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THE ASYMPTOTIC DISTRIBUTION OF INTERMEDIATE SUMS

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Let $X_{1,n} \le \cdots \le X_{n,n}$ be the order statistics of n independent random variables with a common distribution function F and let k_n be positive numbers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, and consider the sums $I_n(a,b) = \sum_{i=\lceil ak_n \rceil+1}^{[bk_n]} X_{n+1-i,n}$ of intermediate order statistics, where $0 < a < b$. We find necessary and sufficient conditions for the existence of constants $A_n > 0$ and C_n such that $A_n^{-1}(I_n(a,b) - C_n)$ converges in distribution along subsequences of the positive integers $\{n\}$ to nondegenerate limits and completely describe the possible subsequential limiting distributions.

1. Introduction and statement of results. Let $X, X_1, X_2, \ldots$ be a sequence of independent nondegenerate random variables with a common distribution function $F(x) = P\{X \le x\}$, $x \in \mathbb{R}$, and for each integer $n \ge 1$ let $X_{1,n} \le \cdots \le X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Let $\{k_n\}$ be a sequence of positive numbers such that

$$k_n \to \infty \quad \text{and} \quad k_n/n \to 0 \quad \text{as } n \to \infty. \tag{1.1}$$

When the k_n are integers, many authors have investigated the asymptotic distribution of the single intermediate order statistic $X_{n+1-k_n,n}$. [See, e.g., Pickands (1975), Balkema and de Haan (1978), Watts, Rootzén and Leadbetter (1982) and Cooil (1985) and the references therein.] In this paper we are interested in the problem of the asymptotic distribution of the intermediate sum

$$I_n(a,b) = I_n(a,b;k_n) = \sum_{i=\lceil ak_n \rceil+1}^{[bk_n]} X_{n+1-i,n}, \qquad 0 < a < b, \tag{1.2}$$

where $\lceil x \rceil$ is the smallest integer not smaller than x, $\{k_n\}$ satisfies (1.1) and the empty sum is always understood as 0.

Consider the inverse or quantile function of F defined as

$$Q(s) = \inf\{x \colon F(x) \ge s\}, \qquad 0 < s \le 1. \tag{1.3}$$

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It will be more convenient in our investigations to work with the left-continuous nondecreasing function

$$H(s) = -Q((1-s)-), \qquad 0 \le s < 1, \tag{1.4}$$

the increments of which

$$\Delta_n(a,b) = H\left(\frac{\lceil bk_n \rceil}{n}\right) - H\left(\frac{\lceil ak_n \rceil}{n}\right), \qquad 0 < a < b, \tag{1.5}$$

will play an important role. Set

$$\Delta_n^*(a,b) = \begin{cases} \Delta_n(a,b), & \text{if } \Delta_n(a,b) > 0, \\ 1, & \text{if } \Delta_n(a,b) = 0, \end{cases} \tag{1.6}$$

and for $c = a, b$ and n large enough define the functions

$$\psi_n(c;x) = \begin{cases} \dfrac{H\left(\dfrac{\lceil ck_n \rceil}{n} + x\dfrac{\sqrt{k_n}}{n}\right) - H\left(\dfrac{\lceil ck_n \rceil}{n}\right)}{\Delta_n^*(a,b)}, & -\dfrac{c\sqrt{k_n}}{2} \le x \le \dfrac{c\sqrt{k_n}}{2}, \\ \psi_n\left(c; \dfrac{-c\sqrt{k_n}}{2}\right), & -\infty < x < \dfrac{-c\sqrt{k_n}}{2}, \\ \psi_n\left(c; \dfrac{c\sqrt{k_n}}{2}\right), & \dfrac{c\sqrt{k_n}}{2} < x < \infty. \end{cases}$$

These are nondecreasing and left-continuous for each n, and satisfy $\psi_n(c;0) = 0$. We also need the functions

$$\varphi_n(x) = \begin{cases} 0, & x < \dfrac{\lceil ak_n \rceil}{k_n}, \\ \left\{H\left(\dfrac{xk_n}{n}\right) - H\left(\dfrac{\lceil ak_n \rceil}{n}\right)\right\} \Big/ \Delta_n^*(a,b), & \dfrac{\lceil ak_n \rceil}{k_n} \le x \le \dfrac{\lceil bk_n \rceil}{k_n}, \\ \varphi_n\left(\dfrac{\lceil bk_n \rceil}{k_n}\right), & x > \dfrac{\lceil bk_n \rceil}{k_n}. \end{cases}$$

Note that $\varphi_n \equiv 0$ on $\mathbb{R}$ whenever $\Delta_n(a,b) = 0$, and otherwise φ_n is a left-continuous distribution function on $\mathbb{R}$ for which $\varphi_n(x) = 1$ if $x \ge \lceil bk_n \rceil / k_n$. In what follows, the symbol $\Rightarrow$ will denote weak convergence of functions, that is, pointwise convergence at every continuity point of the limiting function in the interval to be indicated, and $\to_{\mathscr{D}}$ will denote convergence in distribution. Finally, introduce the centering sequence

$$\mu_n(a,b) = -n\int_{\lceil ak_n \rceil/n}^{\lceil bk_n \rceil/n} H(s)\, ds \tag{1.7}$$

and, as usual, let $W(t)$, $t \ge 0$, denote a standard Wiener process.

THEOREM 1. *Let* $\{k_n\}$ *be a sequence as in* (1.1) *and fix* $0 < a < b$.

