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SPEED OF STOCHASTIC LOCALLY CONTRACTIVE SYSTEMS

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The auto-regressive model on $\mathbb{R}^d$ defined by the recurrence equation $Y_n^y = a_n Y_{n-1}^y + B_n$, where $\{(a_n, B_n)\}_n$ is a sequence of i.i.d. random variables in $\mathbb{R}_+^* \times \mathbb{R}^d$, has, in the critical case $\mathbb{E}[\log a_1] = 0$, a local contraction property, that is, when Y_n^y is in a compact set the distance $|Y_n^y - Y_n^x|$ converges almost surely to 0. We determine the speed of this convergence and we use this asymptotic estimate to deal with some higher-dimensional situations. In particular, we prove the recurrence and the local contraction property with speed for an autoregressive model whose linear part is given by triangular matrices with first Lyapounov exponent equal to 0. We extend the previous results to a Markov chain on a nilpotent Lie group induced by a random walk on a solvable Lie group of $\mathcal{N}\mathcal{A}$ type.

1. Introduction. A dynamical system $(\mathcal{X}, F)$ on a metric space $(\mathcal{X}, d)$ is said to be stable or contractive if, for any couple of points x and y in $\mathcal{X}$, the distance $d(F^n x, F^n y)$ converges to 0, that is, asymptotically the evolution of the system does not depend on the starting point. This is the case, for instance, when F is a function with Lipschitz coefficient strictly less than 1. The same concept may be translated in a random setting. Suppose that we have an iterated function system (IFS), that is, a Markov chain $\{Y_n^y\}_n$ defined recursively as

$$Y_n^y = F_n(Y_{n-1}^y), \qquad Y_0^y = y,$$

where $\{F_n\}_n$ is a sequence of i.i.d. random functions defined on a probability space $(\Omega, \mathbb{P})$. We say that the system is *strongly contractive* if the distance between the trajectories starting from any two fixed points x and y goes to zero $\mathbb{P}$-almost surely:

$$\lim_{n\to\infty} d(Y_n^y, Y_n^x) = \lim_{n\to\infty} d(F_n \circ \cdots \circ F_1 x, F_n \circ \cdots \circ F_1 y) = 0.$$

This happens, for instance, when the functions $\{F_n\}_n$ are Lipschitz and are contractive in the mean (i.e., $\mathbb{E}[\log[F_n]_{\text{lip}}] < 0$, where $[F]_{\text{lip}}$ is the Lipschitz coefficient of F) and in this case it is also easy to see that $d(Y_n^y, Y_n^x)$ goes to zero exponentially fast.

Besides this situation, it is possible to have weaker but still interesting stability properties. Using an idea due to Babillot, Bougerol and Elie (1997), Benda introduced the concept of *locally contractive* IFS when the Markov chain $\{Y_n^y\}_n$ satisfies the following recurrence and local contraction properties:

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(R) For some y the set of accumulation points of the sequence $\{Y_n^y(\omega)\}_n$ is $\mathbb{P}(d\omega)$-almost surely not empty.

(C) For every compact set K of $\mathcal{X}$ and every couple of starting points $x, y \in \mathcal{X}$,

$$\lim_{n\to\infty} \mathbb{1}_K(Y_n^y) d(Y_n^y, Y_n^x) = 0, \qquad \mathbb{P}\text{-almost surely.}$$

If the metric space is locally compact and second countable, the local contraction property reinforces the recurrence of these systems. In fact the set of accumulation points of every trajectory $\{Y_n^y(\omega)\}_n$ does not depend on the chance ω nor on the starting point y; this subset of $\mathcal{X}$ is called the *attractor* of the process. Another remarkable property is that, such a process has a unique invariant Radon measure (with finite or infinite total mass) and its support is the attractor set.

A typical example is given by the autoregressive process Y_n^y defined on $\mathbb{R}^d$ by

$$Y_n^y = a_n Y_{n-1}^y + B_n, \qquad Y_0^y = y,$$

where $\{(a_n, B_n)\}_n$ are i.i.d. random variables with values in $\mathbb{R}_+^* \times \mathbb{R}^d$, that is, an IFS defined from a sequence of random affine transformations. In this case

$$Y_n^y - Y_n^x = e^{S_n}(y - x),$$

where

$$S_n = \sum_1^n \log a_i \quad \text{and} \quad S_0 = 0.$$

If $\mathbb{E}[\log a_1] < 0$, then S_n/n converges to a negative number and $|Y_n^y - Y_n^x|$ goes to 0 with an exponential speed, so that the system is strongly contractive in the above sense. But when $\mathbb{E}[\log a_1] = 0$, that is, when the dilatation coefficient is centered, S_n is a recurrent random walk on $\mathbb{R}$, so that the distance $|Y_n^y - Y_n^x|$ cannot converge to 0. However, it was shown by Babillot, Bougerol and Elie (1997) that even in this case Y_n^y is a locally contractive system, which in this context means that, for every y and for every compact subset K of $\mathbb{R}^d$, $e^{S_n}\mathbb{1}_K(Y_n^y)$ converges almost surely to 0 [see also Brofferio (2002) for an elementary proof of this fact].

One of the goals of this paper is to study in more detail this process and in particular to show that when the events $[Y_n^y \in K]$ occur, the random walk S_n must not only tend toward $-\infty$, but has also to be close to its minimal value. More precisely the first section of this paper is dedicated to the proof of the following theorem.

THEOREM 1.1. *Consider the auto-regressive process $\{Y_n^y\}_{n\in\mathbb{N}}$ defined above. Assume that the following standard irreducibility and moment hypothesis*

$$\forall\, y, \qquad \mathbb{P}[a_1 y + B_1 = y] < 1 \quad \textit{and} \quad \mathbb{P}[a_1 = 1] < 1$$

and

$$\mathbb{E}\big[|\log a_1|^2\big] < \infty \quad \textit{and} \quad \mathbb{E}\big[(\log^+ |B_1|)^{2+\delta}\big] < \infty \qquad \textit{for some } \delta > 0,$$

and suppose that

$$\mathbb{E}[\log a_1] = 0.$$

Set $M_n = \min\{S_k, k = 0, \dots, n\}$. *Then, for any compact set* $K \subset \mathbb{R}^d$ *and any point* $y \in \mathbb{R}^d$, *we have*

$$\lim_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{S_n}{M_n} = 1, \qquad \mathbb{P}\text{-almost surely.}$$

In particular, using classical results on the minimal values of a recurrent random walk on $\mathbb{R}$, this gives a speed for the rate of contraction.

