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AN EXTENDED VERSION OF THE ERDÖS–RÉNYI STRONG LAW OF LARGE NUMBERS

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Consider a sequence $X_1, X_2, \ldots,$ of i.i.d. random variables. For each integer $m \geq 1$ let S_m denote the mth partial sum of these random variables and set $S_0 = 0$. Assuming that $EX_1 \geq 0$ and the moment generating function ϕ of X_1 exists in a right-neighborhood of 0 the Erdös–Rényi strong law of large numbers states that whenever $k(n)$ is a sequence of positive integers such that $\log n/k(n) \sim c$ as $n \to \infty$, where $0 < c < \infty$ then $\max\{(S_{m+k(n)} - S_m)/(\gamma(c)k(n)):\ 0 \leq m \leq n - k(n)\}$ converges almost surely to 1, where $\gamma(c)$ is a constant depending on c and ϕ. An extended version of this strong law is presented which shows that it remains true in a slightly altered form when $\log n/k(n) \to \infty$.

1. Introduction and statement of results. Let $X_1, X_2, \ldots,$ be a sequence of independent random variables with common distribution function F such that

(i) X_1 is nondegenerate with $0 \leq EX_1 < \infty$ and
(ii) $\sup\{t\colon \phi(t) = E\exp(tX_1) < \infty\} > 0$.

Note that (i) implies that

$$0 < \omega = \sup\{x\colon F(x) < 1\} \leq \infty.$$

Here and elsewhere in this paper $k(n)$ will denote a sequence of nondecreasing positive integers such that $1 \leq k(n) \leq n$ for integers $n \geq 1$. For any such sequence $k(n)$ let

$$M_n(k(n)) = \max_{0 \leq m \leq n-k(n)} \{S_{m+k(n)} - S_m\},$$

where $S_0 = 0$ and $S_m = X_1 + \cdots + X_m$ for $m \geq 1$.

The following two functions play an essential role in describing the asymptotic limiting behavior of $M_n(k(n))$ when $k(n) = O(\log n)$. Set for $0 \leq z < \infty$,

$$\zeta(z) = \sup\{zt - \log\phi(t)\colon t \geq 0 \text{ and } \phi(t) < \infty\}.$$

It is well known that ζ is an extended real valued convex function on $[0, \infty)$ with $\zeta(0) = 0$ and $\zeta(z) \to \infty$ as $z \to \infty$, which when $\omega = \infty$, is finite on $[0, \infty)$ and when $0 < \omega < \infty$ is finite on $[0, \omega)$ and infinite on (ω, ∞). For any $0 \leq x < \infty$ let

$$\gamma(x) = \sup\{z\colon \zeta(z) \leq x\}.$$

From the properties of ζ, it is readily deduced that γ is concave, continuous and nondecreasing on $[0, \infty)$ with $\gamma(x) \to \omega$ as $x \to \infty$.

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Erdös and Rényi (1970) proved that if $c(n) = \log n/k(n) \to c$ as $n \to \infty$ for some $0 < c < \infty$, then

$$\lim_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} = 1 \quad \text{a.s.} \tag{1.1}$$

Recently some remarkable refinements of the Erdös–Rényi strong law of large numbers (1.1) have been achieved by Deheuvels, Devroye and Lynch (1986) and Deheuvels and Devroye (1987). Also consult S. Csörgő (1979) for a unifying approach to Erdös–Rényi strong laws in general and to see how $M_n(k(n))$ behaves when $c(n) \to 0$ refer to Theorem 3.1.1 on page 115 of M. Csörgő and Révész (1981).

The purpose of this paper is to present an extension of the Erdös–Rényi strong law of large numbers to sequences $k(n)$ for which $c(n) \to \infty$. It will be seen that essentially (1.1) continues to remain true in this case, except that sometimes lim must be replaced by lim sup. Our main results are stated in the following theorem and their relationship with other work on this problem is discussed in Remark 3 below.

THEOREM. *Let $X_1, X_2, \ldots,$ be a sequence of independent random variables with common distribution function F satisfying* (i) *and* (ii).