(i) *Suppose that there exist a subsequence* $\{n'\}$ *of the positive integers* $\{n\}$ *and nondecreasing, left-continuous functions* ψ_a *and* ψ_b *on* $\mathbb{R}$, *necessarily satisfying the conditions* $\psi_c(0) \le 0$ *and* $\psi_c(0+) \ge 0$, *such that*

$$\psi_{n'}(c;\cdot) \Rightarrow \psi_c(\cdot) \quad \text{on } \mathbb{R} \text{ as } n' \to \infty,\ c = a, b. \tag{1.8}$$

Then there exist a subsequence $\{n''\} \subset \{n'\}$ *and a nonnegative, nondecreasing, left-continuous function* φ *on* $\mathbb{R}$ *such that*

$$\varphi(a) = 0, \quad \text{either } \varphi(\infty) = 0 \text{ or } \varphi(b+) = 1 \text{ and } \varphi_{n''}(\cdot) \Rightarrow \varphi(\cdot) \text{ on } \mathbb{R} \tag{1.9}$$

and

$$\frac{1}{\sqrt{k_{n''}}\,\Delta^*_{n''}(a,b)}\left\{\sum_{i=\lceil ak_{n''}\rceil+1}^{\lceil bk_{n''}\rceil} X_{n''+1-i,n''} - \mu_{n''}(a,b)\right\} \to_{\mathscr{D}} V(\psi_a,\varphi,\psi_b) \tag{1.10}$$

as $n'' \to \infty$, *where*

$$\begin{aligned} V(\psi_a,\varphi,\psi_b) &= \int_0^{-W(a)} \psi_a(x)\,dx + \int_{[a,b]} W(x)\,d\varphi(x) - \int_0^{-W(b)} \psi_b(x)\,dx \\ &= -\int_{-W(a)}^0 \psi_a(x)\,dx + \int_a^b W(x)\,d\varphi(x) + (\varphi(b+) - \varphi(b))W(b) \\ &\quad + \int_{-W(b)}^0 \psi_b(x)\,dx. \end{aligned}$$

Furthermore, we necessarily have

$$\psi_a(x) \le \varphi(a+) \quad \text{and} \quad \psi_b(x) \ge \varphi(b) - 1, \qquad x \in \mathbb{R}, \tag{1.11}$$

and the limiting random variable $V(\psi_a,\varphi,\psi_b)$ *is degenerate if and only if* $\psi_a = \psi_b \equiv 0$ *on* $\mathbb{R}$ *and* $\varphi \equiv 0$ *on* $[a,b]$.

(ii) *If for some subsequence* $\{n'\} \subset \{n\}$ *and constants* $B_{n'} > 0$ *there exist nondecreasing, left-continuous functions* ψ_a *and* ψ_b *on* $\mathbb{R}$ *such that* $\psi_c(0) \le 0$, $\psi_c(0+) \ge 0$ *and*

$$\frac{\Delta^*_{n'}(a,b)}{B_{n'}}\psi_{n'}(c;\cdot) \Rightarrow \psi_c(\cdot) \quad \text{on } \mathbb{R},\ c = a, b, \tag{1.12}$$

and

$$\Delta_{n'}(a,b)/B_{n'} \to 0 \quad \text{as } n' \to \infty, \tag{1.13}$$

then

$$\frac{1}{\sqrt{k_{n'}}\,B_{n'}}\left\{\sum_{i=\lceil ak_{n'}\rceil+1}^{\lceil bk_{n'}\rceil} X_{n'+1-i,n'} - \mu_{n'}(a,b)\right\} \to_{\mathscr{D}} V(\psi_a,0,\psi_b) \tag{1.14}$$

as $n' \to \infty$. Furthermore, we necessarily have

$$\begin{aligned} \psi_a(x) &= 0, \qquad x \geq 0, \\ \psi_b(x) &= 0, \qquad x < 0, \end{aligned} \tag{1.15}$$

and the limiting random variable $V(\psi_a, 0, \psi_b)$ is degenerate if and only if $\psi_a = \psi_b \equiv 0$.

The following converse result shows that Theorem 1 is optimal in general.

THEOREM 2. *Suppose that there exist a subsequence $\{n'\} \subset \{n\}$ and norming and centering constants $A_{n'} > 0$ and $C_{n'}$ such that*

$$\frac{1}{A_{n'}}\left\{\sum_{i=\lceil ak_{n'}\rceil+1}^{\lceil bk_{n'}\rceil} X_{n'+1-i,n'} - C_{n'}\right\} \to_{\mathscr{D}} V^* \quad \text{as } n' \to \infty, \tag{1.16}$$

where V^* is a nondegenerate random variable. Then there exist a subsequence $\{n''\} \subset \{n'\}$ and nondecreasing, left-continuous functions ψ_a and ψ_b on $\mathbb{R}$ such that $\psi_c(0) \leq 0$, $\psi_c(0+) \geq 0$ and

$$\psi_{n''}^*(c;\cdot) = \frac{\sqrt{k_{n''}}\,\Delta_{n''}^*(a,b)}{A_{n''}}\psi_{n''}(c;\cdot) \Rightarrow \psi_c(\cdot) \quad \text{on } \mathbb{R},\ c = a, b \tag{1.17}$$

and, for some $0 \leq \delta < \infty$,

$$\sqrt{k_{n''}}\,\Delta_{n''}(a,b)/A_{n''} \to \delta \quad \text{as } n'' \to \infty. \tag{1.18}$$

For the limiting random variable V^* we necessarily have the distributional equality

$$V^* =_{\mathscr{D}} V(\psi_a, \delta\varphi, \psi_b) + \gamma, \tag{1.19}$$

where $\gamma \in \mathbb{R}$ is some constant and φ is a function satisfying the conditions in and above (1.9). *Furthermore, if $\delta = 0$, then we have* (1.15), *and if $\delta > 0$, then we have* (1.11) *with $\delta\varphi(\cdot)$ replacing $\varphi(\cdot)$.*

