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THE MAXIMUM OF A GAUSSIAN PROCESS WITH NONCONSTANT VARIANCE: A SHARP BOUND FOR THE DISTRIBUTION TAIL

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Let $X(t)$, $0 \le t \le 1$, be a real separable Gaussian process with mean 0 and continuous covariance function and put $\sigma^2(t) = EX^2(t)$. Under the well known conditions of Fernique and Dudley, the sample functions are continuous and there are explicit asymptotic upper bounds for the probability $P(\max_{[0,1]} X(t) \ge u)$ for $u \to \infty$. Suppose that there is a point τ, $0 \le \tau \le 1$, such that $\sigma^2(t)$ has a unique maximum value at $t = \tau$ and put $\sigma = \sigma(\tau)$. The main result is a sharpening of the standard asymptotic upper bounds for $P(\max_{[0,1]} X(t) \ge u)$ to take into account the existence of the unique maximum of $\sigma(t)$. Indeed, when the order of the standard bound exceeds that of the obvious lower bound $P(X(\tau) \ge u)$, the upper asymptotic bound is shown to be reducible by the factor $\int_0^1 \exp(-u^2 g(t))\,dt$, with $g(t) = (1/\sigma)(1/\bar{\sigma}(t) - 1/\sigma)$, where $\bar{\sigma}(t)$ is an arbitrary majorant of $\sigma(t)$ satisfying certain general conditions. For a large class of processes the asymptotic order of the bound obtained in this way cannot be further reduced. The results are illustrated by applications to the ordinary Brownian motion and the Brownian bridge.

1. Introduction and summary. Let $X(t)$, $t \in T$, be a centered Gaussian random field, where T is some space. Define a pseudometric $d_X(s,t) = (E(X(s) - X(t))^2)^{1/2}$ on T and assume that T is compact with respect to the pseudometric, that is, for any $\varepsilon > 0$, it contains a finite ε-net. Under this condition there exists a separable version of the field with respect to any countable dense subset [Fernique (1975)]. We denote by $N_X(S, \varepsilon)$ the minimum cardinality of the set of balls of radius at most ε which cover a subset S of T. Throughout this paper we assume Dudley's condition: Put $H(x) = \max(\sqrt{\log x}, 1)$ and assume

$$Q_D(T, x) = \int_0^{x/2} H(N_X(T, \varepsilon))\,d\varepsilon < \infty \tag{1.1}$$

for $x > 0$. This implies that the sample functions are continuous on T with probability 1 [Dudley (1967)]. When T is an interval on the real line, say $T = [0, 1]$, another condition, due to Fernique (1964), is known. Let ϕ be a nondecreasing continuous function such that $\phi(0) = 0$ and

$$d_X(s,t) \le \phi(|s - t|) \quad \text{for } s, t \in T. \tag{1.2}$$

If

$$Q_F(x) = \int_0^\infty \phi(xe^{-t^2})\,dt < \infty \tag{1.3}$$

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for $x = 1$, then the sample functions are almost surely continuous on T. It is well known that (1.3) implies (1.1). Let Q_D^{-1}, Q_F^{-1} and ϕ^{-1} represent the usual inverses of the corresponding monotonic functions. Define $\sigma^2(t) = EX^2(t)$.

Our main result is

THEOREM 1.1. *Let $X(t)$, $t \in T = [0,1]$, be a centered Gaussian process satisfying* (1.1) *and let ϕ be defined as in* (1.2). *We assume that there is a unique point τ, $0 \le t \le 1$, such that $\sigma(\tau) = \max(\sigma(t), 0 \le t \le 1)$, and we put*

$$\sigma = \sigma(\tau). \tag{1.4}$$

Let $\bar\sigma(t)$ be a continuous function on $[0,1]$ with the following properties:

(1.5)
- (i) *$\bar\sigma(t)$ has a unique maximum at $t = \tau$ and $\bar\sigma(\tau) = \sigma$.*
- (ii) *$\sigma(t) \le \bar\sigma(t)$, for all $0 \le t \le 1$.*
- (iii) *$\bar\sigma(t)$ is nondecreasing for $0 \le t \le \tau$ and nonincreasing for $\tau \le t \le 1$.*

Define

$$g(t) = (1/\sigma)(1/\bar\sigma(t) - 1/\sigma), \qquad 0 \le t \le 1, \tag{1.6}$$

and

$$\Psi(u) = (2\pi)^{-1/2} e^{-u^2/2}/u, \qquad u > 0, \tag{1.7}$$

and

$$\begin{aligned} q(u) &= \phi^{-1}(Q_D^{-1}(1/u)), \qquad u > 0, \quad \textit{under } (1.1) \textit{ alone}, \\ &= \phi^{-1}(Q_D^{-1}(1/u)), \qquad u > 0, \quad \textit{or} = Q_F^{-1}(1/u),\ u > 0, \\ &\qquad\qquad\qquad \textit{under } (1.1) \textit{ and } (1.3). \end{aligned} \tag{1.8}$$

Then

$$\limsup_{u \to \infty} \frac{P(\max_{t \in T} X(t) \ge u)}{\Psi(u/\sigma)\left\{1 + (e/q(u))\int_0^1 e^{-u^2 g(t)}\,dt\right\}} \le 2K(\sigma) \tag{1.9}$$

where the constant $K(\sigma)$ is defined as follows:

$$K_1(\sigma) = \exp(32\sqrt{2}/\sigma^2)(1 + 1/(2\sqrt{\pi})),$$

$$K_2(\sigma) = (\exp(2/\sigma^2) + (\sqrt{2} + 1)/\sqrt{2\pi}\,\sigma)\exp(16(2 + \sqrt{2})/\sigma^2)$$

and

$$K(\sigma) = \max[K_1(\sigma), K_2(\sigma)]. \tag{1.10}$$

The proof of the theorem is given in Section 3.

REMARK 1. The condition (1.2) and the continuity of ϕ imply the continuity of $\sigma^2(t) = EX^2(t)$.

REMARK 2. A function $\bar{\sigma}$ with the properties (1.5) exists; for example, there is

$$\bar{\sigma}(t) = \max_{0 \le s \le t} \sigma(s), \qquad 0 \le t \le \tau,$$
$$= \sigma - \min_{\tau \le s \le t} (\sigma - \sigma(s)), \qquad \tau \le t \le 1.$$

Let $\Phi(x)$ be the standard normal distribution function and $\Phi'(x)$ its density function. We recall the well known inequality

$$1 - \Phi(x) \le x^{-1}\Phi'(x) = \Psi(x) \quad \text{for } x > 0. \tag{1.11}$$

We briefly indicate the relation of Theorem 1.1 to recent work. For any closed subinterval S of $[0,1]$, put $\sigma_S = \max_{t \in S}\sigma(t)$. Berman (1985b) showed that $P(\max_S X(t) > u)$ is of the order $\Psi(u/\sigma_S)/Q_F^{-1}(1/u)$ for $u \to \infty$. Then he showed that this result is sharp by furnishing a class of stationary Gaussian processes for which $P(\max_S X(t) > u)$ is exactly of the given order. More recently, Berman (1985c) showed that the bound can be cut to the factor $\Psi(u/\sigma_S)$ alone if $\sigma(t)$ has a relatively sharp spike at its maximum value. The assumption was that the ratio

$$E(X(t) - X(s))^2/|\sigma(t) - \sigma(s)| \tag{1.12}$$

tends to 0 at a specified rate for $s, t \to \tau$. The reason for the reduction in the asymptotic bound is that the spike of $\sigma(t)$ causes the value of $X(\tau)$ to dominate all other values in a small neighborhood of τ. More recent results in this area have been obtained by Dobric, Marcus and Weber (1988) and Talagrand (1988).