COROLLARY 1.2. *If we reinforce the hypotheses of Theorem* 1.1 *and assume that* $|\log a_1|^3$ *is integrable, then for any* $\varepsilon > 0$.

$$\limsup_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{\log |Y_n^y - Y_n^x|}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0.$$

In particular, for any $\chi < \frac{1}{2}$ *one has*

$$\lim_{\substack{n\to+\infty \\ Y_n^y \in K}} e^{n^\chi} |Y_n^y - Y_n^x| = 0.$$

It is natural and interesting in various applications to consider autoregressive models of higher dimensions, that is, processes on $\mathbb{R}^d$ defined recursively by

$$Y_n^y = A_n Y_{n-1}^y + B_n \quad \text{and} \quad Y_0^y = y,$$

where now $\{(A_n, B_n)\}_n$ is a sequence of i.i.d. random variables in $M(\mathbb{R}^d) \times \mathbb{R}^d$. If the positive part of $\log \|A_1\|$ is integrable, it is possible to define the *first Lyapounov exponent* as the almost-sure limit

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \log \|A_n \cdots A_1\|.$$

This quantity will naturally play the role that $\mathbb{E}[\log a_1]$ had in the one-dimesional case. The case $\lambda_1 < 0$ is well understood [see, e.g., Kesten (1973) and Le Page (1983)] and one knows that, under some moment conditions, Y_n^y is a recurrent and strongly contractive process. Less is known for the critical case $\lambda_1 = 0$. We conjecture that in this situation (at least if λ_1 is simple) Y_n^y is recurrent and satisfies the local contraction property. In the second part of this paper, we shall provide some evidence giving a class of examples where this conjecture holds. For this we shall apply the quantitative estimate of Corollary 1.2 to some cases where the matrices A_n are triangular. A different situation where the conjecture might be easier to check arises when one supposes that the laws of the vector B_1 and of the matrix A_1 have a rotation invariant density. In this situation, using some of the results of the present paper, Bougerol has noticed that the process Y_n^y can be proved to be recurrent, but we do not yet know whether it is a locally contractive system.

Our first example where the conjecture holds is given when the diagonal matrices

$$\begin{bmatrix} a_n & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

in the previous model are perturbed by a unipotent factor; thus we consider an autoregressive scheme $Y_n^y = A_n Y_{n-1}^y + B_n$ obtained recursively by the action on $\mathbb{R}^d$ of i.i.d. random affine transformations whose linear part are upper triangular matrices of the form

$$A_n = \begin{bmatrix} a_n & * & * \\ & \ddots & * \\ 0 & & a_n \end{bmatrix}, \tag{1.1}$$

where $a_n > 0$. We shall obtain that, when $\mathbb{E}[\log a_1] = 0$, this system is still locally contractive.

A more intricate situation arises when the matrices A_n have different eigenvalues. We consider the particular case when

$$A_n = \begin{bmatrix} A_n^1 & & 0 \\ & \ddots & \\ 0 & & A_n^k \end{bmatrix} \tag{1.2}$$

and the submatrices A_n^i are "Jordan blocks" of the previous type (1.1). One of the problems that often arises in higher dimensions is that it is much more difficult to obtain recurrent systems. For instance, we can observe that in the simplest case where the matrices A_n are diagonal,

$$\begin{bmatrix} a_n^1 & & 0 \\ & \ddots & \\ 0 & & a_n^k \end{bmatrix},$$

the translation part is

$$B_n = \begin{pmatrix} b_n^1 \\ \vdots \\ b_n^k \end{pmatrix}$$

and the variables (a_n^i, b_n^i) for $i = 1, \dots, k$ are mutually independent (then we just deal with k independent random walks on the one dimensional affine group), we cannot have recurrence of the autoregressive model if three or more of the eigenvalues a_n^i are centered. Indeed it follows from the local limit theorem on the affine group [Le Page and Peigné (1997)] that, if q is the number of centered eigenvalues, for every compact set the probability $\mathbb{P}[Y_n^y \in K]$ goes to zero as $n^{-q/2}$; thus if $q \geq 3$, the series $\sum \mathbb{P}[Y_n^y \in K]$ converges and the expected number of visits of Y_n^y in K is finite, so that in this case the autoregressive process cannot have any accumulation point.

In order to simplify, we will consider, in the sequel, the case where only one eigenvalue is centered while the others are negative in mean.

THEOREM 1.3. *Let $a_n^i, i = 1, \dots, k$, be the eigenvalues of the matrices A_n of the type* (1.2) *and suppose that suitable irreducibility and moment hypotheses hold. If there exists i_0 such that*

$$\mathbb{E}[\log a_n^{i_0}] = 0 \quad \textit{and} \quad \mathbb{E}[\log a_n^i] < 0 \qquad \textit{for all } i \neq i_0$$

then the Markov chain Y_n^y defined by $Y_n^y = A_n Y_{n-1}^y + B_n$ is a locally contractive system with a speed faster than $e^{-\sqrt{n}(\log n)^{-1-\varepsilon}}$ for any $\varepsilon > 0$.

The case where only two eigenvalues are centered is also interesting, but involves others considerations, which will not be developed here.

Another way to extend the results on the real affine group is to consider a more abstract situation where an Abelian Lie group $\mathcal{A}$ acts on a nilpotent Lie group $\mathcal{N}$. In the third and last section we will be able to deduce from the previous results that the left random walk on the group $\mathcal{N}\mathcal{A}$ induces on $\mathcal{N}$ a Markov chain that is a locally contractive system.

We point out that our results hold without any density assumption on the law of the random variables (A_n, B_n), and thus are applicable to IFS, generated by a finite number of functions chosen at random.

We remark that the hypothesis requiring that the linear part of the affine actions have only real positive eigenvalue is not essential. Indeed it is possible to generalize these results to autoregressive models $Y_n = A_n Y_{n-1} + B_n$ where the matrices A_n are in $\mathbb{R}_+^* \times O(\mathbb{R}^d)$ or are in the "Jordan form" (1.2) but with complex coefficients.

2. Local contraction speed. As said above, we shall first concentrate on the "one-dimensional" situation where the autoregressive model is given by the action of $\mathbb{R}_+^*$ on $\mathbb{R}^d$ and we give a quantitative estimate of the speed of the local contraction when the dilatation coefficient is centered. We first introduce some notation, hypotheses and preliminary results, then we give a proof of Theorem 1.1.

We will denote by $\mathrm{Aff}(\mathbb{R}^d)$ the group of affine transformations of the Euclidean space $\mathbb{R}^d$ of the form

$$g = (a, B) : x \mapsto g \cdot x = ax + B,$$

where a is a positive real number and B a vector in $\mathbb{R}^d$, thus $\mathrm{Aff}(\mathbb{R}^d)$ may be identified with $\mathbb{R}_+^* \times \mathbb{R}^d$. We will denote a and b the projection of $\mathrm{Aff}(\mathbb{R}^d)$ on $\mathbb{R}_+^*$ and $\mathbb{R}^d$, respectively, so that $g = (a(g), b(g))$ for each $g \in \mathrm{Aff}(\mathbb{R}^d)$.