Theorems 1 and 2 provide a complete description of the class of possible subsequential limiting distributions for $I_n(a,b)$. It is straightforward to conjecture that all the limiting types $V(\psi_a, \varphi, \psi_b)$ arise in general. However, we do not address here the nontrivial problem of constructing an F and a subsequence to obtain a prescribed admissible triple $(\psi_a, \varphi, \psi_b)$. Using techniques from Csörgő, Haeusler and Mason (1988, 1991) and the fact that ψ_c can be a constant function only if $\psi_c \equiv 0$, $c = a, b$, it can be shown that $V(\psi_a, \varphi, \psi_b)$ is nondegenerate normal if and only if $\psi_a = \psi_b \equiv 0$ and $\varphi \not\equiv 0$. It then follows from Theorems 1 and 2 that $I_{n'}(a,b)$ is asymptotically nondegenerate normal with some centering and norming sequences along the fixed subsequence $\{n'\}$ if and only if $\psi_{n'}(c;x) \to 0$ as $n' \to \infty$ for all $x \in \mathbb{R}$ and $c = a, b$ and $\varphi_{n'}(\cdot) \Rightarrow \varphi(\cdot)$ on $\mathbb{R}$ as $n' \to \infty$ for some function $\varphi \not\equiv 0$ satisfying the conditions in and above (1.9). In this case (1.10) holds true along $\{n'\}$ with the limiting normal variable

$V(0,\varphi,0)$. An example of the situation when this is true along the whole sequence $\{n\}$ is given below.

The limiting random variable $V(\psi_a,\varphi,\psi_b)$ in Theorem 1 is a functional of the Wiener process W on $[a,b]$. When this functional is linear, that is, when $\psi_a=\psi_b\equiv 0$, the statement can be obtained from a weak convergence result in the function space $C[a,b]$, proved in Csörgő and Mason (1992), but under a necessarily stronger condition.

We close by an example. We say that F is in the domain of attraction of an extreme value distribution if $(X_{n,n}-c_n)/a_n \to_{\mathscr{D}} Y$ as $n\to\infty$, where $a_n>0$ and $c_n\in\mathbb{R}$ are some constants and Y is nondegenerate. As pointed out in Csörgő, Haeusler and Mason (1991), with earlier references, this happens if and only if for some constant $\gamma\in\mathbb{R}$,

$$(1.20)\quad \lim_{s\downarrow 0}\frac{H(sx)-H(sy)}{H(sv)-H(sw)}=\begin{cases}(x^{-\gamma}-y^{-\gamma})/(v^{-\gamma}-w^{-\gamma}), & \text{if } \gamma\neq 0,\\ \log(x/y)/\log(v/w), & \text{if } \gamma=0,\end{cases}$$

for all distinct $0<x,y,v,w<\infty$. In this case we write $F\in\mathscr{D}(\Lambda_\gamma)$, where, with appropriate choices of a_n and c_n,

$$\Lambda_\gamma(y)=P\{Y\le y\}=\begin{cases}\exp(-y^{1/\gamma}), & y>0;\quad \text{if } \gamma>0,\\ \exp(-\exp(-y)), & y\in\mathbb{R};\quad \text{if } \gamma=0,\\ \exp\bigl(-(-y)^{1/\gamma}\bigr), & y<0;\quad \text{if } \gamma<0.\end{cases}$$

It is easily checked that if $F\in\mathscr{D}(\Lambda_\gamma)$ for some $\gamma\in\mathbb{R}$, then for all $x\in\mathbb{R}$, $\psi_n(c;x)\to 0$ for any choice of $c>0$ and $\varphi_n(x)\to\varphi_\gamma(x)$, $a\le x\le b$, for any choice of $0<a<b$ as $n\to\infty$, where

$$\varphi_\gamma(x)=\varphi_{\gamma,a,b}(x)=\begin{cases}(x^{-\gamma}-a^{-\gamma})/(b^{-\gamma}-a^{-\gamma}), & \text{if } \gamma\neq 0,\\ \log(x/a)/\log(b/a), & \text{if } \gamma=0.\end{cases}$$

The last convergence follows, of course, from (1.20). So if $F\in\mathscr{D}(\Lambda_\gamma)$ for some $\gamma\in\mathbb{R}$, then, by Theorem 1,

$$\frac{1}{\sqrt{k_n}\,\Delta_n^*(a,b)}\left\{\sum_{i=\lceil ak_n\rceil+1}^{\lceil bk_n\rceil}X_{n+1-i,n}-\mu_n(a,b)\right\}\to_{\mathscr{D}}\int_a^b W(x)\,d\varphi_\gamma(x)$$

as $n\to\infty$ for any choice of $0<a<b<\infty$ and sequence $\{k_n\}$ as in (1.1).

An unexpected connection between the convergence in distribution of the intermediate sums $I_n(a,b)$ and the stochastic compactness of the maxima $X_{n,n}$ [for the latter see de Haan and Resnick (1984)] will be pointed out in a subsequent note elsewhere.

