The Annals of Probability
1989, Vol. 17, No. 2, 789–808

SUMS OF INDEPENDENT RANDOM VARIABLES IN REARRANGEMENT INVARIANT FUNCTION SPACES

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Let X be a quasinormed rearrangement invariant function space on $(0,1)$ which contains $L_q(0,1)$ for some finite q. There is an extension of X to a quasinormed rearrangement invariant function space Y on $(0,\infty)$ so that for any sequence $(f_i)_{i=1}^\infty$ of symmetric random variables on $(0,1)$, the quasinorm of $\sum f_i$ in X is equivalent to the quasinorm of $\sum \mathbf{f}_i$ in Y, where $(\mathbf{f}_i)_{i=1}^\infty$ is a sequence of disjoint functions on $(0,\infty)$ such that for each i, $\mathbf{f}_i$ has the same decreasing rearrangement as f_i. When specialized to the case $X = L_q(0,1)$, this result gives new information on the quantitative local structure of L_q.

1. Introduction. In this article we formalize one aspect of the general principle that independent random variables behave like disjoint functions. To illustrate the principle, suppose that $(f_n)_{n=1}^\infty$ is a sequence of random variables. Denote by $(\mathbf{f}_n)_{n=1}^\infty$ a *disjointification of* $(f_n)_{n=1}^\infty$ on $(0,\infty)$; for example, $\mathbf{f}_n = f_n(t-[n-1])1_{(n-1,n)}$ if the f_n's are defined on $(0,1)$. The sequence $(\mathbf{f}_n)_{n=1}^\infty$ appears implicitly in many limit theorems because

$$\sum_{n=1}^\infty P[f_n > t] = \operatorname{meas}\left[\sum_{n=1}^\infty \mathbf{f}_n > t\right],$$

where meas denotes Lebesgue measure on $(0,\infty)$. The sequence $(\mathbf{f}_n)_{n=1}^\infty$ also arises naturally in moment inequalities for $\sum_{n=1}^\infty f_n$. Indeed, Rosenthal [19] proved that for $2 \le p < \infty$, if the f_n's are independent and have mean 0, then $\|\sum_{n=1}^\infty f_n\|_p$ is equivalent, up to a constant depending on p, to the expression

$$\max\left[\left(\sum_{n=1}^\infty \|f_n\|_p^p\right)^{1/p}, \left(\sum_{n=1}^\infty \|f_n\|_2^2\right)^{1/2}\right],$$

which is the same as

$$\max\left\{\|\mathbf{f}\|_{L_p(0,\infty)}, \|\mathbf{f}\|_{L_2(0,\infty)}\right\}$$

with $\mathbf{f} = \sum_{n=1}^\infty \mathbf{f}_n$. In the range $2 \le p < \infty$

$$\|f\| = \max\left\{\|f\|_{L_p(0,\infty)}, \|f\|_{L_2(0,\infty)}\right\}$$

is the natural norm on the rearrangement invariant function space

$$L_p(0,\infty) \cap L_2(0,\infty),$$

Received November 1987; revised May 1988.
[1]Supported in part by the U.S.–Israel Binational Science Foundation.
[2]Supported in part by NSF Grants DMS-85-00764 and DMS-87-03815.
AMS 1980 *subject classifications.* Primary 60G50, 46E30.
Key words and phrases. Independent random variables, moment inequalities, Rosenthal's inequality, rearrangement invariant space, uniform approximation property.

so Rosenthal's inequality says, in the language of Banach space theory, that the basic sequence $(f_n)_{n=1}^\infty$ in $L_p(0,\infty)$ is equivalent to the sequence $(\mathbf{f}_n)_{n=1}^\infty$ in $L_p(0,\infty) \cap L_2(0,\infty)$.

Recently Carothers and Dilworth [3] proved an analogous result for some of the Lorentz spaces; namely, if the f_n's are independent symmetric random variables, then for $2 < p < \infty$; $0 < q \le \infty$, $\|\Sigma_{n=1}^\infty f_n\|_{L_{p,q}(0,1)}$ is equivalent to $\|\Sigma_{n=1}^\infty \mathbf{f}_n\|_{L_{p,q}(0,\infty)\cap L_2(0,\infty)}$.

In Section 2 we extend the above results to the appropriate class of quasinormed rearrangement invariant function spaces. [A quasinorm is a function which satisfies the axioms for a norm except that the triangle inequality is replaced by

$$\|x+y\| \le K(\|x\| + \|y\|). \tag{1}$$

We denote by $K(X)$ the smallest constant K which works in (1). A quasinormed rearrangement invariant function space is a complete quasinormed vector space $(X, \|\cdot\|)$ of measurable functions on $(0,1)$ or on $(0,\infty)$ such that $\|1_A\| = 1$ if meas $A = 1$, and if g is in X, then f is in X and $\|f\| \le \|g\|$ if f is a measurable function satisfying $f^* \le g^*$, where h^* denotes the decreasing rearrangement of the function $|h|$.] In order to state the theorems precisely, we need to repeat a definition from Johnson, Maurey, Schechtman and Tzafriri [11]. Given a quasinormed rearrangement invariant space X on $(0,1)$, define two quasinormed spaces $Y = Y_X$ and $Z = Z_X$ on $(0,\infty)$ by saying $f \in Y$ (respectively, $f \in Z$) if and only if $f^*1_{(0,1)} \in X$ and $f^*1_{(1,\infty)} \in L_2(1,\infty)$ [respectively, $f^*1_{(0,1)} \in X$ and $f^*1_{(1,\infty)} \in L_1(1,\infty)$], and set

$$\|f\|_Y = \left(\|f^*1_{(0,1)}\|_X^2 + \|f^*1_{(1,\infty)}\|_2^2\right)^{1/2},$$

$$\|f\|_Z = \|f^*1_{(0,1)}\|_X + \|f^*1_{(1,\infty)}\|_1.$$

Evidently, $\|\cdot\|_Y$ and $\|\cdot\|_Z$ are quasinorms. If X is normed, both are equivalent to norms. Indeed, X then satisfies the inequality $\|\cdot\|_X \ge \|\cdot\|_1$, so $\|f\|_Z$ is equivalent to the norm

$$\|f^*1_{(0,1)}\|_X + \|f\|_1,$$

while $\|f\|_Y$ is equivalent to the norm

$$\|f^*1_{(0,1)}\|_X + \|f\|_{L_1+L_2},$$

where $\|\cdot\|_{L_1+L_2}$ is the classical norm on $Y_{L_1} = L_1(0,\infty) + L_2(0,\infty)$ defined by

$$\|\cdot\|_{L_1+L_2} = \inf\{\|h\|_1 + \|g\|_2 : f = h + g\}.$$

Now we can state the generalizations of the inequalities of Rosenthal and Carothers and Dilworth.

THEOREM 1. *Suppose that X is a quasinormed rearrangement invariant function space on $(0,1)$ such that for some $0 < p < q < \infty$,*

$$\|f\|_p \le \|f\|_X \le \|f\|_q \tag{2}$$

for all $f \in X$. Then there exists a constant $C = C(p, q, K(X))$ such that if $(f_n)_{n=1}^m$ and $(g_n)_{n=1}^m$ are sequences of independent random variables with the

f_n's symmetric and the g_n's nonnegative, then

$$\frac{1}{C}\left\|\sum_{n=1}^{m}\mathbf{f}_n\right\|_Y \le \left\|\sum_{n=1}^{m} f_n\right\|_X \le C\left\|\sum_{n=1}^{m}\mathbf{f}_n\right\|_Y, \tag{3}$$

$$\frac{1}{C}\left\|\sum_{n=1}^{m}\mathbf{g}_n\right\|_Z \le \left\|\sum_{n=1}^{m} g_n\right\|_X \le C\left\|\sum_{n=1}^{m}\mathbf{g}_n\right\|_Z. \tag{4}$$

Moreover, if X is a normed rearrangement invariant space, then (3) *holds for all sequences* $(f_n)_{n=1}^m$ *of mean zero independent random variables on* $(0,1)$.