The contribution of this paper has two parts: 1. It furnishes an asymptotic bound for $P(\max X(t) \ge u)$ in the case where $\sigma(t)$ has a unique maximum without other conditions. 2. While the results of Berman (1985b, c) were proved under the assumption (1.3), the current results are valid under the more general assumption (1.1).

In Section 4 we show that for a large class of Gaussian processes, the asymptotic bound of Theorem 1.1 is actually attained and so the bound cannot, in general, be asymptotically improved.

2. Preliminary results.

First we prove a lemma concerning the bivariate normal distribution which will be used to extend the results in Berman [(1985b), Corollary 3.1] for $n = 1$.

LEMMA 2.1. *Let X and Y be random variables having a bivariate normal distribution with mean* 0, *variance* 1 *and correlation r. We assume that*

$$\varepsilon = \left(E(X - Y)^2\right)^{1/2} \le 1. \tag{2.1}$$

Then, for

$$0 \le \varepsilon x < y, \tag{2.2}$$

we have

$$P(X \geq x + \varepsilon y, Y \leq x) \leq 2\varepsilon\Psi(x + \varepsilon y)\Psi((y - \varepsilon x)/2). \tag{2.3}$$

PROOF. It follows from (2.1) and the definition of r that

$$\varepsilon = (2(1 - r))^{1/2} \quad \text{and} \quad \tfrac{1}{2} \leq r \leq 1. \tag{2.4}$$

Let P represent the left-hand member of (2.3). Then, by the independence of $X - rY$ and Y it follows that

$$\begin{aligned} P &= \int_{-\infty}^{x} P(X \geq x + \varepsilon y | Y = t)\, d\Phi(t) \\ &= \int_{-\infty}^{x} P\left(\frac{X - rY}{(1 - r^2)^{1/2}} \geq \frac{x + \varepsilon y - rt}{(1 - r^2)^{1/2}}\right) d\Phi(t). \end{aligned} \tag{2.5}$$

For $t \leq x$, we have the elementary inequalities

$$\begin{aligned} (x + \varepsilon y - rt)(1 - r^2)^{-1/2} &\geq (\varepsilon y + (1 - r)x)(1 - r^2)^{-1/2} \\ &= (\sqrt{2}\, y + \varepsilon x/\sqrt{2})(1 + r)^{-1/2} \quad [\text{by } (2.4)] \\ &\geq y + \varepsilon x/2 > (y + \varepsilon x)/2 > 0. \end{aligned}$$

It then follows that the last member of (2.5) is, by application of (1.11), at most equal to

$$\int_{-\infty}^{x} \frac{(1 - r^2)^{1/2}}{x + \varepsilon y - rt} \Phi'\left(\frac{x + \varepsilon y - rt}{(1 - r^2)^{1/2}}\right) \Phi'(t)\, dt. \tag{2.6}$$

Since the function $\Phi'((u - rv)(1 - r^2)^{-1/2})\Phi'(v)$ is symmetric in u and v, the integral (2.6) is equal to

$$\int_{-\infty}^{x} \frac{(1 - r^2)^{1/2}}{x + \varepsilon y - rt} \Phi'(x + \varepsilon y)\Phi'\left((t - r(x + \varepsilon y))(1 - r^2)^{-1/2}\right) dt. \tag{2.7}$$

For $t \leq x$ we have the elementary inequalities

$$\begin{aligned} t - r(x + \varepsilon y)(1 - r^2)^{-1/2} &\leq ((1 - r)x - \varepsilon r y)(1 - r^2)^{-1/2} \\ &= (\varepsilon x - 2ry)(2 + 2r)^{-1/2} \quad [\text{by } (2.1)] \\ &\leq (\varepsilon x - y)(2 + r)^{-1/2} \quad \left(\text{for } y > 0 \text{ and } r \geq \tfrac{1}{2}\right) \\ &\leq -(y - \varepsilon x)/2 \quad (\text{for } y > \varepsilon x). \end{aligned}$$

Thus, by another application of (1.11) and the symmetry of $\Phi'(x)$ we find that (2.7) is at most equal to

$$\begin{aligned} &2\Psi(x + \varepsilon y) \int_{-\infty}^{-(y - \varepsilon x)/2} d\Phi(t)(1 - r^2)^{-1/2} \\ &\quad \leq 2(1 - r^2)^{1/2}\Psi(x + \varepsilon y)\Psi((y - \varepsilon x)/2). \end{aligned}$$

By $(1 - r^2)^{1/2} \le (2(1 - r))^{1/2}$ and (2.4), the last member displayed above is at most equal to the right-hand member of (2.3). □

The proof of the following lemma is based on Lemma 2.1 and the idea of the proof of a theorem of Sirao (1960).

LEMMA 2.2. *Let $X(t)$, $t \in T$, be a centered Gaussian random field, where T is compact in the pseudometric $d_X(s, t) = (E(X(s) - X(t))^2)^{1/2}$. We assume that there are constants $\underline{\sigma}^2$ and $\bar{\sigma}^2$ such that*

$$0 < \underline{\sigma}^2 \le EX^2(t) \le \bar{\sigma}^2 \quad \text{for all } t \in T. \tag{2.8}$$

Then the inequality

$$\begin{aligned} &P\left(\sup_{t \in T} X(t) \ge x + \sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) \\ &\quad \le N_X(T, \underline{\sigma}\varepsilon_1)\Psi(x/\bar{\sigma}) + \sum_{n=2}^{\infty} N_X(T, \underline{\sigma}\varepsilon_n) p_n \end{aligned} \tag{2.9}$$

holds under the conditions

$$0 < \cdots \le \varepsilon_2 \le \varepsilon_1 \le 1 \quad \text{and} \quad \varepsilon_1 \sum_{n=1}^{\infty} \varepsilon_n \le \tfrac{1}{2}, \tag{2.10}$$

$$0 < \lambda_1 \le \lambda_2 \le \cdots \quad \text{and} \quad 0 < 2\varepsilon_1 x < \lambda_1, \tag{2.11}$$

where

$$p_n = \frac{4\varepsilon_{n-1}\bar{\sigma}^2 \exp\left(-x^2/2\bar{\sigma}^2 - (\lambda_{n-1} - 2\varepsilon_{n-1}x)^2/(32\bar{\sigma}^2)\right)}{\pi(\lambda_{n-1} + \varepsilon_{n-1}x)(\lambda_{n-1} - 2\varepsilon_{n-1}x)}. \tag{2.12}$$

PROOF. Define a new random field $Y(t) = X(t)/\sigma(t)$ and the corresponding pseudometric $d_Y(s, y) = (E(Y(s) - Y(t))^2)^{1/2}$. Then we have

$$\begin{aligned} d_Y^2(s, t) &= 2 - 2E(X(s)X(t))/(\sigma(s)\sigma(t)) \\ &= \left[d_X^2(s, t) - (\sigma(s) - \sigma(t))^2\right]/(\sigma(t)\sigma(s)) \\ &\le d_X^2(s, t)/\underline{\sigma}^2. \end{aligned}$$

It follows that $N_Y(S, \varepsilon) \le N_X(S, \underline{\sigma}\varepsilon)$ for any $\varepsilon > 0$. Denote by S_n a set of the centers of balls of radius $\le \varepsilon_n$ with respect to the distance d_Y which cover T and such that the cardinality is equal to $N_Y(T, \varepsilon_n)$.