2. Proofs. Let $U_1,\dots,U_n$ be independent random variables uniformly distributed on $(0,1)$ with corresponding order statistics $U_{1,n}\le\cdots\le U_{n,n}$. Consider the uniform empirical and quantile processes $\alpha_n(t)=\sqrt{n}\,(\tilde{G}_n(t)-t)$ and $\beta_n(t)=\sqrt{n}\,(t-\tilde{U}_n(t))$, $0\le t\le 1$, where $\tilde{G}_n(t)=n^{-1}\#\{1\le k\le n: U_k\le t\}$, $0\le t\le 1$, and $\tilde{U}_n(t)=\inf\{0\le s\le 1: \tilde{G}_n(s)\ge t\}$, $0<t\le 1$, $\tilde{U}_n(0)=U_{1,n}$, so

that $\tilde{U}_n(t) = U_{k,n}$ if $(k-1)/n < t \le k/n$, $k = 1, \dots, n$. The tail empirical process [cf. Mason (1988)] and the tail quantile process [cf. Cooil (1985)] pertaining to the given sequence $\{k_n\}$ satisfying (1.1) are defined as $\tilde{w}_n(s) = (n/k_n)^{1/2}\alpha_n(sk_n/n)$ and $\tilde{v}_n(s) = (n/k_n)^{1/2}\beta_n(sk_n/n)$, $0 \le s \le n/k_n$. As Mason (1988) points out, for any $T > 0$, $\tilde{w}_n(\cdot)$ converges weakly in the Skorohod space $D[0,T]$ to a standard Wiener process. Then by a Skorohod construction there exist versions $w_n(\cdot)$ of $\tilde{w}_n(\cdot)$ on a suitable probability space such that for any $T > 0$ we have the distributional equalities

$$\{w_n(s)\colon 0 \le s \le T\} =_{\mathscr{D}} \{\tilde{w}_n(s)\colon 0 \le s \le T\} \tag{2.1}$$

for each n large enough to have $n/k_n \ge T$ and

$$\sup_{0\le s\le T} |w_n(s) - W(s)| \to 0 \quad \text{a.s. as } n \to \infty, \tag{2.2}$$

where $W(s)$, $s \ge 0$, is a standard Wiener process. Then by the obvious left-continuous version of Lemma 1 of Vervaat (1972) we obtain the versions $v_n(\cdot)$ of $\tilde{v}_n(\cdot)$ defined on the same suitable space such that

$$\{v_n(s)\colon 0 \le s \le T\} =_{\mathscr{D}} \{\tilde{v}_n(s)\colon 0 \le s \le T\} \tag{2.3}$$

for all n such that $n/k_n \ge T$ and

$$\sup_{0\le s\le T} |v_n(s) - W(s)| \to 0 \quad \text{a.s. as } n \to \infty. \tag{2.4}$$

From the definition of H in (1.4) and (1.3), we have

$$(X_{1,n}, \dots, X_{n,n}) =_{\mathscr{D}} (-H(U_{n,n}), \dots, -H(U_{1,n})), \qquad n \ge 1.$$

If we integrate with respect to a right-continuous or a left-continuous function, we accordingly use the symbol $\int_x^y$ to mean $\int_{(x,y]}$ or $\int_{[x,y)}$. Using this convention [in fact already used in (1.7) and in the formulation of the theorems], the notation in (1.2) and (1.7), integration by parts and some elementary rearrangements, from the last distributional equality we obtain

$$\begin{aligned} I_n(a,b) - \mu_n(a,b) =_{\mathscr{D}} {} & n\int_{\lceil ak_n\rceil/n}^{\lceil bk_n\rceil/n} \big(\tilde{G}_n(u) - u\big)\, dH(u) \\ & - n\int_{\lceil ak_n\rceil/n}^{\tilde{U}_n(ak_n/n)} \left(\tilde{G}_n(u) - \frac{\lceil ak_n\rceil}{n}\right) dH(u) \\ & + n\int_{\lceil bk_n\rceil/n}^{\tilde{U}_n(bk_n/n)} \left(\tilde{G}_n(u) - \frac{\lceil bk_n\rceil}{n}\right) dH(u). \end{aligned}$$

Substituting $u = sk_n/n$ and going over to the probability space of relations (2.1)–(2.4), we arrive at

$$I_n(a,b) - \mu_n(a,b) =_{\mathscr{D}} M_n - R_n(a) + R_n(b), \tag{2.5}$$

where

$$M_n = \sqrt{k_n}\int_{\lceil ak_n\rceil/k_n}^{\lceil bk_n\rceil/k_n} w_n(s)\, dH\left(\frac{sk_n}{n}\right)$$

and, for $c = a, b$,

$$R_n(c) = \int_{\lceil ck_n \rceil/k_n}^{-(1/\sqrt{k_n})v_n(\lceil ck_n \rceil/k_n) + \lceil ck_n \rceil/k_n} \left\{ \sqrt{k_n}\, w_n(s) + sk_n - \lceil ck_n \rceil \right\} dH\left(\frac{sk_n}{n}\right).$$

Aiming at a proof of Theorem 1, we handle these terms in separate lemmas.

LEMMA 1. *For any subsequence $\{n'\} \subset \{n\}$ there exist a further subsequence $\{n''\} \subset \{n'\}$ and a nonnegative, nondecreasing, left-continuous function φ such that* (1.9) *holds and*

$$\frac{M_{n''}}{\sqrt{k_{n''}}\,\Delta^*_{n''}(a, b)} \to \int_{[a,b]} W(s)\, d\varphi(s)$$

$$= \int_a^b W(s)\, d\varphi(s) + (\varphi(b+) - \varphi(b))W(b)$$

almost surely as $n'' \to \infty$.

PROOF. We distinguish two cases. In the first one there exists a subsequence $\{n''\} \subset \{n'\}$ such that $\Delta_{n''}(a, b) = 0$ for all n''. In this case $\varphi_{n''} \equiv 0$ on $\mathbb{R}$, and hence we have (1.9) with $\varphi \equiv 0$ on $\mathbb{R}$, implying the second statement with zero limit.

