) = 0$. Since $\lim_{\lambda\to\infty}\lambda U^{\lambda}1 = 1$, we have, at the same time,

$$\lim_{\lambda\to\infty}\lambda\int_{G^c}u^{\lambda}(x,y)\,dy = 1, \quad \text{for } x \in K. \tag{2.7}$$

Combining (2.5) with (2.7) and using Fatou's lemma, we get

$$(1-\delta)C^{\nu}(K) \le C^{\mu}(K), \quad \text{for } \nu > \mu. \tag{2.8}$$

We have proved (2.2). Since $\pi_S^{\lambda}(K) \le C^{\lambda}(K)$, (2.1) follows from (2.2). *The statement* (2.1) *together with the monotone property of* π_S^{λ} *with respect to* λ

implies that π_S^λ *has a vague limit, which we denote by* π_S^∞. At the same time we see that π_B^λ, $\hat{\pi}_B^\lambda$ have vague limits π_B^∞, $\hat{\pi}_B^\infty$, respectively, for each Borel subset B of S.

Step 2. Let S and π_S^∞ be as in Step 1. We shall show

$$(1-\delta)\langle f, \pi_S^\infty\rangle \le \langle f, \pi_S^\mu\rangle \le \langle f, \pi_S^\infty\rangle, \tag{2.9}$$

for every nonnegative continuous function f of compact support, where $\langle f, \pi\rangle = \int f(x)\pi(dx)$. For the estimate we prepare

$$\pi_B^\lambda(dy) = \lambda \int_E dx\, \hat{E}_x\big(\exp(-\lambda \hat{T}_B),\ \hat{X}_{\hat{T}_B} \in dy\big), \tag{2.10}$$

for every Borel subset of E.

Using condition (ii) together with the switching identity, we have

$$E_x(\exp(-\lambda T_B)) = E_x\big(\exp(-\lambda T_B)\lambda U^\lambda 1(X_{T_B})\big)$$
$$= \lambda \int u^\lambda(x, z)\left[\int_E dy\, \hat{E}_y\big(\exp(-\lambda \hat{T}_B),\ \hat{X}_{\hat{T}_B} \in dz\big)\right].$$

So (2.10) follows from the uniqueness of measures of potentials. Now set

$$H = \{x;\ P_x(T_S < \infty) > \delta\}.$$

Using (2.3) again, we get

$$\langle f, \pi_S^\lambda\rangle \le \langle f, \pi_S^\mu\rangle + \lambda\delta \int_{H^c} \hat{E}_x\big(\exp(-\lambda \hat{T}_S) f(\hat{X}_{\hat{T}_S})\big)\, dx$$
$$+ \| f \| \lambda \int_H \hat{E}_x\big(\exp(-\lambda \hat{T}_S),\ \hat{X}_{\hat{T}_S} \in B\big)\, dx,$$

where $B = \operatorname{supp}(f) \cap S$. The second term on the right is smaller than $\delta\langle f, \pi_S^\lambda\rangle$ by (2.10). The third term is smaller than $\| f \|\lambda \int_H \hat{E}_x(\exp(-\lambda \hat{T}_B))\, dx$. But

$$\lambda \int_H \hat{E}_x\big(\exp(-\lambda \hat{T}_B)\big)\, dx = \lambda \int U^\lambda 1_H(x) \hat{\pi}_B^\lambda(dx) \le \lambda \int U^\lambda 1_H(x) \hat{\pi}_B^\infty(dx),$$

and we can prove $\lim_{\lambda \to \infty} \lambda U^\lambda 1_H(x) = 0$ in the same way as in the proof of (2.6) for every $x \in S$. Since $\hat{\pi}_B^\infty$ is a bounded measure on E and $\lambda U^\lambda 1_H \le 1$, the bounded convergence theorem assures that $\lim_{\lambda \to \infty} \lambda \int U^\lambda 1_H(x) \hat{\pi}_B^\infty(dx) = 0$. Hence we have $\lim_{\lambda \to \infty} \lambda \int_H \hat{E}_x(\exp(-\lambda \hat{T}_B))\, dx = 0$. Here we remark a bit on the support of the limit measure π_B^∞ and $\hat{\pi}_B^\infty$ in case B is a bounded subset of S. Since $S \subset S^{ir} \cap S^{c0-ir}$, $\operatorname{supp}(\pi_B^\lambda) \subset B$ and $\operatorname{supp}(\hat{\pi}_B^\lambda) \subset B$. So $\lim_{\lambda \to \infty} \pi_B^\lambda(B) =$ the total mass of $\pi_B^\infty = \pi_B^\infty(\overline{B})$, because B is bounded. Since $\pi_B^\lambda(B) \le \pi_B^\infty(\overline{B})$, we must have $\pi_B^\infty(\overline{B}) = \pi_B^\infty(B)$. Note that this fact is used in the preceding. At the same time we get $\lim_{\lambda \to \infty} \pi_B^\lambda(B) = \pi_B^\infty(B)$. The same is true for $\hat{\pi}_B^\infty$. Anyway we finished the proof of the first inequality of (2.9). The second inequality is obvious.