REMARK 2. It is a formal consequence of Theorem 1 that inequality (2) can be weakened to the condition that $L_q(0,1) \subset X \subset L_p(0,1)$, because this condition implies that the corresponding injections are continuous so that (2) is satisfied for an equivalent rearrangement invariant quasinorm on $(0,1)$ (if $\delta\|f\|_p \le \|f\|_X \le C\|f\|_q$, replace $\|\cdot\|_X$ with $\max\{1/C\|\cdot\|_X, \|\cdot\|_p\}$). Kalton pointed out to us that the condition $X \subset L_p(0,1)$ for some $p > 0$ is automatically satisfied: Just apply the Nikisin factorization theorem [18] to the identity operator from X into the space of measurable functions on $(0,1)$; Nikisin's theorem says that $X \cdot 1_E \subset L_p(E)$ for some set E of measure $1/2$ and some $p > 0$. Since X is rearrangement invariant, this implies that $X \subset L_p(0,1)$.

In Section 3 we apply Theorem 1 to the problem of estimating the uniformity function which arises in the study of the approximation property of L_p; indeed, it was this application which led us to the results in Section 2. The result in Section 3 suggests that for $1 < p < \infty$, unlike the cases $p = 1$ and $p = \infty$ [7], there is a polynomial upper estimate on the uniformity function for L_p (see Section 3 for precise statements).

2. Comparing independent sums to disjoint sums. One natural approach to proving Theorem 1 for symmetric random variables is to check inequality (3) for the cases $X = L_p(0,1)$, $0 < p < \infty$, and then use interpolation to extend to the general case. This is how Carothers and Dilworth [3] proved the right-hand side of (3) for $L_{p,q}$, $2 < p < \infty$; however, there are some problems in applying the interpolation theorems we know to the general case we consider [X is only quasinormed, not normed; we only assume that X contains $L_q(0,1)$ for some $q < \infty$ rather than that the upper Boyd index of X is finite]. Our more elementary approach is just to make direct distributional comparisons between Σf_i and $\Sigma \mathbf{f}_i$. In fact, after most of our work was completed, Carothers and Dilworth [4] verified directly the L_p-boundedness for $2 < p < \infty$ of a projection similar to the projection P in Section 3. This gives an alternate proof of inequality (3) for $X = L_p$, $1 < p < 2$. Probably the Boyd interpolation method (see [17]) can be modified to give a proof of inequality (3) for normed rearrangement invariant spaces X whose indices are strictly between 1 and ∞, but we did not try to check this.

Before proceeding to the proof of Theorem 1, we recall the connection between inequalities (3) and (4), which is probably best understood by employing the notion of *s-convexification* (see [17], page 53, for a development of this theory for general Banach lattices): Given a quasinormed rearrangement invariant space X on some measure space, define for $0 < s < \infty$ the s-convexification $X^{(s)}$ of X to be the collection of measurable functions, f, for which $|f|^s \in X$ and set

$$\|f\|_{X^{(s)}} = \||f|^s\|_X^{1/s}.$$

Thus if X is a quasinormed rearrangement invariant space on $(0,1)$, then, even up to equal quasinorms,

$$(Y_X)^{(1/2)} = Z_{X^{(1/2)}}, \qquad (Z_X)^{(2)} = Y_{X^{(2)}}. \tag{5}$$

Now if $(f_i)_{i=1}^n$ is a symmetric independent sequence in a quasinormed rearrangement invariant space on $(0,1)$ which has finite cotype (see [17], page 72, for a discussion of cotype), then the Maurey–Khintchine inequality ([17], page 49) yields that $\|\sum f_i\|_X$ is equivalent to $\|(\sum |f_i|^2)^{1/2}\|_X$. One can then use (5) and inequality (4) for $X^{(1/2)}$ to derive inequality (3) for X.

When X does not have finite cotype, the square function argument still works to derive the left-hand side of (3) from the left-hand side of (4) under a mild additional assumption which is satisfied by all Banach rearrangement invariant spaces as well as the classical nonlocally convex quasinormed rearrangement invariant spaces. (In Section 4 we present an example, due to Kalton, which shows that the extra assumption is not satisfied by all quasinormed rearrangement invariant spaces.) To prove the left-hand side of (3) in the general case, we need to repeat part of the argument for the left-hand side of (4) and use the fact that the left-hand side of (3) is true for L_p, $0 < p < \infty$.

To prove the right-hand side of (3) and (4), we first prove the right-hand side of (4) for L_p, $0 < p < \infty$. Since L_p has finite cotype and $L_p^{(1/2)} = L_{p/2}$, the Maurey–Khintchine equivalence discussed above applies to give the right-hand side of (3) for all the L_p spaces. [Of course, for L_p this equivalence is a well known and classical consequence of Khintchine's inequality, so our proofs for both sides of (3) really use only Khintchine's inequality instead of its modern Maurey–Khintchine version.] A truncation argument then gives the right-hand side of (3) and (4) for general rearrangement invariant spaces X which satisfy (1).

To prove the left-hand side of (4) and (3), we use the following distributional inequality from [8], proof of Lemma 3.2, and include the simple proof for completeness.

LEMMA 3. *Let $(g_i)_{i=1}^n$ be nonnegative independent random variables. Then for all $0 \le t < \infty$,*

$$P\Big[\max_{1\le i\le n} g_i > t\Big] \ge \frac{\sum_{i=1}^n P[g_i > t]}{1 + \sum_{i=1}^n P[g_i > t]}.$$

Consequently, if also $\sum_{i=1}^n P[g_i > 0] \le 1$, then in any quasinormed rearrangement invariant space X on $(0,1)$,

$$\Big\|\sum_{i=1}^n \mathbf{g}_i\Big\|_X \le 2K(X)\Big\|\sum_{i=1}^n g_i\Big\|_X.$$

Proof. Using the inequality $1 - e^{-t} \geq t/(1+t)$, we see that

$$P\Big[\max_{1\leq i\leq n} g_i > t\Big] = 1 - \prod_{i=1}^{n}(1 - P[g_i > t])$$
$$\geq 1 - \exp\left(-\sum_{i=1}^{n} P[g_i > t]\right) \geq \frac{\sum_{i=1}^{n} P[g_i > t]}{1 + \sum_{i=1}^{n} P[g_i > t]}.$$

If also $\sum_{i=1}^{n} P[g_i > t] \leq 1$, we get for all $t > 0$,

$$\operatorname{meas}\left[\sum_{i=1}^{n} \mathbf{g}_i > t\right] = \sum_{i=1}^{n} P[g_i > t] \leq 2P\Big[\max_{1\leq i\leq n} g_i > t\Big] \leq 2P\left[\sum_{i=1}^{n} g_i > t\right],$$

which yields the "consequently" statement. □

We now prove the left-hand side of (4). Assume, without loss of generality, that $P[g_i = t] = 0$ for all $1 \leq i \leq n$ and all real t. We can thus select $\infty = t_0 > t_1 > \cdots > t_n = 0$ so that for each $1 \leq j \leq n$,

$$\sum_{i=1}^{n} P\big[t_j < g_i < t_{j-1}\big] = 1.$$

From Lemma 3

$$\begin{aligned}(6)\qquad \left\|\left(\sum_{i=1}^{n} \mathbf{g}_i\right)^{*} 1_{(0,1)}\right\|_X &= \left\|\sum_{i=1}^{n} \mathbf{g}_i 1_{[\mathbf{g}_i > t_1]}\right\|_X \\ &\leq 2K(X)\left\|\sum_{i=1}^{n} g_i 1_{[g_i > t_1]}\right\|_X \leq 2K(X)\left\|\sum_{i=1}^{n} g_i\right\|_X.\end{aligned}$$