The second case occurs when $\Delta_{n'}(a, b) > 0$ for all n' large enough. In this case, since $\varphi_{n'}(a) = \varphi_{n'}(\lceil ak_{n'} \rceil/k_{n'}) = 0$, $\varphi_{n'}(\lceil bk_{n'} \rceil/k_{n'}) = 1$, $\lceil bk_{n'} \rceil/k_{n'} \geq b$, $\lceil bk_{n'} \rceil/k_{n'} \to b$ as $n' \to \infty$, by a Helly selection we can choose a subsequence $\{n''\} \subset \{n'\}$ such that (1.9) holds with $\varphi(b+) = 1$.

To prove the second statement in the present second case, notice that

$$\frac{M_{n''}}{\sqrt{k_{n''}}\,\Delta^*_{n''}(a, b)} = \int_0^\infty w_{n''}(s)\, d\varphi_{n''}(s)$$

and

$$\left| \int_0^\infty w_{n''}(s)\, d\varphi_{n''}(s) - \int_0^\infty W(s)\, d\varphi_{n''}(s) \right| \leq \sup_{0 \leq s \leq 2b} |w_{n''}(s) - W(s)|$$

for all sufficiently large n'', and this bound goes to 0 almost surely as $n'' \to \infty$ by (2.2). Furthermore, if $a < d < b$ is a continuity point of φ, then, since $\varphi_{n''}(a) = 0$ and $\varphi(a) = 0$ and since for

$$T_n(b) = \int_b^{\lceil bk_n \rceil/k_n} (W(s) - W(b))\, d\varphi_n(s)$$

we have

$$|T_n(b)| \leq \sup_{b \leq s \leq \lceil bk_n \rceil/k_n} |W(s) - W(b)| \to 0$$

almost surely as $n \to \infty$, we obtain that

$$\int_0^\infty W(s)\, d\varphi_{n''}(s) = \int_a^d (W(s) - W(a))\, d\varphi_{n''}(s) + W(a)\varphi_{n''}(d)$$
$$+ \int_d^b (W(s) - W(b))\, d\varphi_{n''}(s) + W(b)(1 - \varphi_{n''}(d))$$
$$+ T_{n''}(b)$$

converges almost surely to

$$\int_a^d (W(s) - W(a))\, d\varphi(s) + W(a)\varphi(d)$$
$$+ \int_d^b (W(s) - W(b))\, d\varphi(s) + W(b)(1 - \varphi(d))$$
$$= \int_a^b W(s)\, d\varphi(s) + W(b)(1 - \varphi(b)) = \int_{[a,b]} W(s)\, d\varphi(s). \qquad \square$$

LEMMA 2. *If* (1.12) *holds for some sequence* $B_{n'} > 0$ [*that can be* $\Delta^*_{n'}(a, b)$] *along* $\{n'\}$ *with limiting functions* ψ_a *and* ψ_b *having the properties listed above* (1.12), *then for* $c = a, b$,

$$\frac{R_{n'}(c)}{\sqrt{k_{n'}}\, B_{n'}} \to \int_{-W(c)}^0 \psi_c(x)\, dx$$

almost surely as $n' \to \infty$.

PROOF. By the change of variables

$$s = \frac{\lceil ck_n \rceil}{k_n} + \frac{x}{\sqrt{k_n}},$$

where c is either a or b, we obtain

$$\frac{R_{n'}(c)}{\sqrt{k_{n'}}\, B_{n'}} = \int_0^{-v_{n'}(\lceil ck_{n'} \rceil / k_{n'})} \left\{ w_{n'}\left(\frac{\lceil ck_{n'} \rceil}{k_{n'}} + \frac{x}{\sqrt{k_{n'}}} \right) + x \right\}$$
$$\times d \frac{H\left(\dfrac{\lceil ck_{n'} \rceil}{n'} + x \dfrac{\sqrt{k_{n'}}}{n'} \right) - H\left(\dfrac{\lceil ck_{n'} \rceil}{n'} \right)}{B_{n'}}.$$

Therefore, denoting the integrator by $\psi^*_{n'}(c; \cdot\,)$, using (2.2) and (2.4) for $T = 2c$, say, and the almost sure uniform continuity of $W(\cdot)$ on $[0, 2c]$, it follows that

$$\frac{R_{n'}(c)}{\sqrt{k_{n'}}B_{n'}} = \int_0^{-W(c)+o(1)} \{W(c) + o(1) + x\}\, d\psi^*_{n'}(c;x)$$

$$= \int_0^{-W(c)} \{W(c) + x\}\, d\psi^*_{n'}(c;x) + o(1)$$

$$= \int_0^{-W(c)} x\, d\psi^*_{n'}(c;x) + W(c)\psi^*_{n'}(c; -W(c)) + o(1)$$

$$= \int_0^{-W(c)} x\, d\psi_c(x) + W(c)\psi_c(-W(c)) + o(1)$$

$$= \int_{-W(c)}^{0} \psi_c(x)\, dx + o(1)$$

almost surely as $n' \to \infty$, where we also used the fact that $-W(c)$ is almost surely a continuity point of $\psi_c(\cdot)$ □

The next lemma is needed for the proof of the degeneracy statements in Theorem 1.

LEMMA 3. *Let g and h be two Borel measurable functions on $\mathbb{R}$ and let Z_1 and Z_2 be two independent nondegenerate normal random variables. If $g(Z_1) = h(Z_1 + Z_2)$ almost surely, then g and h are constant almost everywhere on $\mathbb{R}$.*

PROOF. The condition implies that $P\{g(Z_1) = h(Z_1 + Z_2) | Z_1 = z\} = 1$ for almost all $z \in \mathbb{R}$, which in turn implies that $P\{g(z) = h(z + Z_2)\} = 1$ for almost all $z \in \mathbb{R}$. Thus $g(z) = h(x)$ for almost every z and x, which occurs only if both g and h are the same constant almost everywhere. □

PROOF OF THEOREM 1. All the statements in (1.9), (1.10) and (1.14) follow directly from the distributional equality (2.5) and Lemmas 1 and 2.