Let $g_n = (a_n, B_n)$ be a sequence of independent identically distributed random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathrm{Aff}(\mathbb{R}^d)$. The (left) *random walk* is the Markov chain on $\mathrm{Aff}(\mathbb{R}^d)$ defined recursively by

$$L_{n+1} = g_{n+1} L_n,$$

when L_0 is the identity. The autoregressive model may be seen as the process induced on $\mathbb{R}^d$ by the random walk on $\mathrm{Aff}(\mathbb{R}^d)$; indeed, if Y_n^y denotes the process

$$Y_n^y = L_n \cdot y$$

then it satisfies the equation $Y_n^y = a_n Y_{n-1}^y + B_n$. The starting point y is a random vector independent of the increments g_n and we denote by $\mathbb{P}_\nu$ and $\mathbb{E}_\nu$ the probability and the expectancy under which the distribution of y is ν; we will use the same notation when the measure ν is not a probability, so that $\mathbb{P}_\nu$ may have an infinite mass.

We will assume the following irreducibility conditions

$$\forall\, y : \mathbb{P}[a_1 y + B_1 = y] < 1 \quad \text{and} \quad \mathbb{P}[a_1 = 1] < 1, \tag{2.1}$$

which means that the random walk does not live in the translation group or in the group of homotheties.

We observe that

$$S_n = \log(a(L_n)) = \sum_{k=0}^{n} \log a_k$$

is just a sum of real valued i.i.d. random variables. When the variable $\log a_1$ is integrable and $\mathbb{E}[\log a_1] < 0$, the system is strongly contractive with an exponential speed; furthermore, if $\mathbb{E}[\log^+ |B_1|]$ is finite, then Y_n^y is ergodic and has a unique invariant probability measure. We will deal with the more delicate case where the strong contraction property does not hold; our main assumption in this section is that

$$\mathbb{E}[\log a_1] = 0$$

so that S_n is a recurrent random walk on $\mathbb{R}$. If we reinforce the moment hypotheses as follows

$$\mathbb{E}\big[|\log a_1|^2\big] < \infty \quad \text{and} \quad \mathbb{E}\big[(\log^+ |B_1|)^{2+\delta}\big] < \infty \qquad \text{for some } \delta > 0, \tag{2.2}$$

the Markov chain Y_n^y is still recurrent and it has a unique invariant Radon measure [Babillot, Bougerol and Elie (1997)], but this measure is not a probability [Bougerol and Picard (1992)].

In order to approach the study of a centered random walk we will use a classical technique in this situation and extract a strictly decreasing subrandom walk. We first introduce the sequence of stopping times at which the random walk S_n reaches a new minimum, generally known as *ladder stopping times*:

$$l_n = \inf\{k > l_{n-1} : S_k - S_{l_{n-1}} < 0\}$$

for all $n \geq 1$ and $l_0 = 0$. The sequence $\{L_{l_n}\}_n$ is still a random random walk on $\mathrm{Aff}(\mathbb{R}^d)$ since each element is the product of the independent random variables $L_{l_k} L_{l_{k-1}}^{-1} = g_{l_k} \cdots g_{l_{k-1}+1}$ that have all the same distribution as L_{l_1}. The second moment conditions (2.2) has been shown [see Elie (1982)] to be sufficient to guarantee that

$$\mathbb{E}\big[\big|\log\big(a(L_{l_1})\big)\big|\big] < +\infty \quad \text{and} \quad \mathbb{E}\big[\log^+ \big|b\big(L_{l_1}\big)\big|\big] < \infty. \tag{2.3}$$

Thus L_{l_n} is a random walk with first moment and such that $\log a(L_{l_1}) = S_{l_1} < 0$; the extracted Markov chain $Y_{l_n}^y = L_{l_n} \cdot y$ is then an ergodic Markov chain with a unique invariant probability measure, that we will denote by ν^l.

The recurrence of the subchain $Y_{l_n}^y$ implies that Y_n^y is also recurrent. Even if Y_n^y has no invariant probability measure, it has a unique invariant Radon measure ν. This in particular implies that for, almost every y, the process Y_n^y visits infinitely often every subset K of $\mathbb{R}^d$ of positive ν-measure. Therefore we can consider the sequence of stopping times at which Y_n^y enters in K,

$$t_n^K = t_n^{y,K} = \inf\big\{k > t_{n-1}^{y,K} : Y_k^y \in K\big\},$$

for all $n \geq 1$ and $t_0^{y,K} = 0$. The restricted process to K, $Y_{t_n^{y,K}}^y$, is then a Markov chain with invariant probability $\nu_K(\cdot) = \nu(\cdot \cap K)/\nu(K)$ [see, e.g., Meyn and Tweedie (1993), Theorem 10.4.7].

The uniqueness of the invariant measure of process Y_n^y is a consequence of the local contraction property that was shown by Babillot, Bougerol and Elie (1997); for every x and y in $\mathbb{R}^d$ we have

$$|Y_n^y - Y_n^x| \mathbb{1}_K(Y_n^y) = e^{S_n} |y - x| \mathbb{1}_K(L_n \cdot y) \to 0, \qquad \mathbb{P}\text{-almost surely.} \tag{2.4}$$

Thus we have $\lim_{n\to\infty} S_{t_n^{y,K}} = -\infty$ and the goal of this section is to estimate the speed of this convergence.

Our first step is to prove that the sequence $S_{t_n^{y,K}}$ satisfies a law of large numbers.

PROPOSITION 2.1. *Suppose that the hypotheses* (2.1) *and* (2.2) *hold and that* $\mathbb{E}[\log a_1] = 0$, *then there is a compact set* K_0 *such that for every* $K \supseteq K_0$ *with* $\nu(K) < \infty$ *one has*

$$-\infty < \mathbb{E}_{\nu_K}\big[S_{t_1^K}\big] < 0$$

and

$$\lim_{n\to\infty} \frac{S_{t_n^{y,K}}}{n} = \mathbb{E}_{\nu_K}\big[S_{t_1^K}\big], \qquad \mathbb{P}_{\nu_K}\text{-almost surely.}$$

PROOF. We first observe that under the probability $\mathbb{P}_{\nu_K}$ the sequence

$$S_{t_n^{y,K}} = \sum_{k=1}^{n} S_{t_{k+1}^{y,K}} - S_{t_k^{y,K}}$$

is a sum of stationary real random variables. In fact the process

$$W_n = \left(S_{t_{n+1}^{y,K}} - S_{t_n^{y,K}}, Y^y_{t_n^{y,K}}\right)$$

is a Markov chain on $\mathbb{R} \times K$ whose invariant probability measure is given by

$$m(f) = \mathbb{E}_{\nu_K}\left[f\left(S_{t_1^{y,K}}, y\right)\right].$$

We recall the following ergodic lemma first proved by Kesten [see Lemma 2.3 in Bougerol and Lacroix (1985) for a proof].