Assuming, as we may, that the "p" in (2) is at most 1, we have

$$\begin{aligned}\left\|\left(\sum_{i=1}^{n} \mathbf{g}_i\right)^{*} 1_{(1,\infty)}\right\|_1 &= \left\|\sum_{i=1}^{n} \mathbf{g}_i 1_{[\mathbf{g}_i < t_1]}\right\|_1 \\ &= \left\|\sum_{j=2}^{n}\sum_{i=1}^{n} \mathbf{g}_i 1_{[t_j < \mathbf{g}_i < t_{j-1}]}\right\|_1 \leq \sum_{j=2}^{n} t_{j-1} \\ &\leq \sum_{j=1}^{n}\left\|\sum_{i=1}^{n} \mathbf{g}_i 1_{[t_j < \mathbf{g}_i < t_{j-1}]}\right\|_p \\ &\leq 2K(L_p)\sum_{j=1}^{n}\left\|\sum_{i=1}^{n} g_i 1_{[t_j < g_i < t_{j-1}]}\right\|_p \qquad \text{(by Lemma 3)} \\ &\leq 2^{1/p}\left\|\sum_{j=1}^{n}\sum_{i=1}^{n} g_i 1_{[t_j < g_i < t_{j-1}]}\right\|_p \qquad \text{(since } p \leq 1\text{)} \\ &= 2^{1/p}\left\|\sum_{i=1}^{n} g_i\right\|_p \leq 2^{1/p}\left\|\sum_{i=1}^{n} g_i\right\|_X \qquad \text{[by (2)]}.\end{aligned}$$

Combining this with (6), we get the left-hand side of (4):

$$\left\| \sum_{i=1}^{n} \mathbf{g}_i \right\|_Z \le \left(2K(X) + 2^{1/p}\right) \left\| \sum_{i=1}^{n} g_i \right\|_X ,$$

where it is understood that $p \le 1$.

To prove the left-hand side of (3), note that if X satisfies (2), then $X^{(1/2)}$ satisfies (2) if p and q are replaced by $p/2$ and $q/2$, respectively. Setting $Y = Y_X$, $W = Z_{X^{(1/2)}}$, we thus get

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \mathbf{f}_i \right\|_Y &= \left\| \left(\sum_{i=1}^{n} |\mathbf{f}_i|^2 \right)^{1/2} \right\|_Y = \left(\left\| \sum_{i=1}^{n} |\mathbf{f}_i|^2 \right\|_{Y^{(1/2)}} \right)^{1/2} \\
&= \left\| \sum_{i=1}^{n} |\mathbf{f}_i|^2 \right\|_W^{1/2} \le \left(2K(X^{(1/2)}) + 2^{2/p}\right)^{1/2} \left(\left\| \sum_{i=1}^{n} |f_i|^2 \right\|_{X^{(1/2)}} \right)^{1/2} \\
&\le \left(2K(X) + 2^{1/p}\right) \left\| \left(\sum_{i=1}^{n} |f_i|^2 \right)^{1/2} \right\|_X ,
\end{aligned}$$

where we have assumed $p \le 2$. We now introduce the mild additional hypothesis mentioned earlier. Assume that X is *lattice r-convex* for some $r > 0$; that is, assume that there is a constant M so that for all sequences $(x_i)_{i=1}^n$ in X,

$$\left\| \left(\sum_{i=1}^{n} |x_i|^r \right)^{1/r} \right\|_X \le M \left(\sum_{i=1}^{n} \|x_i\|_X^r \right)^{1/r} .$$

We thus have

$$\begin{aligned}
\left\| \left(\sum_{i=1}^{n} |f_i|^2 \right)^{1/2} \right\|_X &= \left\| \left(\text{Average}_{\pm} \left| \sum_{i=1}^{n} \pm f_i \right|^2 \right)^{1/2} \right\|_X \\
&\le B(r) \left\| \left(\text{Average}_{\pm} \left| \sum_{i=1}^{n} \pm f_i \right|^r \right)^{1/r} \right\|_X \\
&\qquad\qquad \text{(by Khintchine's inequality)} \\
&\le MB(r) \left(\text{Average}_{\pm} \left\| \sum_{i=1}^{n} \pm f_i \right\|_X^r \right)^{1/r} = MB(r) \left\| \sum_{i=1}^{n} f_i \right\|_X .
\end{aligned}$$

This gives the left-hand side of (3) with constant $C = MB(r)(2K(X) + 2^{1/p})$, so for $X = L_p$, $C = 3B(p)2^{1/p}$ works.

We now prove the left-hand side of (3) in the general case. Since the f_i's are symmetric and independent, for each $t > 0$,

$$P\left[\max_{1 \le i \le n} f_i > t \right] \le 2P\left[\sum_{i=1}^{n} f_i > t \right].$$

Thus applying Lemma 3 with $g_i = |f_i|$, we have by the argument for the "consequently" statement in Lemma 3 that if $\sum_{i=1}^n P[|f_i| > 0] \le 1$, then

$$\left\| \sum_{i=1}^n \mathbf{f}_i \right\|_X \le 4K(X) \left\| \sum_{i=1}^n f_i \right\|_X .$$

Just as in the proof of the left-hand side of (4) we choose t_1 so that

$$\sum_{i=1}^n P[|f_i| > t_1] = 1.$$

We thus get

$$\tag{6'} \begin{aligned} \left\| \left(\sum_{i=1}^n \mathbf{f}_i \right)^* 1_{(0,1)} \right\|_X &= \left\| \sum_{i=1}^n \mathbf{f}_i 1_{[|\mathbf{f}_i| > t_1]} \right\|_X \\ &\le 4K(X) \left\| \sum_{i=1}^n f_i 1_{[|f_i| > t_1]} \right\|_X \le 8K(X)^2 \left\| \sum_{i=1}^n f_i \right\|_X . \end{aligned}$$

The last inequality follows from the fact that for all $t > 0$,

$$p\left[\left| \sum_{i=1}^n f_i 1_{[|f_i| > t_1]} \right| > t \right] \le 2P\left[\left| \sum_{i=1}^n f_i \right| > t \right].$$

The other term used to define the quasinorm in Y_X is no problem because we know that the left-hand side of (3) is valid for L_p:

$$\begin{aligned} \left\| \left(\sum_{i=1}^n \mathbf{f}_i \right)^* 1_{(1,\infty)} \right\|_2 &\le \left\| \left(\sum_{i=1}^n \mathbf{f}_i \right)^* \right\|_{Y_{L_p}} \\ &\le 2B(p)4^{1/p} \left\| \sum_{i=1}^n f_i \right\|_p \le 3B(p)2^{1/p} \left\| \sum_{i=1}^n f_i \right\|_X . \end{aligned}$$

This and (6′) give the left-hand side of (3) with $C = 8K(X)^2 + 3B(p)2^{1/p}$.

We turn now to the proof of the right-hand side of (3) and (4). For L_p, $0 < p < 1$, the right-hand side of (4) is an easy consequence of the following lemma.