To prove (1.11) in case (i), let $x \in \mathbb{R}$ be arbitrary and consider a continuity point s of φ in (a, b). Then, since H is nondecreasing, $\psi_{n''}(a; x) \le \varphi_{n''}(s)$ for all n'' large enough. Hence, letting $n'' \to \infty$, $\limsup \psi_{n''}(a; x) \le \varphi(s)$, which, letting now $s \downarrow a$, implies the statement for ψ_a. Similarly, for all $x \in \mathbb{R}$ and n'' large enough,

$$\psi_{n''}(b;x) \ge \{H(sk_{n''}/n'') - H(\lceil bk_{n''}\rceil/n'')\}/\Delta^*_{n''}(a,b)$$
$$= \varphi_{n''}(s) - 1,$$

and the statement for ψ_b follows by letting first $n'' \to \infty$ and then $s \uparrow b$.

The corresponding statement (1.15) in case (ii) follows in exactly the same way, using also (1.13).

Finally, we prove the claims about the degeneracy of the limiting random variables. If all ψ_a, φ and ψ_b are identically 0, then $V(\psi_a, \varphi, \psi_b) = 0$ almost surely. To prove the converse statements, set

$$g_c(y) = \int_0^{-y} \psi_c(x)\,dx, \qquad y \in \mathbb{R}, c = a, b.$$

Assume that in case (ii), $V(\psi_a, 0, \psi_b) = C$ almost surely for some $C \in \mathbb{R}$. Writing $Z_1 = W(a)$, $Z_2 = W(b) - W(a)$, $g = g_a$, and $h = g_b + C$, the degeneracy statement follows directly from Lemma 3.

Assume now that $V(\psi_a, \varphi, \psi_b) = C$ almost surely in case (i). Using the notation just introduced, this means that

$$g_a(Z_1) + \gamma_1 Z_1 + Z_3 + g_b(Z_1 + Z_2) - C + \gamma_2 Z_2 = 0 \tag{2.6}$$

almost surely, where $\gamma_1 = \varphi(b+) - \varphi(a)$,

$$\gamma_2 = \int_{[a,b]} (u - a)\,d\varphi(u)/(b - a)$$

and

$$Z_3 = \int_{[a,b]} W(u)\,d\varphi(u) - \gamma_1 W(a) - \gamma_2(W(b) - W(a))$$

$$= \int_{[a,b]} W(u)\,d\varphi(u) - \gamma_1 Z_1 - \gamma_2 Z_2.$$

so that Z_1, Z_2, and Z_3 are independent. Therefore, (2.6) can happen only if both Z_3 and $g_a(Z_1) + \gamma_1 Z_1 + g_b(Z_1 + Z_2) - C + \gamma_2 Z_2$ are degenerate. However, since $\varphi(a) = 0$, Z_3 can degenerate only if $\varphi \equiv 0$ on $[a, b]$. But in this case $\gamma_1 = \gamma_2 = 0$, and hence Lemma 3 implies again that $\psi_a = \psi_b \equiv 0$ on $\mathbb{R}$ exactly as in case (ii). The theorem is completely proved. □

In the proof of Theorem 2 we will use the representation in (2.5) in the form

$$I_n(a,b) - \mu_n(a,b) =_{\mathscr{D}} M_n + R_n(b) - R_n(a). \tag{2.7}$$

where for $c = a, b$, as seen in the proof of Lemma 2,

$$R_n(c) = \int_0^{-v_n(\lceil ck_n \rceil/k_n)} \left\{ w_n\left(\frac{\lceil ck_n \rceil}{k_n} + \frac{x}{\sqrt{k_n}} \right) + x \right\} dH\left(\frac{\lceil ck_n \rceil}{n} + x\frac{\sqrt{k_n}}{n} \right)$$

and, transforming back the tail empirical process,

$$M_n = \int_{\lceil ak_n \rceil/n}^{\lceil bk_n \rceil/n} n(G_n(u) - u)\,dH(u),$$

where

$$G_n(u) = u + \sqrt{k_n}\, w_n(nu/k_n)/n, \qquad 0 \le u \le 1,$$

for which we have $\{G_n(u)\colon 0 \le u \le 1\} =_{\mathscr{D}} \{\tilde{G}_n(u)\colon 0 \le u \le 1\}$.

We begin by establishing several lemmas needed in the proof.

Lemma 4. *For* $D_n(a,b) = |R_n(b) - R_n(a)|/(\sqrt{k_n}\,\Delta_n^*(a,b))$ *we have*

$$\lim_{M\to\infty}\liminf_{n\to\infty} P\{D_n(a,b) < M\} > 0.$$

Proof. In the course of the proof of (1.11) we have already shown that

$$\limsup_{n\to\infty} \psi_n(a;x) \le 1 \quad \text{for } x > 0 \tag{2.8}$$

and

$$\limsup_{n\to\infty} (-\psi_n(b;x)) \le 1 \quad \text{for } x < 0. \tag{2.9}$$

Also, as in the proof of Lemma 2,

$$\begin{aligned} D_n(a,b) = \bigg| &\int_0^{-W(b)+o(1)} \{W(b) + x + o(1)\}\, d\psi_n(b;x) \\ &- \int_0^{-W(a)+o(1)} \{W(a) + x + o(1)\}\, d\psi_n(a;x) \bigg| \end{aligned} \tag{2.10}$$

almost surely as $n \to \infty$. Hence, introducing the event $A = \{-1 < -W(b) < -1/2,\ 1/2 < -W(a) < 1\}$ and using (2.8) and (2.9), we see that $D_n(a,b)\mathbb{1}_A \le \mathbb{1}_A\{4 + o(1)\}$ almost surely for all n large enough, where $\mathbb{1}_A$ is the indicator of A. Noting that $P\{A\} > 0$, the lemma follows. □