LEMMA 2.2. *Let $\{X_n\}_n$ be a sequence of stationary real variables such that $\mathbb{E}[X_1^-] < +\infty$ and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} X_k = -\infty \qquad \textit{almost surely},$$

then $\mathbb{E}[|X_1|]$ is finite and $\mathbb{E}[X_1] < 0$.

As a direct consequence one obtains

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} X_k}{n} = \mathbb{E}[X_1] \qquad \text{almost surely}.$$

By the local contraction property we already know that $S_{t_n^{y,K}}$ converges to $-\infty$. If we show that $\mathbb{E}_{\nu_K}[S_{t_1^K}] > -\infty$, we will be able to conclude that $\mathbb{E}_{\nu_K}[S_{t_1^K}] < 0$ and

$$\frac{S_{t_n^{y,K}}}{n} \to \mathbb{E}_{\nu_K}[S_{t_1^K}], \qquad \mathbb{P}_{\nu_K}\text{-almost surely}.$$

Let us first notice that the entrance time t_1^K is not integrable, because the random walk is centered. In order to show that $S_{t_1^K}$ is integrable, we shall bound it from below by the strictly decreasing random walk S_{l_n}.

Let $\bar{t}^K$ be the first time at which $Y^y_{l_n}$ enters in K. Since $Y^y_{l_{\bar{t}^K}} \in K$ we have $t_1^K \le l_{\bar{t}^K}$; on the other hand, for every $k \le l_n$, one has $S_k \ge S_{l_n} = M_{l_n}$, and so $S_{t_1^K} \ge S_{l_{\bar{t}^K}}$.

We have remarked that under the moment assumption (2.2) the random walk L_{l_n} is integrable in the sense (2.3) and, since $\log a(L_{l_1}) < 0$, we can apply to L_{l_n} the following lemma. Its proof is rather technical and it is postponed to the Appendix.

LEMMA 2.3. *Let $\{(a_n, B_n)\}_n \subset \mathrm{Aff}(\mathbb{R}^d)$ be a sequence of independent identically distributed random variables. Suppose that the irreducibility hypothesis* (2.1) *holds and that*

$$\mathbb{E}[|\log a_1|] < +\infty \quad \textit{and} \quad \mathbb{E}[\log^+ |B_1|] < \infty.$$

Assume furthermore that $\mathbb{E}[\log a_1] < 0$. Let $Y_n^y = a_n Y_{n-1}^y + B_n$ be the induced process on $\mathbb{R}^d$ and $t_n^K = t_n^{y,K}$ the sequence of entrance times of Y_n^y in K. Then there is a compact set K_0 such that, for every $K \supseteq K_0$ and every compact set C, there exists a constant $M = M(C, K)$ satisfying the following property:

$$\forall x \in C, \qquad \mathbb{E}_x[t_1^K] < M \quad \textit{and} \quad \mathbb{E}_x\big[S_{t_1^K}\big] > M\mathbb{E}[\log a_1].$$

Since $\bar{t}^K$ it is the first time on which $Y_{l_n}^y$ enters in K (that we have chosen sufficiently large), there is a constant M_0 such that for every $x \in K$, $\mathbb{E}_x[S_{l_{\bar{t}^K}}] > M_0$; thus, when y is distributed as ν_K, $\mathbb{E}_{\nu_K}[S_{l_{\bar{t}^K}}] > M_0$. Finally,

$$\mathbb{E}_{\nu_K}\big[S_{t_1^K}\big] \geq \mathbb{E}_{\nu_K}\big[S_{l_{\bar{t}^K}}\big] > -\infty,$$

which completes the proof of Proposition 2.1. $\square$

We are now able to give a rate of escape of S_n restricted to the events $[Y_n^y \in K]$.

THEOREM 2.4. *Suppose that the hypotheses* (2.1) *and* (2.2) *hold and let*

$$M_n = \min_{k=0,\dots,n} S_k,$$

then for every compact set $K \subset \mathbb{R}^d$ and every $x \in \mathbb{R}^d$,

$$\lim_{\substack{n\to+\infty\\ Y_n^x \in K}} \frac{S_n}{M_n} = 1, \qquad \mathbb{P}\textit{-almost surely.} \tag{2.5}$$

In particular, if $\mathbb{E}[|\log a_1|^3] < \infty$ for any $\varepsilon > 0$

$$\limsup_{\substack{n\to+\infty,\\ Y_n^x \in K}} \frac{S_n}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0, \qquad \mathbb{P}\textit{-almost surely .}$$

Figure 1 shows the result of a simulation of the processes S_n and M_n. We have also drawn the trajectory S_n restricted to the events $[Y_n^y \in K]$.

PROOF OF THEOREM 2.4. Before we start with the proof we recall that the unique invariant Radon of the process Y_n^y measure can be obtained by the formula

$$\nu(f) = \frac{\mathbb{E}_{\nu^l}[\sum_{n=0}^{l_1-1} f(Y_n^y)]}{\mathbb{E}[-S_{l_1}]},$$

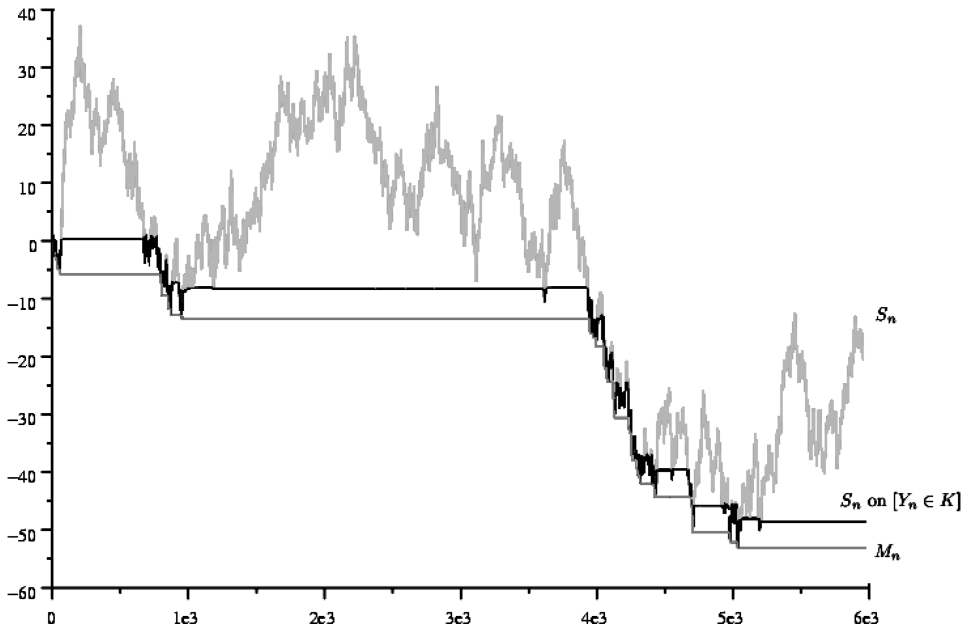

FIG. 1.

where l_1 is the first ladder time and ν^l the invariant probability measure of the extracted chain $Y^y_{l_n}$.