LEMMA 4. *Let $(f_i)_{i=1}^n$ be nonnegative measurable simple functions on $[0, \infty)$. Then $\sum_{i=1}^n \mathbf{f}_i$ is in the convex hull of the set of all measurable simple functions on $[0, \infty)$ which have the same distribution as $\sum_{i=1}^n f_i$.*

PROOF. An obvious iteration argument reduces the lemma to the following trivial comment: Suppose that $0 < t < \infty$, $0 < a, b < \infty$, and h is a function supported on $[2t, \infty)$. Then

$$\begin{aligned} a1_{[0,t)} + b1_{[t,2t)} + h &= \frac{a}{a+b}\left((a+b)1_{[0,t)} + h \right) \\ &\quad + \frac{b}{a+b}\left((a+b)1_{[t,2t)} + h \right). \end{aligned}$$

□

Recall that a quasinormed rearrangement invariant function space X is said to be *1-concave* or simply *concave* provided $X_+ \setminus \mathrm{Ball}(X)$ is convex; equivalently, if

$$\|f+g\|_X \geq \|f\|_X + \|g\|_X$$

for all nonnegative functions f and g in X. Observe that if X is a concave quasinormed rearrangement invariant function space on $[0,1]$; for example, L_p with $p < 1$, then $Z = Z_X$ is also concave (use the easy fact that for f in Z,

$$\|f\|_Z = \inf\{\|f 1_A\|_X + \|f 1_{A^c}\|_1 : \text{meas } A = 1\}).$$

COROLLARY 5. *Let $(g_i)_{n=1}^n$ be a sequence of nonnegative functions in a quasinormed rearrangement invariant function space X on $[0,\infty)$. If X is concave, then*

$$\left\| \sum_{i=1}^n \mathbf{g}_i \right\|_X \geq \left\| \sum_{i=1}^n g_i \right\|_X,$$

while if X is normed, then

$$\left\| \sum_{i=1}^n g_i \right\|_X \geq \left\| \sum_{i=1}^n \mathbf{g}_i \right\|_X.$$

The second conclusion of Corollary 5 was proved in [11], page 171; the argument we give here is a bit simpler.

The first conclusion in Corollary 5 gives the right-hand side of (4) with constant $C = 1$ for $X = L_p$, $0 < p \leq 1$. As mentioned in the Introduction, a square-function argument then gives the right-hand side of (3) for $X = L_p$, $0 < p < 2$. This competes the proof of Theorem 1 for L_p, since Rosenthal's original work [19] takes care of the remaining cases.

We break the proof of the right-hand side of (3) and (4) into two cases. Set $S = |\sum_{i=1}^n f_i|$ or $S = \sum_{i=1}^n g_i$, depending on whether we are proving (3) or (4), respectively, and normalize so that $\|S\|_X = 1$. For some $T = T[p, q, K(X)]$ to be specified later we set $S_1 = S 1_{[S \leq T]}$.

CASE 1. $\|S_1\|_X \geq 1/(2K(X))$. For definiteness, we prove the right-hand side of (4). We have

$$\frac{1}{2K(X)} \leq \|S_1\|_X \leq \|S_1\|_q \leq \|S_1\|_p^{p/q} T^{1-p/q}$$

so that

$$[2K(X)]^{-q/p} T^{1-q/p} \leq \|S\|_p.$$

Thus, using (4) for L_p, we get

$$\begin{aligned}1 = \|S\|_X &\le [2K(X)]^{q/p} T^{q/p-1} \|S\|_p \\ &\le [2K(X)]^{q/p} T^{q/p-1} \left\| \sum_{i=1}^{n} \mathbf{f}_i \right\|_{Z_{L_p}} \\ &\le [2K(X)]^{q/p} T^{q/p-1} \left\| \sum_{i=1}^{n} \mathbf{f}_i \right\|_{Z_X}.\end{aligned}$$

The proof of the right-hand side of (3) is the same; just replace "Z" by "Y."

In the proof of Case 2, we use an iterated version of an inequality due to Hoffmann-Jørgensen [9], presented below in Lemmas 6 and 7. (Lemma 7 is more general than Lemma 6 but gives a worse constant.) Although the arguments are easy variations of those in [9], we present complete proofs for the convenience of the reader.

LEMMA 6. *Let $(g_n)_{n=1}^{\infty}$ be a sequence of nonnegative independent random variables and set $S = \sum_{n=1}^{\infty} g_n$, $M = \sup_{1 \le n < \infty} g_n$. Then, for every positive integer k and positive real t,*

$$P[S > (2k-1)t] \le P[M > t] + P[S > t]^k.$$

PROOF. Define an increasing sequence of stopping times by $\tau_0 = 0$, $\tau_1 = \inf\{m \ge 1: \sum_{i=1}^{m} g_i > t\}$, $\tau_2 = \inf\{m > \tau_1: \sum_{i=\tau_1+1}^{m} g_i > t\}$, $\tau_3 = \inf\{m > \tau_2: \sum_{i=\tau_2+1}^{m} g_i > t\}, \ldots$, with the usual convention $\inf \phi = \infty$. If $S > (2k-1)t$, then either $M > t$ or else $\sum_{i=\tau_{j-1}+1}^{\tau_j} g_i \le 2t$ whenever $\tau_j < \infty$. We thus get

$$P[S > (2k-1)t] \le P[M > t] + P[\tau_k < \infty]. \tag{7}$$

Now for $1 \le j \le k$,

$$\begin{aligned}P[\tau_j < \infty] &= P\left[\tau_{j-1} < \infty \text{ and } \sum_{i=\tau_{j-1}+1}^{\infty} g_i > t\right] \\ &= \sum_{n=0}^{\infty} P\left[\tau_{j-1} = n \text{ and } \sum_{i=n+1}^{\infty} g_i > t\right] \\ &= \sum_{n=0}^{\infty} p[\tau_{j-1} = n] P\left[\sum_{i=n+1}^{\infty} g_i > t\right] \\ &\le \sum_{n=0}^{\infty} P[\tau_{j-1} = n] P[S > t] \\ &= P[\tau_{j-1} < \infty] P[S > t].\end{aligned}$$

Iterating, we get that $P[\tau_k < \infty] \le P[S > t]^k$ and the conclusion follows from (7). $\square$

LEMMA 7. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of independent random variables taking values in some Banach space and set*

$$s^* = \sup_{1 \le j \le m < \infty} \left\| \sum_{i=j}^{m} f_i \right\|, \qquad s_n^* = \sup_{m > n} \left\| \sum_{i=n+1}^{m} f_i \right\|, \qquad M = \sup_n \| f_n \| .$$

Then for every positive integer k and positive real t,

$$P[s^* > 2kt] \le P[M > t] + \sup_{0 \le n < \infty} P[s_n^* > t]^k .$$

Consequently, if the f_n's are also symmetric, then

$$P\left[\left\| \sum_{n=1}^{\infty} f_n \right\| > 2kt\right] \le P[M > t] + \left(2P\left[\left\| \sum_{n=1}^{\infty} f_n \right\| > t\right]\right)^k .$$

PROOF. Define an increasing sequence of stopping times by

$$\tau_0 = 0,$$
$$\tau_1 = \inf\left\{ m \ge 1 : \left\| \sum_{i=1}^{m} f_i \right\| > t \right\},$$
$$\tau_2 = \inf\left\{ m > \tau_1 : \left\| \sum_{i=\tau_1+1}^{m} f_i \right\| > t \right\},$$
$$\tau_3 = \inf\left\{ m > \tau_2 : \left\| \sum_{i=\tau_2+1}^{m} f_i \right\| > t \right\}, \ldots,$$

with the usual convention $\inf \phi = \infty$. Now suppose that $1 \le j \le n < \infty$ and fix a point in the probability space. Let r be the largest integer so that $\tau_r < j$ and let m be the smallest integer for which $n < \tau_m$. Then

$$\begin{aligned} \left\| \sum_{i=j}^{n} f_i \right\| &\le \left\| \sum_{i=j}^{\tau_{r+1}-1} f_i \right\| + \| f_{\tau_{r+1}} \| + \left\| \sum_{i=\tau_{r+1}+1}^{\tau_{r+2}-1} f_i \right\| + \cdots + \left\| \sum_{i=\tau_{m-1}+1}^{n} f_i \right\| \\ &\le 2t + M + t + M + \cdots + t, \end{aligned}$$

which is at most $2mt$ if $M \le t$. Thus

$$P[s^* > 2kt] \le P[M > t] + P[\tau_k < \infty].$$

Now for $1 \le j \le k$,

$$\begin{aligned} P[\tau_j < \infty] &= P\left[\tau_{j-1} < \infty \text{ and } s_{\tau_{j-1}}^* > t\right] \\ &= \sum_{n=0}^{\infty} P[\tau_{j-1} = n] P[s_n^* > t] \le \sum_{n=0}^{\infty} P[\tau_{j-1} = n] \sup_{0 \le n < \infty} P[s_n^* > t] \\ &= P[\tau_{j-1} < \infty] \sup_{0 \le n < \infty} P[s_n^* > t]. \end{aligned}$$

Thus, by iteration,

$$P[\tau_n < \infty] \leq \sup_{0 \leq n < \infty} P[s_n^* > t]^k,$$

which gives the first conclusion of the lemma. When the f_n's are symmetric, a standard computation yields that for every j and every positive real t,

$$P\left[s_j^* > t\right] \leq 2P\left[\left\| \sum_{n=1}^{\infty} f_n \right\| > t\right],$$

so the "consequently" statement follows formally from the first conclusion. □

We return to the proof of the right-hand side inequality in (3) and (4).