Setting

$$S_n' = \bigcup_{j=1}^{n} S_j,$$

$$A = \left\{ \sup_{t \in T} Y(t) \geq x/\bar{\sigma} + \sum_{j=1}^{\infty} \varepsilon_j \lambda_j/\bar{\sigma} \right\},$$

$$A_n' = \left\{ \max_{t \in S_n'} Y(t) \geq x/\bar{\sigma} + \sum_{j=1}^{\infty} \varepsilon_j \lambda_j/\bar{\sigma} \right\},$$

$$A_n = \left\{ \max_{t \in S_n'} Y(t) \geq x/\bar{\sigma} + \sum_{j=1}^{n-1} \varepsilon_j \lambda_j/\bar{\sigma} \right\},$$

where $\sum_{j=1}^{0}$ means 0, we have

$$\left\{ \sup_{t \in T} X(t) \geq x + \sum_{j=1}^{\infty} \varepsilon_j \lambda_j \right\} \subset A$$

and

$$P(A) = \lim_{n \to \infty} P(A_n') \leq \liminf_{n \to \infty} P(A_n).$$

We also have

$$P(A_n) \leq P(A_{n-1}) + P(A_n \cap A_{n-1}^c),$$

$$A_n \cap A_{n-1}^c \subset \bigcup_{t \in S_n} \left\{ Y(t) \geq x/\bar{\sigma} + \sum_{j=1}^{n-2} \varepsilon_j \lambda_j/\bar{\sigma} + \varepsilon_{n-1}\lambda_{n-1}/\bar{\sigma}, \right.$$

$$\left. \text{and } Y(t') < x/\bar{\sigma} + \sum_{j=1}^{n-2} \varepsilon_j \sigma_j/\bar{\sigma} \right\},$$

where $t' \in S_{n-1}$ is taken so that $d_Y(t, t') \leq \varepsilon_{n-1}$ and $n \geq 2$. Applying Lemma 2.1 by setting

$$x_n = x/\bar{\sigma} + \sum_{j=1}^{n-2} \varepsilon_j \lambda_j/\bar{\sigma}, \qquad y_n = \lambda_{n-1}/\bar{\sigma}, \qquad \varepsilon = d_Y(t, t') \leq \varepsilon_{n-1},$$

we have

$$P(Y(t) \geq x_n + \varepsilon y_n, Y(t') < x_n)$$

$$= 2\varepsilon \Psi(x_n + \varepsilon y_n) \Psi((y_n - \varepsilon x_n)/2).$$

Taking account of conditions (2.10) and (2.11), we have

$$\begin{aligned}
&\varepsilon(y_n + \varepsilon x_n)^{-1}(y_n - \varepsilon x_n)^{-1} \\
&\quad \le \varepsilon_{n-1}(y_n + \varepsilon_{n-1}x_n)^{-1}(y_n - \varepsilon_{n-1}x_n)^{-1} \\
&\quad = \varepsilon_{n-1}\bar{\sigma}^2\left\{\lambda_{n-1} + \varepsilon_{n-1}\left(x + \sum_{j=1}^{n-2}\varepsilon_j\lambda_j\right)\right\}^{-1}\left\{\lambda_{n-1} - \varepsilon_{n-1}\left(x + \sum_{j=1}^{n-2}\varepsilon_j\lambda_j\right)\right\}^{-1} \\
&\quad \le \varepsilon_{n-1}\bar{\sigma}^2(\lambda_{n-1} + \varepsilon_{n-1}x)^{-1}\left(\lambda_{n-1} - \varepsilon_{n-1}x - \varepsilon_{n-1}\lambda_{n-1}\sum_{j=1}^{\infty}\varepsilon_j\right)^{-1} \\
&\quad \le 2\varepsilon_{n-1}\bar{\sigma}^2(\lambda_{n-1} + \varepsilon_{n-1}x)^{-1}(\lambda_{n-1} - 2\varepsilon_{n-1}x)^{-1}.
\end{aligned}$$

As a consequence of the preceding estimates we obtain

$$P(A_n \cap A_{n-1}^c) \le N_Y(T, \varepsilon_n)p_n \le N_X(T, \underline{\sigma}\varepsilon_n)p_n$$

and

$$P(A_1) \le N_X(T, \underline{\sigma}\varepsilon_1)\Psi(x/\bar{\sigma}).$$

This completes the proof of the lemma. □

LEMMA 2.3. *Let $X(t)$, $t \in T$, be a centered Gaussian random field satisfying* (1.1). (*Then there is a version with continuous sample functions, so that* "sup" *may be replaced by* "max".) *We also assume the existence of constants $\bar{\sigma}$ and $\underline{\sigma}$ such that* (2.8) *holds. Then, for all numbers a and x such that*

$$0 < a \le 1, \qquad ax \le 4\sqrt{2}\,\bar{\sigma}, \tag{2.13}$$

we have

$$\begin{aligned}
&P\Big(\max_{t\in T} X(t) \ge x + 32\sqrt{2}\,(\bar{\sigma}/\underline{\sigma})Q_D(T, a\underline{\sigma}/2)\Big) \\
&\qquad \le \big(N_X(T, a\underline{\sigma}/2) + 2^{-1}\pi^{-1/2}\big)\Psi(x/\bar{\sigma}).
\end{aligned} \tag{2.14}$$

PROOF. In Lemma 2.2 set $\varepsilon_n = a2^{-n}$, where $0 < a \le 1$ and $\lambda_n = 8\sqrt{2}\,\bar{\sigma}H(N_X(T, \underline{\sigma}\varepsilon_{n+1}))$. Then we have

$$\sum_{n=1}^{\infty}\varepsilon_n\lambda_n \le 32\sqrt{2}\,(\bar{\sigma}/\underline{\sigma})\int_0^{a\underline{\sigma}/4}H(N_X(T, \varepsilon))\,d\varepsilon.$$

By applying Lemma 2.2 under the conditions

$$0 < a \le 1, \qquad 0 < ax \le 4\sqrt{2}\,\bar{\sigma} \le \lambda_1/2,$$

we have

$$\begin{aligned}
&P\Big(\max_{t\in T} X(t) \ge x + 32\sqrt{2}\,(\bar{\sigma}/\underline{\sigma})Q_D(a\underline{\sigma}/2)\Big) \\
&\qquad \le N_X(T, a\underline{\sigma}/2)\Psi(x/\bar{\sigma}) + \sum_{n=2}^{\infty}N_X(T, \underline{\sigma}\varepsilon_n)p_n.
\end{aligned} \tag{2.15}$$

Let us estimate the last term in the inequality above. The preceding definitions imply

$$\lambda_{n-1} = 8\sqrt{2}\,\bar{\sigma}H(N_X(T, \underline{\sigma}\varepsilon_n)) \geq 8\sqrt{2}\,\bar{\sigma}$$

and

$$2\varepsilon_{n-1}x \leq 2\varepsilon_1 x = ax \leq 4\sqrt{2}\,\bar{\sigma} \leq \lambda_{n-1}/2.$$