(a) *If* $0 < \omega < \infty$, *then for all sequences $k(n)$ such that $c(n) \to \infty$ as $n \to \infty$,*

$$\lim_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} = \lim_{n\to\infty} \frac{M_n(k(n))}{k(n)\omega} = 1 \quad a.s. \tag{1.2}$$

(b) *If $\omega = \infty$, then for all sequences $k(n)$ such that $c(n) \to \infty$ as $n \to \infty$,*

$$\limsup_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} = 1 \quad a.s. \tag{1.3}$$

Moreover, the lim sup *in* (1.3) *can be replaced by* lim *for all such sequences $k(n)$ if and only if*

$$\lim_{z\to\infty} \gamma(-\log(1 - F(z)))/z = 1 \tag{1.4}$$

if and only if

$$\lim_{n\to\infty} \max_{1\le m\le n} X_m/\gamma(\log n) = 1 \quad a.s. \tag{1.5}$$

The proof is postponed until Section 2.

REMARK 1. Some examples of distributions which satisfy (1.4) are the normal, geometric, Poisson and Weibull with shape parameter $a \ge 1$. Not all distribution functions, however, satisfying (i) and (ii) with $\omega = \infty$ also fulfill (1.4). Consider for instance

$$-\log(1 - F(x)) = x + 2^{-1}x \sin^2(\log x) \quad \text{for } x \ge 1.$$

Here, $\gamma(z) \sim z$, as $z \to \infty$, so that (1.4) is obviously not satisfied, yet (i) and (ii) with $\omega = \infty$ hold. For this distribution function we necessarily have

$$0 \le \liminf_{n \to \infty} M_n(1)/\gamma(\log n) < 1 \quad \text{a.s.}$$

These examples were worked out by using the fact that when $\omega = \infty$, ζ is asymptotically equivalent as $z \to \infty$ to the greatest convex minorant of $-\log(1 - F(z -))$. This fact is easily derived from Lemma 2.1 below.

REMARK 2. It is shown in Lemma 2.6 below that assuming $\omega = \infty$, (1.4) is equivalent to

$$\lim_{x \to \infty} \gamma(\log x)/Q(1 - 1/x) = 1, \tag{1.6}$$

where for $0 < s < 1$, $Q(s) = \inf\{x\colon\ F(x) \ge s\}$. Whenever $\omega = \infty$, an easily verified sufficient condition for (1.4) is

$$\lim_{x \to \infty} -\log(1 - F(x))/\zeta(x) = 1. \tag{1.7}$$

On the other hand, this condition is not necessary for (1.4). Let

$$-\log(1 - F(x)) = \exp([x] + 1), \qquad 0 \le x < \infty,$$

and equal to 0 for $x < 0$ with $[x]$ denoting the integer part of x. The greatest convex minorant g of $-\log(1 - F(x -))$ is

$$g(x) = \begin{cases} e^{[x]} + (e^{[x]+1} - e^{[x]})(x - [x]), & 1 \le x < \infty, \\ xe, & 0 \le x < 1, \\ 0, & x < 0. \end{cases}$$

We see that

$$\limsup_{x \to \infty} -\log(1 - F(x))/g(x) = \lim_{n \to \infty} -\log(1 - F(n))/g(n) = e,$$

which since $\zeta(x) \sim g(x)$ shows that (1.7) does not hold. Whereas it is easily verified that

$$\gamma(x) \sim \log x \quad \text{and} \quad \log(-\log(1 - F(z))) \sim z,$$

showing that (1.4) is satisfied.

REMARK 3. Book and Shore (1978) and Steinebach (1983) proved a version of this theorem. They assumed among other conditions, a condition that implies that $-\log(1 - F(x)) \sim x^2/(2\sigma^2)$ for some $\sigma^2 > 0$. In this situation, assuming (i), (1.3) of our theorem holds with lim replacing lim sup and $k(n)\gamma(c(n)) \sim (2\sigma^2 k(n)\log n)^{1/2}$. This agrees with Theorem C of Steinebach (1983). de Acosta and Kuelbs (1983) have also obtained Erdös–Rényi type results for the case $c(n) \to \infty$ assuming that X_1 has Weibull-like upper tails with shape parameter $a > 1$.