Step 3. We shall continue to study the set S and π_S^∞. We show

$$\pi_S^\infty(B) = \pi_B^\infty(B), \quad \text{for every bounded Borel subset } B \text{ of } S. \tag{2.11}$$

Choose an arbitrary disjoint pair of bounded Borel subsets A, B of S. Then, noting $S \subset S^{ir}$,

$$\begin{aligned} &E_x\big(\exp(-\lambda T_S),\ X_{T_S} \in A \cup B\big) \\ &\quad = E_x(\exp(-\lambda T_S),\ T_S = T_A) + E_X(\exp(-\lambda T_S),\ T_S = T_B) \\ &\quad = E_x(\exp(-\lambda T_A),\ T_S = T_A) + E_x(\exp(-\lambda T_B),\ T_S = T_B) \\ &\quad = E_x(\exp(-\lambda T_A)) - E_x(\exp(-\lambda T_A),\ T_S < T_A) \\ &\qquad + E_x(\exp(-\lambda T_B)) - E_x(\exp(-\lambda T_B),\ T_S < T_B). \end{aligned}$$

But

$$\begin{aligned} E_x(\exp(-\lambda T_A),\ T_S < T_A) &\le E_x(\exp(-\lambda T_S))E_{X_{T_S}}(\exp(-\lambda T_A)) \\ &= \int \pi_A^\lambda(dz)\hat{E}_z\big(\exp(-\lambda \hat{T}_S),\ \hat{X}_{\hat{T}_S} \in dy\big)u^\lambda(x, y). \end{aligned}$$

A similar inequality holds for $E_x(\exp(-\lambda T_B),\ T_S < T_B)$. Integrating with respect to $\lambda\ dx$ and using condition (ii), it follows that

$$\begin{aligned} &\lambda \int dx\, E_x\big(\exp(-\lambda T_S),\ X_{T_S} \in A \cup B\big) \\ &\quad \ge C^\lambda(A) + C^\lambda(B) - \int \pi_A^\lambda(dz)\hat{E}_z\big(\exp(-\lambda \hat{T}_S)\big) - \int \pi_B^\lambda(dz)\hat{E}_z\big(\exp(-\lambda \hat{T}_S)\big) \\ &\quad \ge C^\lambda(A) + C^\lambda(B) - \int \pi_A^\infty(dz)\hat{E}_z\big(\exp(-\lambda \hat{T}_S)\big) - \int \pi_B^\infty(dz)\hat{E}_z\big(\exp(-\lambda \hat{T}_S)\big). \end{aligned}$$

Using (2.10) as the dual form, we have $\hat{\pi}_S^\lambda(A \cup B) \ge \hat{\pi}_A^\lambda(A) + \hat{\pi}_B^\lambda(B) -$ {the remaining term} and hence $\hat{\pi}_S^\infty(A \cup B) \ge \hat{\pi}_A^\lambda(A) + \hat{\pi}_B^\lambda(B) -$ {the remaining term}. Since $S \subset S^{c0-ir}$, we get $\hat{\pi}_S^\infty(A \cup B) \ge \hat{\pi}_A^\infty(A) + \hat{\pi}_B^\infty(B)$ by tending λ to ∞. (Recall the remark at the end of Step 2.) As the dual form we get $\pi_S^\infty(A \cup B) \ge \pi_A^\infty(A) + \pi_B^\infty(B)$. However, $\pi_S^\lambda(A) \le \pi_A^\lambda(A)$ in general. We must have $\pi_S^\infty(B) = \pi_B^\infty(B)$. The proof of (2.11) is finished.

Step 4. Now we prove the proposition in the case $\varepsilon < 1$. It is sufficient if we prove the statement in case B is compact. If $\pi_S^\lambda(B) = 0$, then $\pi_S^\infty(B) = 0$ by (2.9). Hence $\pi_B^\infty(B) = 0$ by (2.11). Then $C^\lambda(B) = 0$ for all $\lambda > 0$. The converse is obvious.

Step 5. Consider the case $\varepsilon = 1$. Let $S_n = S \cap \{x;\ P_x(T_S < \infty) < 1 - 1/n,\ \hat{P}_x(\hat{T}_S < \infty) < 1 - 1/n\}$. Clearly, $S_n \uparrow S$. It follows from (2.11) that $\pi_{S_n}^\infty|_{S_m} = \pi_{S_m}^\infty$ if $n > m$, where $\pi|_A$ denotes the restriction of the measure π to A. Hence we can define the limit measure π so that $\pi|_{S_n} = \pi_{S_n}^\infty$ for every n. It is clear that $\pi(B) = 0$ if and only if B is polar in case B is a Borel subset of S. The proof of the proposition is complete.

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Note added. After this paper was submitted, the author discovered that (2.2) had been proved for a class of Markov processes by Rao (1987). Rao's proof is different from the one given here.

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