Therefore,

$$\lambda_{n-1} - 2\varepsilon_{n-1}x \geq \lambda_{n-1}/2 \geq 4\sqrt{2}\,\bar{\sigma},$$
$$\lambda_{n-1} + \varepsilon_{n-1}x \geq \lambda_{n-1} \geq 8\sqrt{2}\,\bar{\sigma}$$

and

$$(\lambda_{n-1} - 2\varepsilon_{n-1}x)^2/(32\bar{\sigma}^2) \geq \lambda_{n-1}^2/(128\bar{\sigma}^2) \geq \log N_X(T, \underline{\sigma}\varepsilon_n).$$

Using these inequalities to estimate the coefficients p_n [see (2.12)] in (2.15), we find that the last term is at most equal to

$$\begin{aligned}
&4\pi^{-1}\sum_{n=2}^{\infty} N_X(T, \underline{\sigma}\varepsilon_n)\varepsilon_{n-1}\bar{\sigma}^2(\lambda_{n-1} + \varepsilon_{n-1}x)^{-1}(\lambda_{n-1} - 2\varepsilon_{n-1}x)^{-1}\\
&\qquad\qquad \times\exp\{-x^2/(2\bar{\sigma}^2) - (\lambda_{n-1} - 2\varepsilon_{n-1}x)^2/(32\bar{\sigma}^2)\}\\
&\quad\leq (16\pi)^{-1}a\exp(-x^2/(2\bar{\sigma}^2))\sum_{n=2}^{\infty}2^{-(n-1)}\\
&\quad\leq 2^{-3/2}\pi^{-1}\bar{\sigma}\exp(-x^2/(2\bar{\sigma}^2))/x \quad (\text{for } ax \leq 4\sqrt{2}\,\bar{\sigma})\\
&\quad= (2\sqrt{\pi})^{-1}\Psi(x/\bar{\sigma}).
\end{aligned}$$

It follows from this estimate that (2.15) implies (2.14). □

Corollary 2.1. *Put $T = [b, c]$, a real interval, and assume the conditions of Lemma 2.3. For any subinterval S of T, define $|S|$ = length of S and $\bar{\sigma}^2(S) = \max_{t\in S}E(X(t))^2$. Then*

$$\limsup_{u\to\infty}\ \sup_{S:\,|S|\leq\phi^{-1}(Q_D^{-1}(1/u))}\frac{P(\max_S X(t)\geq u)}{\Psi(u/\bar{\sigma}(S))} \leq K_1(\underline{\sigma}) \tag{2.16}$$

and

$$\limsup_{u\to\infty}\ \sup_{S:\,|S|\geq\phi^{-1}(Q_D^{-1}(1/u))}\frac{\phi^{-1}(Q_D^{-1}(1/u))P(\max_S X(t)\geq u)}{|S|\Psi(u/\bar{\sigma}(S))} \leq K_1(\underline{\sigma}). \tag{2.17}$$

Proof. The hypothesis of a unique maximum and the continuity of $\sigma(t)$ see Section 1, Remark 1) imply that $N(T, \varepsilon)\to\infty$ for $\varepsilon\to 0$. Indeed, $N = \lim N(T, \varepsilon)$ certainly exists for $\varepsilon\to 0$ because $N(T, \varepsilon)$ is nonincreasing. We will suppose that N is finite and deduce a contradiction. Since $N(T, \varepsilon)$ is integer valued, we have

$N(T, \varepsilon) = N$ for all small $\varepsilon > 0$. Thus, for every such ε, there is an ε-net containing N points. Letting $\varepsilon \to 0$ and applying the compactness of $[b, c]$ in the pseudometric d_X, we conclude that $[b, c]$ has a dense finite subset $t_1 \le t_2 \le \cdots \le t_N$. In particular there is a j, $1 \le j \le N$, such that $d_X(t_j, \tau) = 0$. Hence, by the assumed uniqueness of the maximum τ it follows that $\tau = t_j$ and, furthermore, that $d_X(t_k, t_j) > 0$ if $t_k \ne t_j$. Thus, $\tau = t_j$ is an isolated point of $[b,c]$: There exists $\delta > 0$ such that $d_X(s, \tau) \ge \delta > 0$ for all $s \ne \tau$. But this contradicts the continuity of $\sigma(t)$ at τ. Hence, the assumption $N < \infty$ is impossible and so we conclude that $N = \infty$.

The conclusion of the last paragraph implies that there exists $\underline{\varepsilon}$ such that

$$\underline{\sigma}/4 \ge \underline{\varepsilon} > 0 \quad \text{and} \quad H(N_X(T, \underline{\varepsilon})) \ge 1/\left(\sqrt{2}\,\underline{\sigma}^2\right).$$

By Lemma 2.3 for any subinterval S of T, a and x such that

$$0 < a \le 1 \quad \text{and} \quad ax \le 4\sqrt{2}\,\bar{\sigma}(S), \tag{2.18}$$

we have

$$P\Big(\max_{t \in S} X(t) \ge x + 32\sqrt{2}\,(\bar{\sigma}(S)/\underline{\sigma}) Q_D(S, a\underline{\sigma}/2)\Big) \le N_X(S, a\underline{\sigma}/2) + \left(1/(2\sqrt{\pi})\right)\Psi(x/\bar{\sigma}(S)).$$

Since Q_D is nondecreasing in sets S, it follows that

$$\begin{aligned} &P\Big(\max_{t \in S} X(t) \ge x + 32\sqrt{2}\,(\bar{\sigma}(S)/\underline{\sigma}) Q_D(T, a\underline{\sigma}/2)\Big) \\ &\quad\le \left[N_X(S, a\underline{\sigma}/2) + \left(1/(2\sqrt{\pi})\right)\right]\Psi(x/\bar{\sigma}(S)). \end{aligned} \tag{2.19}$$

Now for $x \ge 1/Q_D(T, 2\underline{\varepsilon})$, choose a such that $Q_D(T, a\underline{\sigma}/2) = 1/x$. Since $Q_D(T, \cdot\,)$ is continuous and nondecreasing, it follows from the definition of $\underline{\varepsilon}$ that $a \le 4\underline{\varepsilon}/\underline{\sigma} \le 1$ and, in addition,

$$1/x = Q_D(T, a\underline{\sigma}/2) \ge a\underline{\sigma} H(N_X(T, a\underline{\sigma}/4))/4 \ge a/\left(4\sqrt{2}\,\underline{\sigma}\right),$$

which implies that $ax \le 4\sqrt{2}\,\underline{\sigma} \le 4\sqrt{2}\,\bar{\sigma}(S)$. This implies that the conditions in (2.18) are satisfied.

In (2.19) above set

$$u = x + 32\sqrt{2}\,(\bar{\sigma}(S)/\underline{\sigma})/x. \tag{2.20}$$

Then $x \le u$ and so $Q_D^{-1}(1/u) \le Q_D^{-1}(1/x) = a\underline{\sigma}/2$. Thus, if $|S| \le \phi^{-1}(Q_D^{-1}(1/u))$, then $\phi(|S|) \le a\underline{\sigma}/2$ and so $N_X(S, a\underline{\sigma}/2) = 1$. It then follows from (2.19) that

$$P\Big(\max_{t \in S} X(t) \ge u\Big) \le \left(1 + (2\sqrt{\pi})^{-1}\right)\Psi(x/\bar{\sigma}(S)). \tag{2.21}$$

From (2.20) and the fact that $x \le u$ it follows that

$$\Psi(x/\bar{\sigma}(S))/\Psi(u/\bar{\sigma}(S)) \le (u/x)\exp\left[32\sqrt{2}/\underline{\sigma}^2 + \left(32\sqrt{2}/\underline{\sigma}\right)^2/x^2\right]$$

and so (2.21) implies

$$
\begin{aligned}
P\Big(\max_{t\in S} X(t) \ge u\Big) &\le \Big(1 + (2\sqrt{\pi})^{-1}\Big)\Psi\left(\frac{u}{\bar\sigma(S)}\right) \\
&\quad \times \exp\left[32\sqrt{2}/\underline{\sigma}^2 + \left(\frac{32\sqrt{2}}{\underline{\sigma}x}\right)^2\right]\left(\frac{u}{x}\right).
\end{aligned}
\tag{2.22}
$$

The relation above continues to hold if, in (2.20), $\bar\sigma(S)$ is replaced by $\bar\sigma$. The relation (2.16) now follows.