2. Proof of Theorem. First we consider part (a). Obviously, proving (1.2) is equivalent to showing that

$$\liminf_{n\to\infty} \frac{M_n(k(n))}{k(n)\omega} \geq 1 \quad \text{a.s.} \tag{2.1}$$

Choose any $\varepsilon > 0$ such that $\omega - \varepsilon > 0$ and $1 > P(X_1 > \omega - \varepsilon) = \delta > 0$. Notice that for each integer $n \geq 1$,

$$P(S_{k(n)} \leq (\omega - \varepsilon)k(n)) \leq 1 - P(X_1 > \omega - \varepsilon)^{k(n)} = 1 - \delta^{k(n)}, \tag{2.2}$$

and

$$P(M_n(k(n)) \leq (\omega - \varepsilon)k(n)) \leq P(S_{k(n)} \leq (\omega - \varepsilon)k(n))^{n/k(n)-1},$$

which by (2.2) is less than or equal to

$$(1 - \delta^{k(n)})^{n/k(n)-1} \leq \exp(-n\delta^{k(n)}/k(n))/(1-\delta).$$

Now since $c(n) = \log n/k(n) \to \infty$ as $n \to \infty$ implies

$$(\log n - \log k(n) + k(n)\log\delta)/\log n \to 1 \quad \text{as } n \to \infty,$$

we have

$$\sum_{n=1}^{\infty} \exp(-n\delta^{k(n)}/k(n)) < \infty.$$

This gives by the Borel–Cantelli lemma

$$P(M_n(k(n)) \leq (\omega - \varepsilon)k(n), \text{i.o.}) = 0,$$

which finishes the proof of (2.1) and thus of part (a).

Before proceeding with the proof of part (b), we must establish a couple of facts about ζ.

Let

$$H = \{h\colon h \text{ is convex and nondecreasing on } (-\infty, \infty) \text{ with } E\exp(h(X_1)) < \infty\},$$

and set for $0 \leq z < \infty$,

$$\zeta^*(z) = \sup_{h\in H} \{h(z) - \log E\exp(h(X_1))\}.$$

LEMMA 2.1. $\zeta = \zeta^*$.

PROOF. First observe that $\zeta^* \geq \zeta$. Next choose any $0 \leq z < \infty$ and $h \in H$. Since h is convex and nondecreasing we can find a $0 \leq t < \infty$ and $-\infty < b < \infty$ such that the function $l(x) = tx + b$, satisfies $l(z) = h(z)$ and $l(x) \leq h(x)$ for all $-\infty < x < \infty$. Obviously $l \in H$ with $e^b\phi(t) = E\exp(l(X_1)) \leq E\exp(h(X_1)) < \infty$, which implies that

$$h(z) - \log E\exp h(X_1) \leq tz - \log\phi(t).$$

This shows that $\zeta \geq \zeta^*$ and hence $\zeta = \zeta^*$. □

LEMMA 2.2. *Whenever* $\omega = \infty$,

$$\liminf_{z\to\infty} -\log(1 - F(z-))/\zeta(z) = 1. \tag{2.3}$$

PROOF. First note that $-\log(1 - F(z-)) \geq \zeta(z)$. Suppose that for some $\varepsilon > 0$ and $0 < a < \infty$, for all $z \geq a$,

$$-\log(1 - F(z-))/\zeta(z) > 1 + \varepsilon. \tag{2.4}$$

Since ζ is a finite, nonnegative, nondecreasing and convex function on $[0, \infty)$ with $\zeta(0) = 0$, we can write

$$\zeta(z) = \int_0^z f(x)\, dx \quad \text{for } 0 \leq z < \infty,$$

where f is nonnegative and nondecreasing. Set for $0 \leq z < \infty$,

$$\psi(z) = \int_0^z f(x)\{1 + \varepsilon I(x > a)\}\, dx$$

and let $\psi(z) = 0$ for $z < 0$. Obviously, ψ is convex and nondecreasing on $(-\infty, \infty)$. Note that on account of (2.4) and $\zeta(z) \to \infty$ as $z \to \infty$ we have for all large enough z,

$$\zeta(z) < \psi(z) = \zeta(z) + \varepsilon(\zeta(z) - \zeta(a)) < -\log(1 - F(z-)). \tag{2.5}$$