Lemma 5. *Whenever there exists a subsequence* $\{n'\} \subset \{n\}$ *such that* $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough and*

$$-\psi_{n'}(a;x) \to \infty \qquad \textit{for some } x < 0 \textit{ as } n' \to \infty \tag{2.11}$$

or

$$\psi_{n'}(b;x) \to \infty \quad \textit{for some } x > 0 \textit{ as } n' \to \infty \tag{2.12}$$

and a sequence of positive constants $A_{n'}$ *such that*

$$D_{n'}(a,b)\sqrt{k_{n'}}\,\Delta_{n'}(a,b)\Big/\left(\sqrt{k_{n'}}\,\Delta_{n'}(a,b) \vee A_{n'}\right) = O_P(1) \tag{2.13}$$

as $n' \to \infty$, *where* $D_n(a,b)$ *is as in Lemma* 4 *and* $x \vee y = \max(x,y)$, *then*

$$\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/A_{n'} \to 0 \quad \textit{as } n' \to \infty \tag{2.14}$$

and for all $x \in \mathbb{R}$,

$$\limsup_{n'\to\infty} \sqrt{k_{n'}}\,\Delta_{n'}^*(a,b)\{|\psi_{n'}(a;x)| + |\psi_{n'}(b;x)|\}\big/A_{n'} < \infty. \tag{2.15}$$

Proof. First assume that (2.11) holds for some $x < 0$. [By (2.8) we only have to consider negative arguments.] Choose any $y < x < 0$ and consider the event $B = \{-W(a) < 2y,\ -1 \le -W(b) < -1/2\}$. Using (2.10), (2.9) and

(2.11), we see that with probability 1 for all n' large enough,

$$D_n(a,b)\,\mathbb{I}_B \geq \mathbb{I}_B\left\{\int_y^0 (-2y + z + o(1))\, d\psi_{n'}(a;z) - 2(1+o(1))\right\}$$
$$\geq \mathbb{I}_B\{(y + o(1))\psi_{n'}(a;y) - 2(1+o(1))\}.$$

Since $P\{B\} > 0$ and $y\psi_{n'}(a;y) \to \infty$ as $n' \to \infty$, in order for (2.13) to hold we obviously must have

$$\limsup_{n'\to\infty}\left\{-\psi_{n'}(a;x)\sqrt{k_{n'}}\,\Delta_{n'}(a,b)\Big/\left(\sqrt{k_{n'}}\,\Delta_{n'}(a,b) \vee A_{n'}\right)\right\} < \infty$$

for all $x \in \mathbb{R}$ by the monotonicity of $\psi_{n'}(a;\cdot)$, which of course can only happen if (2.14) holds and

$$\limsup_{n'\to\infty} |\psi_{n'}(a;x)|\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/A_{n'} < \infty \quad \text{for all } x \in \mathbb{R}.$$

A similar argument shows that if (2.12) holds, then again (2.14) must be true and

$$\limsup_{n'\to\infty} |\psi_{n'}(b;x)|\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/A_{n'} < \infty \quad \text{for all } x \in \mathbb{R}.$$

Putting everything together and noting also that by (1.6) we presently have $\Delta_{n'}(a,b) = \Delta^*_{n'}(a,b)$ for all large n', it is now routine to argue that whenever (2.11) or (2.12) hold along with (2.13), we must have (2.14) and (2.15). □

A slight variation of the proof of Lemma 5 also gives the following.

LEMMA 6. *Whenever there exists a subsequence* $\{n'\} \subset \{n\}$ *such that* $\Delta_{n'}(a,b) = 0$ *for every* n' *and positive constants* $A_{n'}$ *such that* $\{R_{n'}(b) - R_{n'}(a)\}/A_{n'} = O_P(1)$ *as* $n' \to \infty$, *then we have* (2.15) *for all* $x \in \mathbb{R}$.

Next we look at the term M_n in (2.7). Since, using the definition in (1.5) and the notation $x \wedge y = \min(x,y)$, we clearly have

$$EM_n^2 = \int_{\lceil ak_n\rceil/n}^{\lceil bk_n\rceil/n}\int_{\lceil ak_n\rceil/n}^{\lceil bk_n\rceil/n} n(u \wedge v - uv)\, dH(u)\, dH(v) \leq \lceil bk_n\rceil\, \Delta_n^2(a,b)$$

for any $n \geq 1$, we obtain the following.

LEMMA 7. *On any subsequence* $\{n'\} \subset \{n\}$ *on which* $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough,*

$$M_{n'}\Big/\left(\sqrt{k_{n'}}\,\Delta^*_{n'}(a,b)\right) = O_P(1) \quad \text{as } n' \to \infty.$$

Finally, we quote a variant of Lemma 2.10 of Csörgő, Haeusler and Mason (1988), which is obtained by the original proof.

LEMMA 8. *Let $Y_{1,n}$ and $Y_{2,n}$ be two sequences of random variables such that $Y_{1,n} + Y_{2,n} = O_P(1)$ as $n \to \infty$, the sequences $|Y_{1,n}|$ and $|Y_{2,n}|$ are asymptotically independent, and for at least one of $i = 1$ or $i = 2$,*

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\{|Y_{i,n}| < M\} > 0. \tag{2.16}$$

Then both sequences $Y_{1,n}$ and $Y_{2,n}$ are stochastically bounded.