CASE 2. $\|S_1\|_X < 1/(2K(X))$. Setting $S_2 = S1_{[S>T]}$, we have in Case 2 that $\|S_2\|_X \geq 1/(2K(X))$. Therefore we get from Lemma 7 for $t > (2k)^{-1}T$ and hence for all positive t,

$$P[S_2 > 2kt] \leq P[M > t] + (2P[S_3 > t])^k, \tag{8}$$

where $k = k(p, q)$ will be specified later, $S_3 = S1_{[S>(2k)^{-1}T]}$, M is the maximum of the g_n's in the proof of (4) and the maximum of the $|f_n|$'s in the proof of (3). Let S_4 be a nonnegative random variable such that for all positive t,

$$P[S_4 > t] = (2P[S_3 > t])^k.$$

Now for all positive t, $t^p P[S > t] \leq \|S\|_p^p \leq 1$, so if $pk > q$ we get

$$\|S_4\|_X \leq \|S_4\|_q \leq \left(q \int_{(2k)^{-1}T}^{\infty} t^{q-1} 2^k t^{-pk}\, dt\right)^{1/q} = \left(\frac{q2^k}{pk-q}\right)^{1/q} \left(\frac{2k}{T}\right)^{pk/q-1}.$$

Now choose k so that $2q \leq pk \leq 3q$. Then if $T > 2k$, we have

$$\|S_4\|_X \leq 2^{k/q} 2kT^{-1}.$$

Therefore, from (8) we get

$$\frac{1}{4kK(X)} \leq \frac{1}{2k}\|S_2\|_X \leq K(X)\left(\|M\|_X + 2^{1+k/q}kT^{-1}\right)$$

so that

$$\|M\|_X \geq \frac{1}{4kK(X)^2} - 2^{1+k/q}kT^{-1}.$$

Finally, setting $T = 2^{4+k/q}kK(X)^2$, we have

$$\|M\|_X \geq \frac{1}{8kK(X)^2} \geq \frac{p}{24qK(X)^2}.$$

This completes the proof of Case 2 and hence of the theorem, because in the proof of (3) we obviously have $\|M\|_X \leq \|\Sigma_{n=1}^m \mathbf{f}_n\|_Y$ and in the proof of (4) $\|M\|_X \leq \|\Sigma_{n=1}^m \mathbf{g}_n\|_Z$.

REMARK 8. In inequality (3) of Theorem 1, if X is a normed space, then the assumption that the f_i's are symmetric can be replaced by the weaker assumption that the f_i's all have mean 0. Indeed, just apply (3) to $(h_i)_{i=1}^n$, where $h_i = f_i - f_i^{\#}$ and $(f_i^{\#})_{i=1}^n$ is a sequence independent of the f_i's having the same joint distribution as $(f_i)_{i=1}^n$, and note that in Y, $(h_i)_{i=1}^n$ [respectively, $(\mathbf{h}_i)_{i=1}^n$] is equivalent to $(f_i)_{i=1}^n$ [respectively, $(\mathbf{f}_i)_{i=1}^n$.]

3. An application to the local theory of L_p. First we recall [7] the uniformity function for the bounded approximation property of L_p. Given a finite-dimensional subspace E of L_p and $K > 1$, let

$$k_p(E, K) = \inf\{\operatorname{rank}(T):\ T: L_p \to L_p, \|T\| \le K, Tx = x \text{ for } x \in E\}$$

and set

$$k_p(n, K) = \sup\{k_p(E, K): E \subset L_p, \dim E = n\}.$$

In [7] it was shown that there is a positive constant δ so that if E is the span of n independent Gaussian or Rademacher random variables, then $k_1(E, K) \ge \exp(\delta K^{-2} n)$, while $k_1(E, K) \ge \exp(\delta K^{-2} n^{2/p^*})$ if E is spanned by n independent symmetric p-stable random variables ($1 < p < 2$; p^* conjugate index to p). In particular, the uniformity function $k_1(n, K)$ admits for any fixed K a lower estimate which is exponential in n. However, it may be that for $1 < p < \infty$, there is a $K = K(p)$ so that $k_p(n, K)$ has an upper estimate which is polynomial in n. Here we prove a result which supports this conjecture:

THEOREM 9. *For each $1 < p < \infty$ there is $K = K(p) < \infty$ so that if E is a subspace of L_p which is spanned by n independent random variables, then there is a projection P on L_p so that $\|P\| \le K$, $Px = x$ for $x \in E$, and $\dim PL_p \le Cn\log(n+1)$, where C is a constant independent of n and p.*

Theorem 1 allows us to reduce Theorem 9 to proving a similar assertion about the span of disjoint functions in $Y_p \equiv Y_{L_p}$. To make this precise, let Y_p^0 be the subspace of Y_p consisting of those functions which have mean 0 on $[n-1, n]$ for each $n = 1, 2, \ldots$. Define the *independent stacking operator* T_p from Y_p^0 into $L_p([0,1]^\infty)$ by

$$(T_p f)(t_1, t_2, \ldots) = \sum_{n=1}^{\infty} f(t_n + n - 1).$$

Theorem 1 and Remark 8 say that T_p is an isomorphism for each $1 \le p < \infty$. Moreover, for $1 < p < \infty$, $T_p Y_p^0$ is a complemented subspace of $L_p([0,1]^\infty)$ because T_{p^*} is also an isomorphism. Indeed, it is a classical fact that for $2 < p < \infty$, $(L_p(0,\infty) \cap L_2(0,\infty))^*$ can be identified with $Y = L_{p^*}(0,\infty) + L_2(0,\infty)$ via the bilinear form

$$\langle f, g\rangle = \int_0^\infty f(t) g(t)\, dt$$

and Y_p (respectively, Y_{p^*}) is easily seen [11] to be $L_p(0,\infty) \cap L_2(0,\infty)$ [respectively, $L_p(0,\infty) + L_2(0,\infty)]$ under an equivalent norm. Now define a projection Q_p from Y_p onto Y_p^0 by setting

$$Q_p f = f - \sum_{n=1}^{\infty}\left(\int_{n-1}^{n} f(t)\,dt\right)1_{(n-1,n)};$$

$\|Q_p\| \le 2$ because $I - Q_p$ is the usual averaging projection onto the closed span of $(1_{(n-1,n)})_{n=1}^{\infty}$. Note also that Q_p^* can be identified with Q_{p^*}, so Y_p^0 and $Y_{p^*}^0$ are duals to each other for $1 < p < \infty$. Finally, (T_p, T_{p^*}) preserves the relevant bilinear form on $(Y_p^0, Y_{p^*}^0)$:

$$\begin{aligned}\langle T_p f, T_{p^*} g\rangle &= \int T_p f(\tilde t)T_{p^*}g(\tilde t)\mu_\infty(d\tilde t)\\ &= \int \sum_{n=1}^{\infty} f(t_n + n - 1)\sum_{n=1}^{\infty} g(t_n + n - 1)\,dt_1\,dt_2\dots\\ &= \sum_{n=1}^{\infty}\int_0^1 f(t_n + n - 1)g(t_n + n - 1)\,dt_n\\ &= \sum_{n=1}^{\infty}\int_{n-1}^{n} f(s)g(s)\,ds = \langle f, g\rangle.\end{aligned}$$

In view of the reflexivity of L_p, this means that $T_pY_p^0 \oplus (T_{p^*}Y_{p^*}^0)^{\perp}$ is a direct sum decomposition of $L_p([0,1]^\infty)$. (One can of course write down explicitly the projection onto $T_pY_p^0$ and use Theorem 1 to check its boundedness but we prefer to emphasize the general principles involved.)