Since for $s < 1$,

$$\int_0^\infty (1 - F(x-))^{-s}\, dF(x) < \infty,$$

it is easy to show using inequality (2.5) that for any $0 \leq s < 1$, the function $s\psi$ is a member of H. Hence for all $0 \leq z < \infty$ and $0 \leq s < 1$,

$$\zeta(z) \geq s\psi(z) - \log E \exp(s\psi(X_1)),$$

which implies by the fact that $\zeta(z) \to \infty$ as $z \to \infty$,

$$\limsup_{z\to\infty} \psi(z)/\zeta(z) \leq 1. \tag{2.6}$$

However from (2.5) we have

$$\lim_{z\to\infty} \psi(z)/\zeta(z) = 1 + \varepsilon,$$

which contradicts (2.6). Therefore (2.3) must be true. □

We are now prepared to prove (1.3) for any sequence $k(n)$ satisfying $c(n) \to \infty$. We shall show first that for all such sequences

$$\limsup_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} \leq 1 \quad \text{a.s.} \tag{2.7}$$

LEMMA 2.3. *For any integer* $k \geq 1$, $m \geq 0$ *and* $\varepsilon > 0$,

$$P(S_k \geq (1 + \varepsilon)k\gamma(m/k)) \leq \exp(-(1 + \varepsilon)m). \tag{2.8}$$

PROOF. Notice that for all $x \geq 0$,

$$P(S_k \geq x) \leq \exp(-k\zeta(x/k)).$$

Thus the left side of inequality (2.8) is less than or equal to

$$\exp(-k\zeta((1+\varepsilon)\gamma(m/k))),$$

which by convexity of ζ and $\zeta(0) = 0$ is less than or equal to

$$\exp(-(1+\varepsilon)k\zeta(\gamma(m/k))).$$

Since $\zeta(\gamma(m/k)) \geq m/k$, we have (2.8). □

LEMMA 2.4. *Whenever $\omega = \infty$, for any sequence $k(n)$ such that $c(n) \to \infty$ and for all $\varepsilon > 0$,*

$$P(M_n(k(n)) \geq (1+\varepsilon)k(n)\gamma(c(n)) \text{ i.o.}) = 0. \tag{2.9}$$

PROOF. For any integer $i \geq 1$, let $\Omega_i = \{n\colon k(n) = i\}$ and for any integer $j \geq 0$ let $\Delta_j = \{n\colon 2^j \leq n < 2^{j+1}\}$. Also for integers $i \geq 1$ and $j \geq 0$ set $\Lambda_{i,j} = \Omega_i \cap \Delta_j$ and $m_j = \#\{i\colon \Delta_{i,j} \neq \varnothing\}$. If $\Lambda_{i,j} \neq \varnothing$, let $m_{i,j} = \min\{n \in \Lambda_{i,j}\}$.

Notice that if $\Lambda_{i,j} \neq \varnothing$,

$$\begin{aligned} P\Big(\max_{n\in\Lambda_{i,j}} M_n(i)/\gamma(\log n/i) \geq (1+\varepsilon)i\Big) &\leq P\big(M_{2^{j+1}}(i) \geq (1+\varepsilon)i\gamma(\log m_{i,j}/i)\big) \\ &\leq 2^{j+1}P\big(S_i \geq (1+\varepsilon)i\gamma(\log m_{i,j}/i)\big), \end{aligned}$$

which by Lemma 2.3 and $m_{i,j} \geq 2^j$ is

$$\leq 2^{j+1}\exp(-(1+\varepsilon)\log m_{i,j}) \leq 2^{j+1}\exp(-(1+\varepsilon)j\log 2).$$

Therefore

$$\begin{aligned} P\Big(\max_{n\in\Lambda_j} M_n(k(n))/(k(n)\gamma(c(n))) \geq 1+\varepsilon\Big) &\leq m_j 2^{j+1}\exp(-(1+\varepsilon)j\log 2) = 2m_j 2^{-\varepsilon j}. \end{aligned}$$

Noting that $m_j \leq k(2^{j+1})$ and thus $c(n) \to \infty$ implies $m_j \leq (j+1)\log 2$ for all sufficiently large j, we see that the Borel–Cantelli lemma completes the proof of (2.9). □

Assertion (2.7) is a direct consequence of Lemma 2.4. The following lemma when combined with (2.7) completes the proof of the first part of (b).