We also know [Babillot, Bougerol and Elie (1997), Theorem 4.1] that for every set K such that $\nu(K)$ is positive and finite

$$\frac{\sum_{k=0}^{n} \mathbb{1}_K(Y^y_k)}{M_n} \to -\nu(K), \qquad \mathbb{P}_{\nu^l}\text{-almost surely,}$$

thus $\mathbb{P}_{\nu^l}$-almost surely

$$\frac{\sum_{k=0}^{t_n^{y,K}} \mathbb{1}_K(Y^y_k)}{M_{t_n^{y,K}}} = \frac{n}{M_{t_n^{y,K}}} \to -\nu(K).$$

Since ν^l is absolutely continuous with respect to ν, the previous proposition implies that, for any sufficiently large compact set K, the sequence $S_{t_n^{y,K}}/n$ converges $\mathbb{P}_{\nu^l_K}$-almost surely to $\mathbb{E}_{\nu_K}[S_{t_1^K}]$ and then

$$\lim_{\substack{n\to+\infty \\ Y^y_n \in K}} \frac{S_n}{M_n} = \lim_{n\to+\infty} \frac{S_{t_n^{y,K}}}{M_{t_n^{y,K}}} = -\nu(K)\mathbb{E}_{\nu_K}[S_{t_1^K}], \tag{2.6}$$

$\mathbb{P}_{\nu^l_K}$-almost surely.

If $\nu^l(K) > 0$, the events $[Y_{l_n}^y \in K]$ occur infinitely often so that

$$\left\{ \frac{S_{l_n}}{M_{l_n}} \,\Big|\, n \in \mathbb{N}, Y_{l_n}^y \in K \right\} \subset \left\{ \frac{S_n}{M_n} \,\Big|\, n \in \mathbb{N}, Y_n^y \in K \right\}$$

is a true subsequence. Then since $S_{l_n}/M_{l_n} = 1$ a.s., for all n, we have

$$-\nu(K)\mathbb{E}_{\nu_K}[S_{t_1^K}] = 1.$$

So that the limit (2.5) holds $\mathbb{P}_{\nu_K^l}$-almost surely.

To conclude we just need to show that we have the limit (2.5) for all starting points and this will be a consequence of the local contraction property. Let $x \in \mathbb{R}^d$ be fixed and $K_\varepsilon = \{z \in \mathbb{R}^d : d(z, K) \le \varepsilon\}$. From (2.6) it follows, in particular, that there is at least a point $y \in \mathbb{R}^d$ such that on the events $Y_n^y \in K_\varepsilon$, the sequence S_n/M_n converges $\mathbb{P}$-almost surely to 1. On the other hand, the local contraction property (2.4) ensures that, for $\mathbb{P}$-almost all $\omega \in \Omega$, there is $N(\omega) \in \mathbb{N}$ such that for every $n \ge N$,

$$Y_n^x(\omega) \in K \quad \Longrightarrow \quad Y_n^y(\omega) \in K_\varepsilon$$

so that

$$\lim_{\substack{n \to +\infty \\ Y_n^x \in K}} \frac{S_n}{M_n} = \lim_{\substack{n \to +\infty \\ Y_n^y \in K}} \frac{S_n}{M_n} = 1.$$

Finally, classical results on the minima of a real random walk [Hirsch (1965), Theorem 2] give the behavior of the superior limit of M_n and then of S_n

$$\begin{aligned} \limsup_{\substack{n \to +\infty \\ Y_n^x \in K}} \frac{S_n}{\sqrt{n}(\log n)^{-1-\varepsilon}} &= \limsup_{\substack{n \to +\infty \\ Y_n^x \in K}} \frac{S_n}{M_n} \frac{M_n}{\sqrt{n}(\log n)^{-1-\varepsilon}} \\ &= \limsup_{\substack{n \to +\infty \\ Y_n^x \in K}} \frac{M_n}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0 \end{aligned}$$

for any $\varepsilon > 0$. $\square$

3. "Jordan blocks." This section is dedicated to a class of higher-dimensional examples, when the dilatation coefficients are matrix valued. The goal is to show that an autoregressive model whose linear parts are matrices composed of "Jordan blocks" (1.2) is a locally contractive system, but before dealing with this general situation, we consider the case where we have just one block.

We consider the group $\mathcal{A}$ of upper triangular matrices in $\mathrm{GL}(\mathbb{R}^d)$ with all diagonal coefficients equal to the same real positive number. We will denote by $\alpha : \mathcal{A} \to \mathbb{R}$ the additive homomorphism such that for every $A \in \mathcal{A}$,

$$A = \begin{bmatrix} e^{\alpha(A)} & * & * \\ & \ddots & * \\ 0 & & e^{\alpha(A)} \end{bmatrix}. \tag{3.1}$$

Every element of $\mathcal{A}$ may be written in the form $A = \exp(\alpha(A)I + N)$ for some N in the algebra $\mathfrak{N}(d)$ of nilpotent matrices of order d.

We consider the group $\mathcal{A} \rtimes \mathbb{R}^d$, semidirect product of $\mathcal{A}$ acting on $(\mathbb{R}^d, +)$ by canonical action; it can be represented as the group of matrices of the form

$$\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\alpha} & & * & & * & \\ & & \ddots & & * & B \\ 0 & & & & e^{\alpha} & \\ 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.2}$$

Let $g_n = (A_n, B_n)$ be a sequence of independent and identically distributed random variables with values in $\mathcal{A} \rtimes \mathbb{R}^d$; set $L_n = g_n \cdots g_1$ the left random walk and

$$Y_n^y = A_n Y_{n-1}^y + B_n = L_n \cdot y, \qquad Y_0^y = y,$$

the associated Markov chain on $\mathbb{R}^d$.

Under a first order moment hypothesis, the Lyapounov exponent of the matrices A_n is $\mathbb{E}[\alpha(A_1)]$, so that if $\mathbb{E}[\alpha(A_1)] < 0$ the system Y_n^y is strongly contractive and it is recurrent as soon as $\mathbb{E}[\log^+ |B_1|]$ is finite. We will now focus on the situation

$$\mathbb{E}[\alpha(A_1)] = 0$$

and show that also in that case we have a local contraction property.

We set $\alpha_n = \alpha(A_n)$ so that $A_n = \exp(\alpha_n I + N_n)$ where $\{(\alpha_n, N_n)\}_n$ is a sequence of i.i.d. random variables with values in $\mathbb{R} \times \mathfrak{N}(d)$.