Now suppose that E is a subspace of L_p which is spanned by n independent random variables $(f_i)_{i=1}^n$, which we can assume all have mean 0 since we are not concerned with absolute constants (the span of E and $1_{(0,1)}$ is spanned by $1_{(0,1)}$ and n independent random variables of mean 0). We can also assume that L_p is isometrically embedded into $L_p([0,1]^\infty)$ in such a way that $f_i(\tilde t)$ depends only on the ith coordinate t_i of $\tilde t$, say, $f_i(\tilde t) = h_i(t_i)$. Let $\mathbf{f}_i$ in Y_p^0 be the natural disjointification of the f_i's; that is,

$$\mathbf{f}_i(s) = h_i(t_i - i + 1).$$

Thus $T_p\mathbf{f}_i = f_i$.

In Theorem 10 we shall prove that there is a projection R on Y_p so that $R\mathbf{f}_i = \mathbf{f}_i$ for $i = 1, 2, \dots, n$, $\|R\| < 3$; and $\dim RY_p < Cn\log(n+1)$. Thus if S is a projection from $L_p([0,1]^\infty)$ onto $T_pY_p^0$, then $U \equiv T_pQ_pRT_p^{-1}S$ is an operator on $L_p([0,1]^\infty)$ with rank at most $Cn\log(n+1)$; $Uf_i = f_i$ for $i = 1, 2, \dots, n$; and $\|U\|$ depends only on p. Denoting by V the norm one projection from $L_p([0,1]^\infty)$ onto the original L_p space, we have that $VU_{|L_p}$ is the identity on the f_i's, has rank at most $Cn\log(n+1)$ and $\|VU_{|L_p}\|$ depends only on p. This shows that $k_p(E, K) \le Cn\log(n+1)$ for $K > \|VU_{|L_p}\|$; however, $VU_{|L_p}$ need not be a projection even if U is, and U need not be a projection if RY_p^0 is not contained in Y_p^0. Rather than work harder to guarantee that the above constructions produce projections for U and $VU_{|L_p}$, we prefer to use the "abstract nonsense" argument of [16] to complete the proof: Since $(I - Q_p)Y_p$ is isometrically isomorphic to l_2, we can in an obvious way extend T_p to an isomorphism $\tilde{T}_p$ from Y_p into $L_p([0,1]^\infty) \oplus l_2$ (which is another isomorphic representation of L_p [12]). Let $\tilde{S}$ be the natural extension of S to a projection from $L_p([0,1]^\infty) \oplus l_2$ onto $\tilde{T}_p Y_p$ and set $\tilde{U} = \tilde{T}_p R \tilde{T}_p^{-1} \tilde{S}$. Thus $\tilde{U}$ is a projection on $L_p([0,1]^\infty) \oplus l_2$ whose range, call it Z, contains E and has dimension $\leq Cn \log(n+1)$, and $\|\tilde{U}\|$ depends only on p. As in [16], we can choose a projection W on L_p with norm at most $2\|\tilde{U}\|$ so that $WVZ = \{0\}$ and WL_p is 2-isomorphic to Z; say, $\tau\colon Z \to WL_p$ satisfies $\|\tau\| = 1$ and $\|\tau^{-1}\| \leq 2$. Define $U_1\colon L_p \to Z$, $T\colon Z \to L_p$ by

$$U_1 = \tilde{U}(1 - W) + \tau^{-1}W, \qquad T = V_{|Z} + \tau(1 - UV_{|Z}).$$

Then $\|U_1\|$, $\|T\|$ depend only on p and $U_1 T = I_Z$, so TU_1 is a projection on L_p with rank at most $\dim Z \leq Cn \log(n+1)$ which is the identity on E. This completes the reduction of Theorem 9 to Theorem 10, which we now state:

THEOREM 10. *Let Y be a normed rearrangement invariant function space on $(0, \infty)$ such that for some $1 \leq p < \infty$, $\|f\|_Y = \|f\|_p$ if $\operatorname{supp} f \subset [0,1]$. Suppose that $(f_i)_{i=1}^\infty$ are disjointly supported functions in Y with $\operatorname{meas}[|f| \neq 0] \leq 1$ for each $i = 1,2,\ldots,n$. Then for each $\varepsilon > 0$ there is a projection P on Y so that $Pf_i = f_i$ for $i = 1,2,\ldots,n$, $\dim PY \leq Cn\log(n+1)$ and $\|P\| < 2 + \varepsilon$, where $C = C(\varepsilon)$ is a constant which is independent of n and p.*

PROOF. Assume, without loss of generality, that $\|f_i\| = 1$ for each $i = 1,2,\ldots,n$. Then

$$\|f_i 1_{[|f_i| \leq \varepsilon/n]}\|_Y \leq \frac{\varepsilon}{n}\|1_{[0,1]}\|_Y = \frac{\varepsilon}{n},$$

so by replacing f_i with the normalization of $f_i 1_{[|f_i| > \varepsilon/n]}$, we can via a standard perturbation argument assume that $[0 < |f_i| \leq \varepsilon/n] = \phi$ for each $i = 1,2,\ldots,n$. Let $g_i = f_i 1_{[|f_i| > n]}$, set $h_i = f_i - g_i$ and let $G = \bigcup_{i=1}^n \operatorname{supp} g_i$. Since $1 \geq \|g_i\|_Y = \|g_i\|_p \geq n(\operatorname{meas}[|f_i| > n])^{1/p}$, $\operatorname{meas} G \leq 1$ and thus $\|\cdot\|_Y$ agrees with $\|\cdot\|_p$ for functions supported on G. Therefore, there is a norm one projection Q from Y onto $\operatorname{span}(g_i)_{i=1}^n$ such that $Qf = 0$ if $G \cap \operatorname{supp} f = \phi$.

Next we chop up each h_i into $m \approx 2\log(n/\varepsilon)/\log(1+\varepsilon)$ pieces whose absolute values are "almost two-valued." Formally, let m be the smallest positive integer for which $n(1+\varepsilon)^{-m} \leq \varepsilon/n$ and for $i = 1,2,\ldots,n$, $k = 1,2,\ldots,m$, set

$$A_{i,k} = \left[n(1+\varepsilon)^{-k} < |f_i| \leq n(1+\varepsilon)^{-k+1}\right].$$

Recall that if $(A_i)_{i=1}^n$ are disjoint sets of finite measure, then $\operatorname{span}(1_{A_i})_{i=1}^n$ is norm one complemented in any normed rearrangement invariant function space by the averaging projection. From this it follows easily that if g_i is supported on A_i and $|g_i(s)/g_i(t)| \leq 1 + \varepsilon$ for all s and t in A_i, then in any normed rearrangement invariant space there is a projection W onto $\operatorname{span}(g_i)_{i=1}^n$ with $\|W\| \leq 1 + \varepsilon$ so that $Wf = 0$ if f is supported off $\bigcup_{i=1}^n A_i$. Thus in our situation there is a projection R from Y onto $\operatorname{span}(f_i 1_{A_{i,k}})_{i=1}^n{}_{k=1}^m$ with $\|R\| \leq 1 + \varepsilon$ and $Rf = 0$ if

$\operatorname{supp} f \cap \operatorname{supp}(f_i - g_i) = \phi$ for all $i = 1, 2, \ldots, n$. Thus $P = Q + R$ is a projection from Y onto $\operatorname{span}(f_i 1_{[|f_i| > n]})_{i=1}^n \cup (f_i|_{A_{i,k}})_{i=1}^n{}_{k=1}^m$ which contains $\operatorname{span}(f_i)_{i=1}^n$ and has dimension at most $n(m+1) \approx C(\varepsilon) n \log(n+1)$, and $\|P\| \le 2 + \varepsilon$ because Q and R operate on complementary bands of Y. This completes the proof of Theorem 10. □

REMARK 11. We do not know whether the projection P in Theorem 10 can be constructed to have norm less than $1 + \varepsilon$ instead of $2 + \varepsilon$.