PROOF OF THEOREM 2. Assume (1.16). To get rid of the centering, for each n' let $M'_{n'} + R'_{n'}(b) - R'_{n'}(a)$ be an independent copy of $M_{n'} + R_{n'}(b) - R_{n'}(a)$ in the representation (2.7). Then, since (1.16) can be written as

$$A_{n'}^{-1}\{I_{n'}(a,b) - \mu_{n'}(a,b)\} + A_{n'}^{-1}(\mu_{n'}(a,b) - C_{n'}) \to_{\mathscr{D}} V^* \tag{2.17}$$

as $n' \to \infty$, it implies that the sequence

$$\{M_{n'} + R_{n'}(b) - R_{n'}(a) - (M'_{n'} + R'_{n'}(b) - R'_{n'}(a))\}\Big/\Big(A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b)\Big)$$

is stochastically bounded. Since by Lemmas 4 and 7 the sequence

$$Y_{1,n'} = \{M_{n'} + R_{n'}(b) - R_{n'}(a)\}\Big/\Big(A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b)\Big)$$

obviously satisfies (2.16), Lemma 8 therefore forces

$$\{M_{n'} + R_{n'}(b) - R_{n'}(a)\}\Big/\Big(A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b)\Big) = O_P(1) \quad \text{as } n' \to \infty,$$

which by Lemma 7 implies that

$$\{R_{n'}(b) - R_{n'}(a)\}\Big/\Big(A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b)\Big) = O_P(1) \quad \text{as } n' \to \infty. \tag{2.18}$$

We shall now show that (2.17) and (2.18) imply the existence of a subsequence $\{n''\} \subset \{n'\}$ such that (1.17) and (1.18) hold along $\{n''\}$. We must consider three separate cases.

CASE 1. $\Delta_{n'}(a,b) > 0$ *for all n' large enough and*

$$\limsup_{n'\to\infty} \big(|\psi_{n'}(a;x)| + |\psi_{n'}(b;x)|\big) < \infty \quad \textit{for all } x \in \mathbb{R}. \tag{2.19}$$

In this case, by a Helly selection one can choose a subsequence $\{n'''\} \subset \{n'\}$ such that

$$\psi_{n'''}(c;\cdot) \Rightarrow \psi_c^*(\cdot) \quad \text{on } \mathbb{R} \text{ as } n''' \to \infty,\ c = a, b, \tag{2.20}$$

for some functions ψ_a^* and ψ_b^* satisfying the usual conditions. By Lemma 1 and its proof we know that for a function φ, along perhaps a further subsequence $\{n''\} \subset \{n'''\}$, we have (1.9),

$$Y_{n''} := \frac{M_{n''}}{\sqrt{k_{n''}}\, \Delta^*_{n''}(a,b)} \to Z(\varphi) := \int_{[a,b]} W(s)\, d\varphi(s)$$

and

$$Y_{n''} = \int_a^{\lceil bk_{n''}\rceil/k_{n''}} W(s)\, d\varphi_{n''}(s) + o(1) =: Z_{n''} + o(1)$$

almost surely as $n'' \to \infty$, where $Z_{n''}$ is easily seen to be a normal random variable with mean 0 and

$$EZ_{n''}^2 = \int_a^{\lceil bk_{n''}\rceil/k_{n''}} \int_a^{\lceil bk_{n''}\rceil/k_{n''}} (u \wedge v)\, d\varphi_{n''}(u)\, d\varphi_{n''}(v) \geq a.$$

This implies that the normal random variable $Z(\varphi)$ is nondegenerate, which in turn implies that $\varphi \not\equiv 0$. Then by part (i) of Theorem 1 we have (1.10) with the nondegenerate limit $V(\psi_a^*, \varphi, \psi_b^*)$ and this, (2.17) and the convergence of types theorem yield (1.18) and

$$\{\mu''_n(a,b) - C_{n''}\}/A_{n''} \to \gamma \quad \text{as } n'' \to \infty \tag{2.21}$$

for some constant $\gamma \in \mathbb{R}$. Now (1.18) and (2.20) imply (1.17) with $\psi_c(\cdot) = \delta\psi_c^*(\cdot)$, $c = a, b$, and we see that for V^* in (1.16) we have (1.19) with the γ from (2.21).

CASE 2. $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough and we have* (2.11) *or* (2.12). In this case (2.18) means exactly condition (2.13) of Lemma 5 and hence we have (2.14) and (2.15). Therefore, by (2.15) and a Helly selection we see that there exist an $\{n''\} \subset \{n'\}$ and two functions ψ_a and ψ_b with the usual properties such that (1.17) holds. Now (2.14) means that in fact we have (1.18) with $\delta = 0$ along the original $\{n'\}$. Thus, by part (ii) of Theorem 1,

$$\{I_{n''}(a,b) - \mu_{n''}(a,b)\}/A_{n''} \to_{\mathscr{D}} V(\psi_a, 0, \psi_b) \quad \text{as } n'' \to \infty,$$

which in conjunction with (2.17) gives (2.21) for some $\gamma \in \mathbb{R}$ by convergence of types. Hence we also have (1.19) with $\delta = 0$.

CASE 3. *There exists a subsequence* $\{n'''\} \subset \{n'\}$ *such that* $\Delta_{n'''}(a,b) = 0$ *for all* n'''. Then by (2.18) and Lemma 6 there exist an $\{n''\} \subset \{n'''\}$ and two functions ψ_a and ψ_b with the usual properties such that (1.17) holds, and since (1.18) now trivially holds with $\delta = 0$, the proof can be finished exactly as in Case 2. This completes the proof of the theorem. □

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