LEMMA 2.5. *Whenever $\omega = \infty$, for any sequence $k(n)$ such that $c(n) \to \infty$,*

$$\limsup_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} \geq 1 \quad a.s. \tag{2.10}$$

Proof. Notice that when $\omega = \infty$, ζ is continuous on $[\gamma(0), \infty)$ [note $\gamma(0) = EX_1$] so that γ is strictly increasing on $[0, \infty)$. By Lemma 2.2 we can find a strictly increasing sequence x_r such that $x_r \geq r$ for integers $r \geq 1$ satisfying

$$\lim_{r \to \infty} -\log(1 - F(x_r -))/\zeta(x_r) = 1. \tag{2.11}$$

Choose any $0 < \varepsilon < 1$ and for integers $r \geq 1$ let

$$n_r = \max\left\{n: \gamma\left(\log n/\left((1+\varepsilon)^3 k(n)\right)\right) \leq x_r\right\}.$$

Note that necessarily

$$\gamma\left(\log(n_r + 1)/\left((1+\varepsilon)^3 k(n_r + 1)\right)\right) > x_r$$

and recalling that $k(n)$ is nondecreasing, we see that for all large enough r, $k(n_r) = k(n_r + 1)$. Also since γ is concave, for all large r, $n_r \geq r$. By continuity of γ we can find a

$$0 \leq \theta_r \leq \log(n_r + 1)/\log n_r - 1$$

such that with

$$\xi_r = (1 + \theta_r)\log n_r/\left((1+\varepsilon)^3 k(n_r)\right),$$

$\gamma(\xi_r) = x_r$. Thus for all large r and using concavity of γ

$$\begin{aligned} P\left(X_1 \geq (1+\varepsilon)^{-3}\gamma(c(n_r))\right)^{k(n_r)} &\geq P\left(X_1 \geq \gamma\left((1+\varepsilon)^{-3}c(n_r)\right)\right)^{k(n_r)} \\ &\geq P\left(X_1 \geq \gamma(\xi_r)\right)^{k(n_r)} = \left(1 - F(x_r -)\right)^{k(n_r)}, \end{aligned}$$

which by (2.11) is greater than or equal to

$$\begin{aligned} \exp(-k(n_r)(1+\varepsilon)\zeta(x_r)) &= \exp\left(-(1+\theta_r)\log n_r/(1+\varepsilon)^2\right) \\ &\geq \exp\left(-(1+\varepsilon)^{-1}\log n_r\right). \end{aligned}$$

Therefore for all large r,

$$\begin{aligned} &P\left(M_{n_r}(k(n_r)) < (1+\varepsilon)^{-3}k(n_r)\gamma(c(n_r))\right) \\ &\quad \leq \left\{1 - P\left(S_{k(n_r)} \geq (1+\varepsilon)^{-3}k(n_r)\gamma(c(n_r))\right)\right\}^{n_r/k(n_r)-1} \\ &\quad \leq \left\{1 - P\left(X_1 \geq (1+\varepsilon)^{-3}\gamma(c(n_r))\right)^{k(n_r)}\right\}^{n_r/k(n_r)-1} \\ &\quad \leq 2\left(1 - n_r^{-1/(1+\varepsilon)}\right)^{n_r/k(n_r)} \leq 2\exp\left(-n_r^{\varepsilon/(1+\varepsilon)}/k(n_r)\right). \end{aligned}$$

Since eventually $n_r \geq r$ and $c(n_r) \to \infty$, it is easy to show that

$$2\sum_{r=1}^{\infty} \exp\left(-n_r^{\varepsilon/(1+\varepsilon)}/k(n_r)\right) < \infty,$$

which by the arbitrary choice of $0 < \varepsilon < 1$ implies (2.10).