We introduce the irreducibility conditions

$$\begin{aligned} \forall\, y \in \mathbb{R}^d, \qquad \mathbb{P}\big[(A_1 y + B_1)^{(d)} = y^{(d)}\big] &< 1, \\ \mathbb{P}[\alpha_1 = 0] &< 1, \end{aligned} \tag{3.3}$$

where $B_1^{(d)}$ is the d-component of the vector B_1 and the following moment hypotheses:

$$\begin{aligned} \mathbb{E}\big[|\alpha_1{}^3|\big] &< +\infty, \\ \mathbb{E}[|N_1|] &< +\infty, \\ \mathbb{E}\big[|\log(B_1)|^{2+\delta}\big] &< +\infty \qquad \text{for some } \delta > 0. \end{aligned} \tag{3.4}$$

PROPOSITION 3.1. *Assume that the hypotheses* (3.3) *and* (3.4) *hold and* $\mathbb{E}[\alpha_1] = 0$. *Then the Markov chain* Y_n^y *satisfies the local contraction property with a speed faster than* $e^{-\sqrt{n}(\log n)^{-1-\varepsilon}}$ *for any* $\varepsilon > 0$.

PROOF. Denote by $v^{(d)}$ the Dth component of the vector v. We observe that

$$(Ay + B)^{(d)} = e^{\alpha(A)} y^{(d)} + B^{(d)}.$$

The mapping

$$\mathcal{A} \rtimes \mathbb{R}^d \to \mathrm{Aff}(\mathbb{R}),$$
$$(A, B) \mapsto \big(e^{\alpha(A)}, B^d\big)$$

is then a homomorphism of groups and the sequence

$$(Y_n^y)^{(d)} = e^{\alpha_n}(Y_{n-1}^y)^{(d)} + B_n^{(d)}$$

is a Markov chain on the real line induced by a random walk on the affine group. We can apply to $(Y_n^y)^{(d)}$ the results of the previous section, so that we obtain

$$\limsup_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{\alpha_n + \cdots + \alpha_1}{\sqrt{n}(\log n)^{-1-\varepsilon}} \le \limsup_{\substack{n\to+\infty \\ (Y_n^y)^{(d)} \in K^{(d)}}} \frac{\alpha_n + \cdots + \alpha_1}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0,$$

$\mathbb{P}$-almost surely.

Denote by $\|\cdot\|$ the canonical norm on $\mathrm{GL}(\mathbb{R}^d)$, so that

$$|Y_n^y - Y_n^x| = |A_n \cdots A_1(x-y)| \le \|A_n \cdots A_1\| |y-x|.$$

As $A_i = \exp(\alpha_i I + N_i)$ with $\alpha_i \in \mathbb{R}$, $N_i \in \mathfrak{N}(d)$ and $\mathfrak{N}(d)$ commute with $\mathbb{R}$, we have

$$A_n \cdots A_1 = \exp(\alpha_n + \cdots + \alpha_1) \exp(N_n) \cdots \exp(N_1).$$

Since $\mathfrak{N}(d)$ is a nilpotent algebra it is possible [Elie (1982)] to find a polynomial function Q from $\mathbb{R}$ in $\mathbb{R}_+$ such that, for every $n \in \mathbb{N}$ and every $A_1, \dots, A_n \in \mathcal{A}$,

$$\begin{aligned}\|A_n \cdots A_1\| &= \exp(\alpha_n + \cdots + \alpha_1) \|\exp(N_n) \cdots \exp(N_1)\| \\ &\le \exp(\alpha_n + \cdots + \alpha_1) Q(\|N_1\| + \cdots + \|N_n\|).\end{aligned}$$

On the other hand, by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{\|N_1\| + \cdots + \|N_n\|}{n} = \mathbb{E}[\|N_1\|],$$

so that $\limsup_{n\to\infty} \log Q(\|N_1\| + \cdots + \|N_n\|)/\sqrt{n}(\log n)^{-1-\varepsilon} = 0$. Then we can conclude that

$$\begin{aligned}\limsup_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{\log|Y_n^y - Y_n^x|}{\sqrt{n}(\log n)^{-1-\varepsilon}} &\le \limsup_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{\log(\|A_n \cdots A_1\| |y-x|)}{\sqrt{n}(\log n)^{-1-\varepsilon}} \\ &\le \limsup_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{(\alpha_n + \cdots + \alpha_1) + \log Q(\|N_1\| + \cdots + \|N_n\|)}{\sqrt{n}(\log n)^{-1-\varepsilon}} \\ &< 0.\end{aligned}$$

□

We will now consider the situation where the matrices A_n have more than one "Jordan block." We show that, when one of the eigenvalues is centered, while the others are contractive, the autoregressive model $Y_n^y = A_n Y_{n-1}^y + B_n$ has a local contraction property and it is recurrent, hence it defines a locally contractive system.

Let us fix some notation. Let now $\mathcal{A}$ be the group of the matrices

$$A = \begin{bmatrix} A^1 & & 0 \\ & \ddots & \\ 0 & & A^k \end{bmatrix} \tag{3.5}$$

where the diagonal blocks A^i are of the form (3.1) and let $\alpha^1, \dots, \alpha^k$ be the homomorphisms from $\mathcal{A}$ to $\mathbb{R}$ such that

$$A^i = \exp\bigl(\alpha^i(A) + N^i\bigr), \tag{3.6}$$

where N^i are nilpotent matrices of order d^i.

Then the group $\mathcal{G}$ of the matrices

$$\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix},$$

where B is a vector on $\mathbb{R}^d$ and A is a matrix of the form (3.5), is the direct product of groups $\mathcal{A}^i \ltimes \mathbb{R}^{d^i}$ of the type we have studied in the previous subsection. We denote by $A(\cdot)$ and $B(\cdot)$ the projections of $\mathcal{G}$ on $\mathcal{A}$ and $\mathbb{R}^d$, respectively, so that

$$g = \begin{bmatrix} A(g) & B(g) \\ 0 & 1 \end{bmatrix}.$$

From now on, $g_n = (A_n, B_n)$ will be a sequence of $\mathcal{G}$-valued independent and identically distributed random variables, $L_n = g_n \cdots g_1$ will denote the left random walk on $\mathcal{G}$ and

$$Y_n^y = A_n Y_{n-1}^y + B_n = L_n \cdot y$$

the associated autoregressive scheme. For every n, we will denote by $\alpha_n^i = \alpha^i(A_n)$ the logarithms of the eigenvalues of the matrices A_n and N_n^i the nilpotent matrices of the decomposition (3.6). We need the following moment hypotheses:

$$\begin{aligned} \sum_{i=1}^k \mathbb{E}\bigl[|\alpha_1^i|^{3+\delta}\bigr] &< +\infty \qquad \text{for some } \delta > 0, \\ \sum_{i=1}^k \mathbb{E}\bigl[\|N_1^i\|\bigr] &< +\infty, \\ \mathbb{E}\bigl[(\log^+ |B_1|)^{2+\delta}\bigr] &< +\infty \qquad \text{for some } \delta > 0. \end{aligned} \tag{3.7}$$