For the proof of (2.17), it suffices to make a small alteration in the preceding argument. If $|S| \ge \phi^{-1}(Q_D^{-1}(1/u))$, then S may be written as the union of at most

$$|S|/\phi^{-1}\big(Q_D^{-1}(1/u)\big) + 1$$

intervals, each of length at most $\phi^{-1}(Q_D^{-1}(1/u))$. Therefore, $N_X(S, a\underline{\sigma}/2) \le |S|/\phi^{-1}(Q_D^{-1}(1/u)) + 1$ and so, in the previous estimate (2.22), we replace the factor $(1 + (2\sqrt{\pi})^{-1})$ by

$$|S|/\phi^{-1}\big(Q_D^{-1}(1/u)\big) + 2 + (2\sqrt{\pi})^{-1}.$$

The passage to the limit in (2.17) is now analogous to that in (2.16). This completes the proof of the corollary. □

The next lemma is a modification of Berman [(1985b), Theorem 3.1].

LEMMA 2.4. *Under the same conditions as Lemma* 2.3, *but with the assumption* (1.3) *in place of* (1.1): *For all numbers a and x such that*

$$
\begin{aligned}
&\phi\big(ae^{-2/\bar\sigma^2}\big) \le \underline{\sigma}, \qquad 32(2+\sqrt{2})Q_F(a)\phi\big(ae^{-2/\bar\sigma^2}\big) \le \underline{\sigma}^2/\bar\sigma, \\
&\qquad\qquad\qquad \phi\big(ae^{-2/\bar\sigma^2}\big)x \le 8\sqrt{2}\,\bar\sigma,
\end{aligned}
\tag{2.23}
$$

we have

$$
\begin{aligned}
&P\Big(\max_{t\in T} X(t) \ge x + 16(2+\sqrt{2})(\bar\sigma/\underline{\sigma})^2 Q_F(a)\Big) \\
&\quad \le N_X\big(T, \phi\big(ae^{-2/\bar\sigma^2}\big)\big)\Psi(x/\bar\sigma) \\
&\qquad + \pi^{-1/2}\underline{\sigma}\Psi(x/\bar\sigma)\sum_{n=2}^{\infty} 2^{-n/2}e^{-2^n/\bar\sigma^2}N_X\big(T, \phi\big(ae^{-2^n/\bar\sigma^2}\big)\big).
\end{aligned}
\tag{2.24}
$$

PROOF. We aim to apply Lemma 2.2 with

$$\varepsilon_n = \phi(ae^{-2^n/\bar{\sigma}^2})/\underline{\sigma} \quad \text{and} \quad \lambda_n = 8\sqrt{2}\, 2^{(n+1)/2}(\bar{\sigma}/\underline{\sigma}).$$

With the exception of the second relation in (2.10), the conditions (2.10) and (2.11) hold for (ε_n) and (λ_n) as direct consequences of the assumptions in (2.23). The second condition in (2.10) requires the following verifying calculation:

$$\begin{aligned}(\bar{\sigma}/\underline{\sigma}) \sum_{n=1}^{\infty} \varepsilon_n &< \sum_{n=1}^{\infty} \varepsilon_n \lambda_n \\ &= 16(2+\sqrt{2}) \sum_{n=1}^{\infty} \bar{\sigma}\phi(ae^{-2^n/\bar{\sigma}^2})(2^{n/2} - 2^{(n-1)/2})/\underline{\sigma}^2 \\ &\le 16(2+\sqrt{2})(\bar{\sigma}/\underline{\sigma})^2 Q_F(a).\end{aligned}$$

The conditions in (2.23) also imply $2\varepsilon_1 x \le \lambda_1/2$ and so $2\varepsilon_{n-1}x \le \lambda_{n-1}/2$, and hence $\lambda_{n-1} - 2\varepsilon_{n-1}x \ge \lambda_{n-1}/2$. It then follows that

$$(\lambda_{n-1} - 2\varepsilon_{n-1}x)^2/(32\bar{\sigma}^2) \ge 2^n/\underline{\sigma}^2 \ge 2^n/\bar{\sigma}^2$$

and

$$\begin{aligned}\varepsilon_{n-1}(\lambda_{n-1} + \varepsilon_{n-1}x)^{-1}(\lambda_{n-1} - 2\varepsilon_{n-1}x)^{-1} &\le x^{-1}(\lambda_{n-1} - 2\varepsilon_{n-1}x)^{-1} \\ &\le 2/(x\lambda_{n-1}) = 2^{-(n+5)/2}\underline{\sigma}/(\bar{\sigma}x)\end{aligned}$$

and so p_n in (2.12) satisfies

$$p_n \le \pi^{-1/2}\underline{\sigma}\Psi(x/\bar{\sigma})2^{-n/2}e^{-2^n/\bar{\sigma}^2}.$$

This completes the proof. □

As a corollary we have

COROLLARY 2.2. *Assume the same conditions as in the hypothesis of Lemma 2.4. Then*

$$\limsup_{u\to\infty} \sup_{S:\, |S| \le Q_F^{-1}(1/u)} \frac{P(\max_{t\in S} X(t) \ge u)}{\Psi(u/\bar{\sigma}(S))} \le K_2(\underline{\sigma}) \tag{2.25}$$

and

$$\limsup_{u\to\infty} \sup_{S:\, |S| \ge Q_F^{-1}(1/u)} \frac{Q_F^{-1}(1/u)P(\max_{t\in S} X(t) \ge u)}{|S|\Psi(u/\bar{\sigma}(S))} \le K_2(\underline{\sigma}). \tag{2.26}$$

PROOF. Let us show that the conclusion (2.24) holds also for any subinterval S of T after the replacement of $\bar{\sigma}$ by $\bar{\sigma}(S)$ for the appropriate values of a and x. Choose a: $Q_F(a) = 1/x$. For x satisfying $x^2 \ge 32(\sqrt{2}+1)(\bar{\sigma}/\underline{\sigma})^2 \ge 32(\sqrt{2}+1)(\bar{\sigma}(S)/\underline{\sigma})^2$, we have

$$(1/x) = Q_F(a) \ge \int_0^{\sqrt{2}/\bar{\sigma}(S)} \phi(ae^{-t^2})\,dt \ge (\sqrt{2}/\bar{\sigma}(S))\phi(ae^{-2/\bar{\sigma}(S)^2}).$$

It follows that

$$\phi\big(ae^{-2/\bar\sigma(S)^2}\big) \le \bar\sigma(S)/(\sqrt{2}\,x) \le \underline{\sigma},$$

$$32(2+\sqrt{2})\phi\big(ae^{-2/\bar\sigma(S)^2}\big)/x \le 32(2+\sqrt{2})\bar\sigma(S)/(\sqrt{2}\,x^2) \le \underline{\sigma}^2/\bar\sigma(S)$$

and

$$\phi\big(ae^{-2/\bar\sigma(S)^2}\big)x = \bar\sigma(S)/\sqrt{2} < 8\sqrt{2}\,\bar\sigma(S).$$

Thus, the inequalities in (2.23) with $\bar\sigma(S)$ in the place of $\bar\sigma$ are satisfied and so (2.24) holds with the same replacement.