Now we turn to the proof of the second part of (b). First assume that (1.4) holds. We must show that for all $k(n)$ such that $c(n) \to \infty$,

$$\liminf_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} \geq 1 \quad \text{a.s.}$$

By (1.4) for all sufficiently large n,

$$\gamma\left\{-\log\left[1 - F\left((1+\varepsilon)^{-2}\gamma(c(n))\right)\right]\right\} < (1+\varepsilon)^{-1}\gamma(c(n)),$$

which gives

$$\begin{aligned} P\left(X_1 > (1+\varepsilon)^{-2}\gamma(c(n))\right) &= \exp\left\{\log\left[1 - F\left((1+\varepsilon)^{-2}\gamma(c(n))\right)\right]\right\} \\ &> \exp\left\{-\zeta\left[(1+\varepsilon)^{-1}\gamma(c(n))\right]\right\} \\ &\geq \exp\left(-(1+\varepsilon)^{-1}\log n/k(n)\right). \end{aligned}$$

The proof now proceeds much as the proof of Lemma 2.5. This in combination with (2.7) implies (1.3) with lim sup replaced by lim.

Next assume that for all sequences $k(n)$ such that $c(n) \to \infty$,

$$\lim_{n\to\infty} \frac{M_n(k(n))}{k(n)\gamma(c(n))} = 1 \quad \text{a.s.} \tag{2.12}$$

In particular, we then have for the choice $k(n) = 1$ for $n \geq 1$,

$$\lim_{n\to\infty} \frac{M_n(1)}{\gamma(\log n)} = \lim_{n\to\infty} \max_{1\leq m\leq n} \frac{X_m}{\gamma(\log n)} = 1 \quad \text{a.s.} \tag{2.13}$$

Set for $n \geq 2$, $\mu_n = Q(1 - 1/n)$. The limit in (2.13) holding implies that $\mu_n \sim \gamma(\log n)$ as $n \to \infty$; cf. Corollary 4.4.1 on page 227 of Galambos (1978) (it is easy to show that his continuity assumption can be removed). □

The following lemma completes the proof of the theorem.

LEMMA 2.6. *Whenever* $\omega = \infty$,

$$\gamma(\log n)/\mu_n \to 1 \quad \text{as } n \to \infty \tag{2.14}$$

if and only if (1.4) *holds.*

PROOF. From the fact that (1.4) implies (2.13), which by the above discussion, in turn implies (2.14), we see that the "if" part of the assertion is proven. Now assume that (2.14) holds. First, since γ is nonnegative, nondecreasing and concave, we can write

$$\gamma(x) = \int_0^x g(y)\,dy + \gamma(0),$$

where g is nonnegative and nonincreasing. Therefore

$$\gamma(\log(n+1)) - \gamma(\log n) = \int_{\log n}^{\log(n+1)} g(y)\,dy \leq n^{-1}g(\log n),$$

which necessarily converges to 0 as $n \to \infty$. This implies that

$$\gamma(\log(n+1))/\gamma(\log n) \to 1 \quad \text{as } n \to \infty. \tag{2.15}$$

Hence for all $x \geq 2$ for which $Q(1 - 1/[x]) > 0$,

$$\begin{aligned}\gamma(\log[x])/Q(1 - 1/([x]+1)) &\leq \gamma(\log x)/Q(1 - 1/x) \\ &\leq \gamma(\log([x]+1))/Q(1 - 1/[x]),\end{aligned}$$

which by (2.14) and (2.15) gives

$$\gamma(\log x)/Q(1 - 1/x) \to 1 \quad \text{as } x \to \infty. \tag{2.16}$$

Since $\gamma(-\log(1 - F(z))) \geq z$ and $Q(F(z)) \leq z$, we see that

$$\gamma(-\log(1 - F(z)))/Q(F(z)) \geq \gamma(-\log(1 - F(z)))/z \geq 1,$$

which by (2.16) yields (1.4). □

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