We can now show the following:

THEOREM 3.2. *Suppose that* (3.7) *is satisfied and that it exists i_0 such that*

$$\mathbb{E}[\alpha_n^{i_0}] = 0 \quad \textit{and} \quad \mathbb{E}[\alpha_n^i] < 0 \qquad \textit{for all } i \neq i_0.$$

If the sequence of random transformations $(A_n^{i_0}, B_n^{i_0})$ *satisfies the irreducibility hypothesis* (3.3), *then the Markov chain* $\{Y_n^y\}_n$ *is a locally contractive system with a local contraction speed faster than* $e^{-\sqrt{n}(\log n)^{-1-\varepsilon}}$ *for any* $\varepsilon > 0$.

PROOF. The local contraction property is more or less a direct consequence of the Proposition 3.1. In fact,

$$\begin{aligned}
&\limsup_{\substack{n\to\infty\\ Y_n^y\in K}} \frac{\log|Y_n^y - Y_n^x|}{\sqrt{n}(\log n)^{-1-\varepsilon}} \\
&\qquad \le \limsup_{\substack{n\to\infty\\ Y_n^y\in K}} \frac{\log k + \max_{i=1,\dots,k}\log(\|A_n^i\cdots A_1^i\||y-x|)}{\sqrt{n}(\log n)^{-1-\varepsilon}}.
\end{aligned} \tag{3.8}$$

For $i = i_0$ we can apply Proposition 3.1 to the random walk on $\mathcal{A}^{i_0} \rtimes \mathbb{R}^{d^{i_0}}$ obtained by projection from $\mathcal{G}$ and we have

$$\limsup_{\substack{n\to\infty\\ Y_n^y\in K}} \frac{\log\|A_n^{i_0}\cdots A_1^{i_0}\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} \le \limsup_{\substack{n\to\infty\\ Y_n^{i_0,y}\in K^{i_0}}} \frac{\log\|A_n^{i_0}\cdots A_1^{i_0}\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0,$$

where K^{i_0} and $Y_n^{i_0,y}$ are the projections of K and Y_n^y on the subspace of $\mathbb{R}^d$ on which acts $\mathcal{A}^{i_0}$.

When $i \ne i_0$ the strong law of large number readily implies that $(\alpha_1^i + \cdots + \alpha_n^i)/n$ converges almost surely to $\mathbb{E}[\alpha_1^i]$; since $\mathbb{E}[\alpha_1^i]$ is strictly negative, the diagonal part of the product $A_n^i \cdots A_1^i$ converges exponentially fast to 0. On the other hand, the norm nilpotent part can be dominated by a polynomial function Q^i in the sum of the norm so that, for the law of the large number, it has at most a polynomial growth; in conclusion we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log\|A_n^i\cdots A_1^i\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} \\
&\qquad \le \limsup_{n\to\infty} \frac{(\alpha_n^i + \cdots + \alpha_1^i) + \log Q^i(\|N_1^i\| + \cdots + \|N_n^i\|)}{\sqrt{n}(\log n)^{-1-\varepsilon}} = -\infty.
\end{aligned}$$

We shall now prove the recurrence property, showing that the set of accumulation points of the sequence $\{Y_n^y(\omega)\}_n$ is $\mathbb{P}(d\omega)$-almost surely not empty.

As in the first section, we consider the ladder times for the greatest eigenvalue

$$l_n = \inf\{k > l_{n-1} : \alpha_k^{i_0} + \cdots + \alpha_{l_{n-1}+1}^{i_0} < 0\}, \qquad l_0 = 0,$$

and the increments $L_{l_{n+1}}^{l_n} := g_{l_{n+1}} \cdots g_{l_n+1}$. These increments form a sequence of i.i.d. random variables in $\mathcal{G}$ so that $L_{l_n} = L_{l_n}^{l_{n-1}} \cdots L_{l_1}^0$ may be seen as a left random walk whose steps are distributed as L_{l_1}. If we show that L_{l_1} has a first moment, then the infinite sum

$$\sum_{n=1}^{+\infty} A(L_{l_1}^0)\cdots A(L_{l_n}^{l_{n-1}})B(L_{l_{n+1}}^{l_n}) = \lim_{n\to\infty} L_{l_1}^0 \cdots L_{l_{n+1}}^{l_n}\cdot 0$$

converges because $|A(L_{l_n}^{l_0})\cdots A(L_{l_n}^{l_{(n-1)}}-1)B(L_{l_{n+1}}^{l_n})|$ goes to 0 exponentialy fast [in fact, the mean of the eigenvalues of $A(L_{l_n}^{l_{(n-1)}}-1)$ is negative, so that the diagonal part of their products converges to zero exponentially fast; thus it dominates the norm of nilpotent part that is at most polynomial and the B terms whose logarithms are integrable]. The distribution of this sum is an invariant probability measure for the chain $Y_{l_n}^y$ and it is the only one because the system is contractive. Thus $Y_{l_n}^y$ has a unique invariant probability measure and it is recurrent. We then conclude that the set of accumulation points of $\{Y_n^y(\omega)\}_n$ is not empty since it contains the set of accumulation points of $\{Y_{l_n}^y(\omega)\}_n$.

Thus it remains to check the moment condition on L_{l_1}. Let

$$L_{l_1} = \begin{bmatrix} A_{l_1}\cdots A_1 & B_{0,l_1} \\ 0 & 1 \end{bmatrix},$$

then we have to show that the positive parts of $\log\|A_{l_1}\cdots A_1\|$ and $\log|B_{0,l_1}|$ are integrable.