Define

$$u = x + 16(2+\sqrt{2})\big(\bar\sigma(S)/\underline{\sigma}\big)^2/x. \tag{2.27}$$

Then note that for every $t > 0$, we have $N_X(S, \phi(t)) \le |S|/t + 1$. From this and from the definition of a it follows that

$$\begin{aligned} N_X\big(S, \phi\big(ae^{-2^n/\bar\sigma(S)^2}\big)\big) &\le |S|e^{2^n/\bar\sigma(S)^2}/a + 1 \\ &\le |S|e^{2^n/\bar\sigma(S)^2}/Q_F^{-1}(1/u) + 1. \end{aligned} \tag{2.28}$$

If $|S| \le Q_F^{-1}(1/u)$, then $N_X(S, \phi(ae^{-2^n/\bar\sigma(S)^2})) \le e^{2^n/\bar\sigma(S)^2} + 1$. Hence, by the version of (2.24) for S in the place of T, it follows by a simple calculation that

$$P\Big(\max_{t\in S} X(t) \ge u\Big) \le \Psi(x/\bar\sigma(S))\left[e^{2/\bar\sigma^2} + 1 + 2\underline{\sigma}\pi^{-1/2}\sum_{n=2}^{\infty} 2^{-n/2}\right].$$

Then (2.25) follows by taking account of the relation

$$\lim_{u\to\infty} \Psi(x/\bar\sigma(S))/\Psi(u/\bar\sigma(S)) = \exp\big(16(2+\sqrt{2})/\bar\sigma^2\big),$$

uniformly for $\underline{\sigma} \le \bar\sigma(S) \le \bar\sigma$. This completes the proof of (2.25). The proof of (2.26) follows similarly from (2.24) and (2.28). □

3. Proof of Theorem 1.1.

We begin with the following lemma.

LEMMA 3.1. *For the proof of* (1.9) *it suffices to replace the interval* $T = [0,1]$ *by any closed subinterval* J *such that*:

(i) *If* $\tau = 0$ *or* 1, *then* J *is nondegenerate and contains* τ.
(ii) *If* $0 < \tau < 1$, *then* τ *is contained in the interior of* J.

PROOF. For simplicity, consider just the case $0 < \tau < 1$, as the other case is similar. Decompose $[0,1]$ into three subintervals, $[0,\tau] \cap J^c$, J and $[\tau, 1] \cap J^c$. Assume that the first of these is not empty, so that its closure is an interval $[0, b]$ for some $b < \tau$.

A result of Fernique (1971) implies that

$$E\left[\exp\left\{\alpha \sup_{[0,\,b]} |X(t)|^2\right\}\right] \le K_\alpha < \infty,$$

for $0 < \alpha < 1/(2\bar{\sigma}([0, b]))$; hence, by Chebyshev's inequality, for $u > 0$, $P(\sup_{[0,\,b]}|X(t)| \ge u) \le K_\alpha \exp(-\alpha u^2)$. Therefore, if we choose α such that $(2\sigma)^{-1} < \alpha < (2\bar{\sigma}([0, b]))^{-1}$, then

$$\limsup_{u\to\infty} P\left(\sup_{[0,\,b]} X(t) \ge u\right) \Big/ \Psi(u/\sigma) = 0.$$

Similarly, if $[\tau, 1] \cap J^c$ is not empty, then its closure is an interval $[c, 1]$ for some $\tau < c$ and we have

$$\limsup_{u\to\infty} P\left(\sup_{[c,1]} X(t) \ge u\right) / \Psi(u/\sigma) = 0.$$

From these two relations and the elementary inequality

$$P\left(\sup_{[0,1]} X(t) \ge u\right) \le P\left(\sup_{[0,\,b]} X(t) \ge u\right) + P\left(\sup_{J} X(t) \ge u\right) + P\left(\sup_{[c,1]} X(t) \ge u\right),$$

it follows that the lim sup of the ratio in (1.9) is unchanged if the interval $[0, 1]$ in the numerator is replaced by J. Since $g(t)$ has a unique minimum 0 at $t = \tau$, the domain of integration $[0, 1]$ in the denominator may also be replaced by J. This completes the proof. □

In the next several lemmas, the interval J is chosen in the following way. For arbitrary ε, $0 < \varepsilon < \sigma$, choose J satisfying the requirement in Lemma 3.1, so that

$$\min_{t\in J} \sigma(t) \ge \sigma - \varepsilon. \tag{3.1}$$

This can be done because $\sigma(t)$ is continuous. Let $\theta > 1$ be arbitrary.

LEMMA 3.2.

$$\limsup_{u\to\infty} \frac{P\left(\max_{\{t:\, t\in J,\, u^2 g(t)\ge 1\}} X(t) \ge u\right)}{\Psi(u/\sigma)\left\{1 + e(q(u))^{-1}\int_{J\cap\{t:\, u^2 g(t)\ge 1\}} e^{-u^2 g(t)}\,dt\right\}} \le \theta K(\sigma - \varepsilon). \tag{3.2}$$

PROOF. Suppose first that $\tau = 1$, so that $g(t)$ is nonincreasing and $g(1) = 0$. Define

$$B_j = B_j(u) = \text{closure of } \{t\colon t \in J,\ j \le u^2 g(t) < j + 1\}, \qquad j = 1, 2, \dots. \tag{3.3}$$

B_j is a closed interval because g is monotonic; we also have

$$1/\sigma(t) \geq 1/\bar{\sigma}(t) \geq \sigma j/u^2 + 1/\sigma \quad \text{for } t \in B_j. \tag{3.4}$$

By subadditivity,

$$P\Big(\max_{\{t:\, t \in J,\, u^2 g(t) \geq 1\}} X(t) \geq u\Big) \leq \sum_{j=1}^{\infty} P\Big(\max_{B_j} X(t) \geq u\Big), \tag{3.5}$$

and the latter series may be represented as the sum of the two series

$$\sum_{j:\, \mathrm{mes}(B_j) \leq q(u)} P\Big(\max_{B_j} X(t) \geq u\Big) \tag{3.6}$$

and

$$\sum_{j:\, \mathrm{mes}(B_j) > q(u)} P\Big(\max_{B_j} X(t) \geq u\Big). \tag{3.7}$$

Corollaries 2.1 and 2.2 imply

$$\limsup_{u \to \infty} \sup_{j:\, \mathrm{mes}(B_j) \leq q(u)} \frac{P\big(\max_{B_j} X(t) \geq u\big)}{\Psi\big(u/\big(\max_{B_j}\sigma(t)\big)\big)} \leq K(\sigma - \varepsilon).$$

Hence, for arbitrary $\theta > 1$, the series (3.6) is, for sufficiently large u, at most equal to

$$\theta K(\sigma - \varepsilon) \sum_{j=1}^{\infty} \Psi\left(\frac{u}{\max_{B_j}\sigma(t)}\right). \tag{3.8}$$

By virtue of (3.4),

$$\Psi\left(\frac{u}{\max_{B_j}\sigma(t)}\right) \leq \Psi\left(\frac{\sigma j}{u} + \frac{u}{\sigma}\right). \tag{3.9}$$