REMARK 12. The proof of Theorem 9 yields a complemented superspace PL_p to $\operatorname{span}(f_i)_{i=1}^n$ which has a "good" unconditional basis (the superspace is even an $X_{p,w}$ space in the sense of Rosenthal [19]), but PL_p need not be well isomorphic to an l_p^d-space. However, Bourgain, Lindenstrauss and Milman [2] recently proved that every well-complemented d-dimensional subspace of L_p embeds into l_p^k, $k \approx d^{r/2}$ ($r \equiv \max\{p, p^*\}$), as a well-complemented subspace. (Actually, the case of $X_{p,w}^d$ had already been dealt with in [20], where a different polynomial estimate in d was obtained.) One can then choose a well-complemented subspace W of $(1-P)L_p$ so that $PL_p + W$ is well isomorphic to l_p^k, so $PL_p + W$ is a well-complemented copy of l_p^k which contains $\operatorname{span}(f_i)_{i=1}^n$ whose dimension is polynomial in n. In contradistinction to this, Arias [1] has recently proved that for some $\varepsilon = \varepsilon(p) > 0$, if the f_i's in Theorem 9 are symmetric and satisfy a mild flatness condition (e.g., they have Gaussian distribution or s-stable distribution for some $p < s < 2$), then if E is a superspace in L_p of $\operatorname{span}(f_i)_{i=1}^n$ which is $1 + \varepsilon$-isomorphic to l_p^k, then $k > a^n$ for some constant $a > 1$ (a depends on the degree of flatness of the f_i's, on ε, and on p).

4. Appendix. In [13], Example 2.4, Kalton gave an example of a quasinormed lattice which is not lattice p-convex for any $p > 0$. By taking l_2-sums of finite-dimensional sublattices of his example, it follows that there are quasinormed spaces with a 1-unconditional boundedly complete basis which are not lattice p-convex for any $p > 0$. Recently, Kalton pointed out to us that there are rearrangement invariant spaces exhibiting the same phenomena which contain $L_2(0,1)$. This is a formal consequence of his earlier example and the following theorem, which is a generalization to the quasinormed setting of a result proved in [11] for normed spaces.

THEOREM 13 (Kalton). *Let E be a quasinormed space with a 1-unconditional basis $(e_i)_{i=1}^\infty$. Then E is isomorphic to a sublattice F of a quasinormed rearrangement invariant space X on $(0,1)$ such that $L_r \subset X \subset L_1$ for all $r > 2$. If the unconditional basis for F is boundedly complete, then F is complemented in X.*

PROOF. By a theorem of Aoki and Rolewicz (cf. [15], page 7) there exist $0 < p < 1$ and a constant M so that if Y is any quasinormed space for which $K(Y) \leq 4K(E)$, then

$$\|y_1 + y_2 + \cdots + y_n\| \leq M\left(\sum_{i=1}^{n} \|y_i\|^p\right)^{1/p}.$$

Choose $1 = a_1 > a_2 > \cdots > 0$ with $a_{k+1} < 2^{-1}a_k$ and

$$\left(\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\left(a_k a_j^{-1}\right)^p + \sum_{j=k+1}^{\infty}\left(a_j a_k^{-1}\right)^p\right)^{1/p} = A < [8MK(E)]^{-1}.$$

For a measurable function f on $(0,1)$, define

$$\|f\|_X = \sup_n \left\|\sum_{k=1}^{n} b_k e_k\right\|_E,$$

where

$$b_k = a_k^{-1}\int_0^{a_k^2} f^*(t)\,dt.$$

If $h = f + g$, then $h^*(t) \leq 2[f^*(t/2) + g^*(t/2)]$ so $\|f\|_X$ is a quasinorm and $K(X) \leq 4K(E)$. Clearly, X is a complete quasinormed rearrangement invariant function space on $(0,1)$ with $L_r \subset X \subset L_1$ for all $r > 2$.

Set $g_k = a_k^{-1}1_{(0,a_k^2]}$ and suppose that $c_k \geq 0$. Then

$$\left\|\sum_{k=1}^{\infty} c_k g_k\right\|_X = \left\|\sum_{k=1}^{\infty} d_k e_k\right\|_E,$$

where

$$d_k = c_k + \sum_{j=1}^{k-1} a_k a_j^{-1} c_j + \sum_{j=k+1}^{\infty} a_j a_k^{-1} c_j.$$

So

$$0 \leq d_k - c_k \leq \left(\sum_{j=1}^{k-1} a_k a_j^{-1} + \sum_{j=k+1}^{\infty} a_j a_k^{-1}\right)\left\|\sum_{k=1}^{\infty} c_k e_k\right\|_E$$

and

$$\left\|\sum_{k=1}^{\infty} d_k e_k\right\|_E \leq K(E)(1 + MA)\left\|\sum_{k=1}^{\infty} c_k e_k\right\|_E.$$

Set $h_k = a_k^{-1} 1_{(a_{k+1}^2, a_k^2]}$ and suppose that $c_k \geq 0$. Then

$$\begin{aligned} K(X)\left\| \sum_{k=1}^{\infty} c_k h_k \right\|_X &\geq \left\| \sum_{k=1}^{\infty} c_k g_k \right\|_X - K(X)\left\| \sum_{k=1}^{\infty} c_k (g_k - h_k) \right\|_X \\ &\geq \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E \\ &\quad - K(X)\left\| \sum_{k=1}^{\infty} c_k \left(\sum_{j=1}^{k} a_j^{-1} a_k^{-1} a_{k+1}^2 e_j + \sum_{j=k+1}^{\infty} a_k^{-1} a_j e_j \right) \right\|_E \\ &= \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E \\ &\quad - K(X)\left\| \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j-1} c_k a_k^{-1} a_j + \sum_{k=j}^{\infty} c_k a_j^{-1} a_k^{-1} a_{k+1}^2 \right) e_j \right\|_E \\ &\geq \left(1 - K(X) M \left[\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j-1} a_k^{-1} a_j + \sum_{k=j}^{\infty} a_j^{-1} a_k^{-1} a_{k+1}^2 \right)^p \right]^{1/p} \right) \\ &\quad \times \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E \\ &\geq [1 - K(X) M A] \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E \geq 2^{-1} \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E . \end{aligned}$$

Therefore, for all $c_k \geq 0$, and hence for all scalars c_k we have

$$[8K(E)]^{-1} \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E \leq \left\| \sum_{k=1}^{\infty} c_k h_k \right\|_X \leq K(E)[1 + MA] \left\| \sum_{k=1}^{\infty} c_k e_k \right\|_E ;$$

that is, the map $e_k \to h_k$ extends to an isomorphism from E onto the closed sublattice of X generated by $(h_k)_{k=1}^{\infty}$.

Since it is easily verified that X has Kalton's property (d) [14], there is a continuous averaging projection from X onto the closed span of $(h_k)_{k=1}^{\infty}$ if $(e_k)_{k=1}^{\infty}$ is a boundedly complete basic sequence. □

REMARK 14. It is evident that the above construction can be modified to guarantee that $L_r \subset X \subset L_1$ for all $r > 1$.