Since

$$\begin{aligned}\Psi\left(\frac{\sigma j}{u} + \frac{u}{\sigma}\right) &\leq \frac{\sigma}{u\sqrt{2\pi}} e^{-(u^2/\sigma^2)/2 - j} \\ &= e^{-j}\Psi\left(\frac{u}{\sigma}\right),\end{aligned} \tag{3.10}$$

(3.9) implies that the series (3.6) is at most equal to

$$\frac{\theta K(\sigma - \varepsilon)}{e - 1} \Psi\left(\frac{u}{\sigma}\right). \tag{3.11}$$

Next we estimate the series (3.7). By Corollaries 2.1 and 2.2, with B_j and J in the places of S and T, respectively, we have for all sufficiently large u, for arbitrary $\theta > 1$,

$$P\Big(\max_{B_j} X(t) \geq u\Big) \leq \theta K(\sigma - \varepsilon)\frac{\operatorname{mes}(B_j)}{q(u)}\Psi\bigg(\frac{u}{\max_{B_j}\sigma(t)}\bigg),$$

uniformly, for $j \geq 1$. By (3.9) and (3.10), the right-hand member above is dominated by

$$\theta K(\sigma - \varepsilon)(q(u))^{-1}\Psi(u/\sigma)\operatorname{mes}(B_j)e^{-j}$$

and so the series (3.7) is dominated by

$$\theta K(\sigma - \varepsilon)(q(u))^{-1}\Psi(u/\sigma)\sum_{j=1}^{\infty}\operatorname{mes}(B_j)e^{-j}. \tag{3.12}$$

Since, by the definition (3.3) of B_j, we have

$$\operatorname{mes}(B_j)e^{-j} \leq e\int_{B_j} e^{-u^2 g(t)}\,dt,$$

the bound (3.12) and the relation

$$\sum_{j=1}^{\infty}\int_{B_j} e^{-u^2 g(t)}\,dt = \int_{\{t:\, t\in J,\, u^2 g(t)\geq 1\}} e^{-u^2 g(t)}\,dt \tag{3.13}$$

yield the following bound for the series (3.7) for large u:

$$\theta K(\sigma - \varepsilon)(q(u))^{-1}\Psi(u/\sigma)e\int_{\{t:\, t\in J,\, u^2 g(t)\geq 1\}} e^{-u^2 g(t)}\,dt. \tag{3.14}$$

Combining the bounds (3.11) and (3.14) for the series (3.6) and (3.7), respectively, we find that the series in (3.5) has the asymptotic bound

$$\theta K(\sigma - \varepsilon)\Psi(u/\sigma)\bigg\{1 + \varepsilon(q(u))^{-1}\int_{\{t:\, t\in J,\, u^2 g(t)\geq 1\}} e^{-u^2 g(t)}\,dt\bigg\},$$

and this establishes (3.2).

This completes the proof of the lemma in the case $\tau = 1$. The case $\tau = 0$ is entirely analogous: The function $g(t)$ is nondecreasing instead of nonincreasing. The third case is reducible to the other two cases. Indeed, in the proof of this case the set B_j is the union of at most two disjoint intervals, and $\operatorname{mes}(B_j)$ is the sum of their lengths. This follows from the fact that $g(t)$ is monotonic on each of the intervals $[0, \tau]$ and $[\tau, 1]$. □

LEMMA 3.3.

$$\limsup_{u\to\infty}\frac{P\big(\max_{\{t:\, t\in J,\, u^2 g(t)\leq 1\}} X(t) \geq u\big)}{\Psi(u/\sigma)\big\{1 + \big(\operatorname{mes}(t:\, t\in J,\, u^2 g(t)\leq 1)\big)/(q(u))\big\}} \leq K(\sigma - \varepsilon). \tag{3.15}$$

PROOF. Put

$$U_1 = \{u\colon u > 0, \mathrm{mes}(t\colon t \in J, u^2 g(t) \le 1) < q(u)\}$$

and

$$U_2 = \{u\colon u > 0, \mathrm{mes}(t\colon t \in J, u^2 g(t) \le 1) \ge q(u)\}.$$

Corollaries 2.1 and 2.2 imply [since $(t\colon u^2 g(t) \le 1)$ is an interval]

$$\limsup_{u \in U_1, u \to \infty} (\Psi(u/\sigma))^{-1} P\Bigl(\max_{\{t\colon t \in J, u^2 g(t) \le 1\}} X(t) \ge u\Bigr) \le K(\sigma - \varepsilon)$$

and

$$\limsup_{u \in U_2, u \to \infty} \frac{q(u) P(\max_{\{t\colon t \in J, u^2 g(t) \le 1\}} X(t) \ge u)}{\Psi(u/\sigma) \mathrm{mes}(t\colon t \in J, u^2 g(t) \le 1)} \le K(\sigma - \varepsilon).$$

These two inequalities imply (3.15). □

LEMMA 3.4. *The inequality* (3.15) *continues to hold when* $\mathrm{mes}(t\colon t \in J, u^2 g(t) \le 1)$ *is replaced by*

$$e \int_J e^{-u^2 g(t)}\, dt.$$

PROOF. This is a simple consequence of the inequality

$$\mathrm{mes}(t\colon t \in J, u^2 g(t) \le 1) \le \int_J e^{1 - u^2 g(t)}\, dt. \qquad \square$$

PROOF OF THEOREM 1.1. For arbitrary ε, $0 < \varepsilon < \sigma$, choose a closed subinterval J of $[0,1]$ so that (3.1) holds. By Lemma 3.1, $P(\max_{[0,1]} X(t) \ge u)$ is asymptotically equal to $P(\max_J X(t) \ge u)$. The latter is at most equal to

$$P\Bigl(\max_{\{t\colon t \in J, u^2 g(t) \ge 1\}} X(t) \ge u\Bigr) + P\Bigl(\max_{\{t\colon t \in J, u^2 g(t) \le 1\}} X(t) \ge u\Bigr).$$

By Lemmas 3.2, 3.3 and 3.4, the latter sum has an asymptotic bound

$$2\theta K(\sigma - \varepsilon)\left\{1 + (e/q(u)) \int_0^1 e^{-u^2 g(t)}\, dt\right\} \Psi(u/\sigma).$$

Since $\varepsilon > 0$ is arbitary and $K(\sigma)$ is continuous, we may replace $K(\sigma - \varepsilon)$ above by $K(\sigma)$. Similarly, since $\theta > 1$ is arbitrary, we may replace θ by 1. Thus we obtain the right-hand member of (1.9). □

4. Sharpness of the asymptotic bound. In this section we assume the condition (1.3) and take $q(u)$ in (1.8) as the function $Q_F^{-1}(1/u)$. From the trivial inequality $\max X(t) \ge X(\tau)$ it follows that $P(\max X(t) \ge u)$ has the lower bound $\Psi(u/\sigma)$. Hence, if

$$\lim_{u\to\infty} (q(u))^{-1} \int_0^1 e^{-u^2 g(t)}\, dt = 0 \tag{4.1}$$

for $g(t)$ defined in (1.6), then the asymptotic order of the bound provided by Theorem 1.1 clearly cannot be reduced. The result extends to the case where (4.1) is replaced by

$$\limsup_{u\to\infty} (q(u))^{-1} \int_0^1 e^{-u^2 g(t)}\, dt < \infty. \tag{4.2}$$

For example, if $X(t)$ is the standard Brownian motion, then the function $\phi(t)$ in (1.2) is $\sqrt{t}$, so that $Q_F(t) = c\sqrt{t}$ for some $c > 0$. Define $\bar{\sigma}(t)$ as $\max(\sqrt{t}, \sqrt{\frac{1}{2}})$, so that $g(t) = \min(\sqrt{2}, t^{-1/2}) - 1$. The limit in (4.1) exists and is equal to $2c^2$. The asymptotic order of the bound furnished by Theorem 1.1 is equal to that of the well known value of $P(\max X(t) \geq u)$ in this case, namely, $2\Psi(u)$.

Now we consider the remaining case, namely, where (4.2) is not assumed. We describe a class of Gaussian processes satisfying the conditions of Theorem 1.1 and for which

$$\liminf_{u\to\infty} \frac{P\left(\max_{[0,1]} X(t) \geq u\right)}{\Psi(u/\sigma)\left\{1 + (e/q(u))\int_0^1 e^{-u^2 g(t)}\, dt\right\}} > 0 \tag{4.3}$$

for a suitable majorant $\bar{\sigma}(t)$.

THEOREM 4.1. *Let $X(t)$, $0 \leq t \leq 1$, satisfy the following conditions*:

(i) (1.3) *holds.*

(ii) $\sigma(t)$ *is nondecreasing on* $[0, \tau]$ *and nonincreasing on* $[\tau, 1]$.

(iii) *There is a nonnegative function* $\gamma(t)$, $0 \leq t \leq 1$, *of regular variation for* $t \to 0$, *of index* α, *for some* $0 < \alpha \leq 2$, *such that*

$$\lim_{s\neq t,\, s,\, t\to\tau} \frac{E(X(t) - X(s))^2}{\gamma(|t-s|)} = 1 \tag{4.4}$$

and

$$\lim_{s\neq t,\, s,\, t\to\tau} \frac{|\sigma(t) - \sigma(s)|}{\gamma(|t-s|)} = 0. \tag{4.5}$$

Then there is a suitable function $\bar{\sigma}(t)$ *such that* (4.3) *holds and*

$$(q(u))^{-1} \int_0^1 e^{-u^2 g(t)}\, dt \to \infty. \tag{4.6}$$

PROOF. According to Lemma 3.1,

$$P\left(\max_{[0,1]} X(t) \geq u\right) \sim P\left(\max_{t\in J} X(t) \geq u\right)$$

for any closed nondegenerate interval J containing τ. Hence if J is of sufficiently small length, then for the process $X(t)$, $t \in J$, the function ϕ in (1.2) may without loss of generality be replaced by $\sqrt{2\gamma(h)}$, which dominates it. Hence, $\phi(h)$ may be assumed to be regularly varying of index $\alpha/2$ for $h \to 0$. By an application of the Karamata representation of regularly varying functions and by the definition (1.3) of Q_F, it follows that

$$(4.7) \quad \lim_{h\to 0} Q_F(h)/\gamma^{1/2}(h) = \lim_{h\to 0} \sqrt{2}\, Q_F(h)/\phi(h) = \sqrt{2}\int_0^\infty e^{-(\alpha/2)t^2}\, dt.$$

Since, by assumption, $\sigma(t)$ may play the role of $\bar{\sigma}(t)$, the function $g(t)$ in (1.6) may be taken as

$$g(t) = (1/\sigma)(1/\sigma(t) - 1/\sigma).$$

Let $v = v(u)$ be a positive function satisfying

$$(4.8) \quad \lim_{u\to\infty} u^2\gamma(1/v) = 1.$$

[If γ^{-1} represents the asymptotic inverse of γ, then $v \sim 1/\gamma^{-1}(1/u^2)$.] Let L_u be the sojourn time of $X(t)$ above the level u: $L_u = \text{mes}(t\colon 0 \le t \le 1,\ X(t) > u)$. According to Berman [(1987), Theorem 6.1], we have

$$(4.9) \quad \lim_{u\to\infty} \frac{\int_0^x y\, dP(vL_u \le y)}{vEL_u} = G(x)$$

at all continuity points x, where G is the distribution function of a random variable ξ identified as follows:

Let $W(t)$, $-\infty < t < \infty$, be a Gaussian process with mean 0 and covariance function $EW(s)W(t) = \frac{1}{2}(|s|^\alpha + |t|^\alpha - |s-t|^\alpha)$, and let Z be a random variable independent of the process $W(\cdot)$ and having a standard exponential distribution. Then define

$$(4.10) \quad \xi = \text{mes}\big(t\colon -\infty < t < \infty,\ W(t) - |t|^\alpha/2\sigma^2 + \sigma Z > 0\big).$$

ξ is almost surely positive because $W(t)$ is almost surely continuous, $W(0) = 0$ and $Z > 0$. Hence,

$$(4.11) \quad G(x) > 0 \quad \text{for } x > 0.$$

According to Berman [(1985a), Lemma 2.2], the relation (4.9) implies

$$(4.12) \quad \lim_{u\to\infty} \frac{P(vL_u > x)}{vEL_u} = \int_x^\infty y^{-1}\, dG(y)$$

at all continuity points x of G. By (4.11), the right-hand member of (4.12) is positive at least for all sufficiently small $x > 0$.

The relations (4.7) and (4.8) imply

$$(4.13) \quad \lim_{u\to\infty} v(u)q(u) = C$$

for some $0 < C < \infty$. Furthermore, by Berman [(1987), Theorem 3.1],

$$(4.14) \quad EL_u \sim \Psi(u/\sigma)\int_0^1 e^{-u^2 g(t)}\, dt,$$

with $g(t)$ as defined here. Thus, (4.11), (4.12), (4.13) and (4.14) imply

$$(4.15) \quad \liminf_{u\to\infty} \frac{q(u)P(vL_u > x)}{\Psi(u/\sigma)\int_0^1 e^{-u^2 g(t)}\, dt} > 0.$$

Furthermore, by (4.13) and Lemma 4.4 of Berman (1987), the assertion (4.6) holds. From this and (4.15) it follows that

$$\liminf_{u\to\infty} \frac{P(vL_u > x)}{\Psi(u/\sigma)\left\{1 + (e/q(u))\int_0^1 e^{-u^2 g(t)}\,dt\right\}} > 0 \tag{4.16}$$

for all sufficiently small $x > 0$. Since $vL_u > x$ for any $x > 0$ implies $\max_{[0,1]} X(t) > u$, (4.16) implies (4.3) and the conclusion of the theorem holds. □

EXAMPLE 4.1. Let $X(t)$, $0 \le t \le 1$, be the Brownian bridge, the Gaussian process with mean 0 and covariance function $\min(s, t) - st$. Here $\sigma^2(t) = t(1 - t)$, so that we may take $\bar{\sigma}^2(t)$ as $\sigma^2(t)$ with $\tau = \sigma = \frac{1}{2}$. It is easy to see that $E(X(t) - X(s))^2 \sim |t - s|$, for $s,\ t \to \frac{1}{2}$. The upper asymptotic bound for $P(\max X(t) > u)$, based on Theorem 1.1, is of the order $u\Psi(2u)$. The conditions of Theorem 4.1 also hold: We take $\gamma(t) = t$ and note that

$$g(t) \sim 2\left[(t(1 - t))^{-1/2} - 2\right] \sim 4\left(t - \tfrac{1}{2}\right)^2 \quad \text{for } t \to \tfrac{1}{2},$$

and so the lower asymptotic bound for $P(\max X(t) > u)$ is also of order $u\Psi(2u)$.

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