5. Concluding remarks. After the first version of this article was submitted, we came across a comparison result of de la Peña [5] (since replaced by [6]) which can be deduced from Theorem 1 and Corollary 5. Suppose that $(h_n)_{n=1}^{m}$ is a sequence of nonnegative random variables on $(0,1)$ and $(g_n)_{n=1}^m$ is a sequence of independent random variables on $(0,1)$ such that for each n, g_n has the same distribution as h_n. Let X be a quasinormed rearrangement invariant function space on $(0,1)$ which satisfies condition (2) of Theorem 1. Corollary 5 and Theorem 1 imply that if X is 1-concave, then

$$\left\| \sum_{n=1}^m h_n \right\|_X \le \left\| \sum_{n=1}^m \mathbf{g}_n \right\|_Z \le C \left\| \sum_{n=1}^m g_n \right\|_X , \tag{9}$$

while if X is normed, then

$$\left\| \sum_{n=1}^m h_n \right\|_X \ge \left\| \sum_{n=1}^m \mathbf{g}_n \right\|_Z \ge \frac{1}{C} \left\| \sum_{n=1}^m g_n \right\|_X . \tag{10}$$

If $(d_n)_{n=1}^m$ is a sequence of mean zero random variables on $(0,1)$ and $(f_n)_{n=1}^m$ is a sequence of independent random variables on $(0,1)$ such that for each n, f_n has the same distribution as d_n, then by applying the left-hand sides of (9) and (10) to $(|d_n|^2)_{n=1}^m$ in the 1/2-convexification of X we get from Theorem 1 that if X is 2-concave (i.e., $X^{(1/2)}$ is concave) and normed, then

$$\|S(d)\|_X \le \left\| \sum_{n=1}^m \mathbf{f}_n \right\|_Y \le \frac{1}{C} \left\| \sum_{n=1}^m f_n \right\|_X , \tag{11}$$

while if X is 2-convex (i.e., $X^{(1/2)}$ is a normed space), then

$$\|S(d)\|_X \ge \left\| \sum_{n=1}^m \mathbf{f}_n \right\|_Y \ge C \left\| \sum_{n=1}^m f_n \right\|_X , \tag{12}$$

where $S^2(d) \equiv \sum_{n=1}^m |d_n|^2$ defines the usual square function of the sequence $(d_n)_{n=1}^m$. Now let ϕ be a symmetric convex Orlicz function such that $\phi(|s|^{1/q})$ is concave for some $q < \infty$ (it is well known and easy to see that every convex Orlicz function which satisfies the Δ_2 condition is equivalent to such a function) and for $t > 0$ define the unit ball of a normed space $\tilde{X}_t$ on $(0,1)$ to be all those random variables, f, on $(0,1)$ for which $E\phi(|f|) \le t$. Turn $\tilde{X}_t$ into a normed rearrangement invariant space X_t by dividing the $\tilde{X}_t$-norm of f by the $\tilde{X}_t$-norm of $1_{(0,1)}$ (recall that our definition of rearrangement invariant space requires that the norm of $1_{(0,1)}$ be 1). All the spaces X_t for $t > 0$ satisfy (4) with any q for which $\phi(|s|^{1/q})$ is concave and, of course, with $p = 1$. Therefore, when $q = 2$, (11) holds with the same constant C for all the spaces X_t. Translated back to expectations, this means that when $\phi(|s|^{1/2})$ is concave,

$$E\phi(S(d)) \le CE\phi\left(\sum_{n=1}^m f_n \right). \tag{13}$$

Similarly, when $\phi(|s|^{1/2})$ is convex, we get from (12)

$$E\phi(S(d)) \geq \frac{1}{C} E\phi\left(\sum_{n=1}^{m} f_n \right). \tag{14}$$

The inequalities (13) and (14) are what de la Peña [5] proved, except that he assumed that $(d_n)_{n=1}^m$ is a martingale difference sequence. This allowed him to apply the Burkholder–Davis–Gundy convex function inequality and state (13) and (14) with the square function of the martingale replaced by the maximal function. It turns out that there is a generalization of the convex function inequality for normed rearrangement invariant function spaces: We checked recently [10] that the norm of the square function of every martingale is equivalent to the norm of the maximal function in a normed rearrangement invariant function space if and only if the upper Boyd index (cf. [17]) of the space is finite.

Acknowledgments. This research was begun while the authors were visiting Université Paris VI. We thank N. Carothers, S. Dilworth, N. Kalton, B. Maurey, G. Pisier and J. Zinn for discussions on various aspects of this work.

REFERENCES

[1] ARIAS, A. (1988). l_n^p superspaces of spans of independent random variables. *Israel J. Math.* To appear.

[2] BOURGAIN, J., LINDENSTRAUSS, J. and MILMAN, V. (1988). Approximation of zonoids by zonotopes. *Acta Math.* To appear.

[3] CAROTHERS, N. L. and DILWORTH, S. J. (1985/1986). Geometry of Lorentz spaces via interpolation. *The University of Texas Functional Analysis Seminar Longhorn Notes* 107–134.

[4] CAROTHERS, N. L. and DILWORTH, S. J. (1988). Inequalities for sums of independent random variables. *Proc. Amer. Math. Soc.* **104** 221–226.

[5] DE LA PEÑA, V. H. (1987). L-bounds for martingales and U-statistics in terms of L-norms of sums of independent random variables. Unpublished manuscript.

[6] DE LA PEÑA, V. H. (1988). L-bounds for martingales and sums of positive rv's in terms of L-norms of sums of independent random variables. To appear.

[7] FIGIEL, T., JOHNSON, W. B. and SCHECHTMAN, G. (1988). Factorizations of natural embeddings of l_p^n into L_r. I. *Studia Math.* **89** 79–103.

[8] GINÉ, E. and ZINN, J. (1983). Central limit theorems and weak laws of large numbers in certain Banach spaces. *Z. Wahrsch. verw. Gebiete* **62** 323–354.

[9] HOFFMAN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 158–186.

[10] JOHNSON, W. B. and SCHECHTMAN, G. (1988). Martingale inequalities in rearrangement invariant function spaces. *Israel J. Math.* To appear.

[11] JOHNSON, W. B., MAUREY, B., SCHECHTMAN, G. and TZAFRIRI, L. (1979). *Symmetric Structures in Banach Spaces. Mem. Amer. Math. Soc.* **217**. Amer. Math. Soc., Providence, R.I.

[12] KADEC, M. I. and PELCZYNSKI, A. (1962). Bases, lacunary sequences and complemented subspaces in the spaces L_p. *Studia Math.* **21** 161–176.

[13] KALTON, N. J. (1984). Convexity conditions for non-locally convex lattices. *Glasgow Math. J.* **25** 141–152.

[14] KALTON, N. J. (1984). Compact and strictly singular operators on certain function spaces. *Arch. Math.* (*Basel*) **43** 66–78.

[15] KALTON, N. J., PECK, N. T. and ROBERTS, J. W. (1984). An F-space sampler. *London Math. Soc. Lecture Note Ser.* **89**. Cambridge Univ. Press, Cambridge.

[16] LINDENSTRAUSS, J. and ROSENTHAL, H. P. (1969). The $\mathscr{L}_p$ spaces. *Israel J. Math.* **7** 325–349.

[17] LINDENSTRAUSS, J. and TZAFRIRI, L. (1979). *Classical Banach Spaces. II. Function Spaces.* Springer, Berlin.

[18] NIKISIN, E. M. (1970). Resonance theorems and superlinear operators. *Uspekhi Mat. Nauk* **25** 129–191.

[19] ROSENTHAL, H. P. (1970). On the subspaces of L_p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.* **8** 273–303.

[20] SCHECHTMAN, G. (1987). Embedding X_p^m spaces into ℓ_r^n. *Geometrical Aspects of Functional Analysis, Israel Seminar, 1985–86. Lecture Notes in Math.* **1267** 53–74. Springer, Berlin.

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