We will first show that $\max_{j=l_1,\dots,1}\log^+\|A_{l_1}\cdots A_j\|$ is integrable or equivalently that

$$\mathbb{E}\Big[\max_{j=l_1,\dots,1}\log^+\|A_{l_1}^i\cdots A_j^i\|\Big] < +\infty \qquad \text{for all } i=1,\dots,k. \tag{3.9}$$

We observe that there exist k polynomial functions Q^i from $\mathbb{R}$ to $\mathbb{R}^+$ such that

$$\begin{aligned}\|A_{l_1}^i\cdots A_j^i\| &= e^{\alpha^i(A_{l_1}^i\cdots A_j^i)}\|\exp(N_{l_1})\cdots\exp(N_j)\| \\ &\le e^{\alpha^i(A_{l_1}^i\cdots A_j^i)} Q^i(\|N_{l_1}^i\|+\cdots+\|N_j^i\|).\end{aligned}$$

We use now the following property often used in this context [cf. Elie (1982) and Grincevičius (1975)] whose proof may be found in Cartwright, Kaĭmanovich and Woess [(1994), Proposition 4]:

LEMMA 3.3. *Let $\{X_n\}_n$ be a sequence of identically distributed positive real random variables adapted to a filtration $\{\mathcal{F}_n\}_n$ such that, for every n, X_n is independent from $\mathcal{F}_{n-1}$ and let τ be a $\mathcal{F}_n$-stopping time. If τ^α and $X_1^{1/\alpha}$ are integrable for some $\alpha<1$, then* $\max_{k=1,\dots,\tau} X_k$ *is integrable.*

Since $(\log^+\|N_1^i\|)^{2+\varepsilon}$ and $l_1^{1/(2+\varepsilon)}$ are integrable [cf. Spitzer (1974)], we conclude at once that for all $i=1,\dots,k$,

$$\mathbb{E}\Big[\max_{j=l_1,\dots,1}\log^+ Q^i(\|N_{l_1}^i\|+\cdots+\|N_j^i\|)\Big] < +\infty.$$

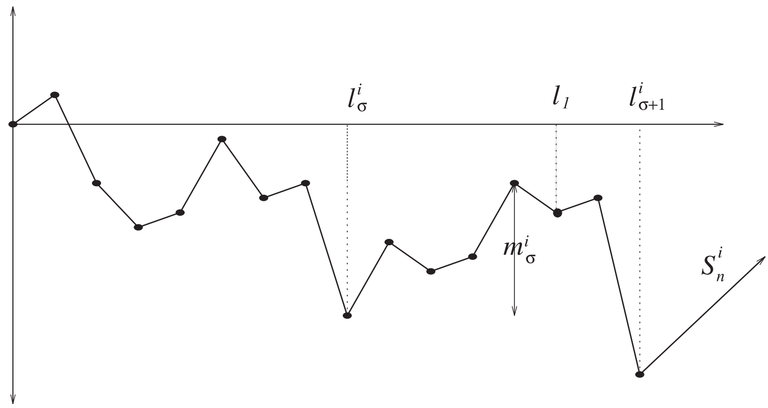

FIG. 2.

In order to prove (3.9), we still have to show that, for every i, the positive part of $\max_{j=1,\dots,l_1} \alpha^i(A_{l_1}\cdots A_j)$ is integrable.

Set $S^i_n = \alpha^i_n + \cdots + \alpha^i_1$. By definition of the ladder time l_1, for all $j = 1,\dots,l_1$, we have

$$\alpha^{i_0}(A_{l_1}\cdots A_j) = S^{i_0}_{l_1} - S^{i_0}_{j-1} < 0.$$

Let now analyze the behavior of the contractive part. For every $i \neq i_0$, let l^i_n be the sequence of ladder times where S^i_n reaches its minima, then the random variables

$$m^i_n = \max_{l^i_n \le h < l^i_{n+1}} \left(\alpha^i_{l^i_n} + \cdots + \alpha^i_h\right),$$

that represent the maximum of the excursion between two minima, are independent and identically distributed.

If we denote by σ^i the number of minima of S^i_n before l_1, then $S^i_{l^i_{\sigma^i}}$ is the last minimum and

$$\max_{j=1,\dots,l_1} \alpha^i(A_{l_1}\cdots A_j) = \max_{j=1,\dots,l_1} S^i_{l_1} - S^i_{j-1} = S^i_{l_1} - S^i_{l^i_{\sigma^i}} \le m^i_{\sigma^i} \le \max_{j=1,\dots,l_1} m^i_j$$

since $\sigma^i \le l_1$. If we suppose that α^i_1 has a moment of order $3+\delta$, then classical results on real random walks [Kiefer and Wolfowitz (1956), Theorem 5] ensure that $\mathbb{E}[(m^i_j)^{2+\delta}] < +\infty$; thus we can apply Lemma 3.3 and conclude that

$$\mathbb{E}\left[\max_{j=1,\dots,l_1} \alpha^i(A_{l_1}\cdots A_j)\right] \le \mathbb{E}\left[\max_{j=1,\dots,l_1} m^i_j\right] < +\infty.$$

We have shown that $\max_{j=l_1,\dots,1} \log^+ \|A_{l_1}\cdots A_j\|$ is integrable and, in particular, that

$$\mathbb{E}[\log^+ \|A_{l_1}\cdots A_1\|] < +\infty. \tag{3.10}$$

To achieve the proof it remains to show that the logarithm of the B component of L_{l_1} is integrable. We observe that

$$\log^+ |B_{0,l_1}| = \log^+ \left| \sum_{j=1}^{l_1} A_{l_1} \cdots A_{j+1} B_j \right|$$
$$\leq \log l_1 + \max_{j=l_1,\dots,1} \log^+ \|A_{l_1} \cdots A_j\| + \max_{j=1,\dots,l_1} \log^+ |B_j|.$$

Thus, using the fact that $\log l_1$ is integrable, applying (3.9) and once more Lemma 3.3, we conclude that $\mathbb{E}[\log^+(|B_{0,l_1}|)]$ is finite. $\square$

REMARK. When two or more eigenvalues are centered the recurrence of the process is strongly related to the dependence between the coordinates. For instance, let us consider the case where we have just two blocks,

$$A_n = \begin{bmatrix} A_n^1 & 0 \\ 0 & A_n^2 \end{bmatrix}.$$

If $A_n^1 = A_n^2$, the previous theorem directly prove that we have a locally contractive system. On the other side, if $A_n^1 = (A_n^2)^{-1}$, we cannot have any kind of recurrence. In fact if we apply the theorem to the systems Y_n^i defined by the matrices

$$\begin{bmatrix} A_n^i & B_n^i \\ 0 & 1 \end{bmatrix}$$

for $i = 1, 2$, the local contraction property says that when

$$Y_n = \begin{pmatrix} Y_n^1 \\ Y_n^2 \end{pmatrix}$$

lies in a compact set both $\|A_n^1\|$ and $\|A_n^2\|$ go to 0, and this is impossible because $\|A_n^2\| \geq 1/\|(A_n^2)^{-1}\| = 1/\|A_n^1 t\|$.

It is still an open question to determine which sort of correlation hypothesis between the blocks we need to assume in order to have a recurrent or a transient process and in particular what happens if we have two independent blocks with centered eigenvalues.

4. $\mathcal{N}\mathcal{A}$ groups. In this section we apply the previous results to study a Markov chain that lives in a more abstract group. We consider now a real Lie group $\mathcal{G} = \mathcal{A} \ltimes \mathcal{N}$ which is the semidirect product of an Abelian Lie group $\mathcal{A}$ isomorphic to $\mathbb{R}^d$ acting on a nilpotent simply connected Lie group $\mathcal{N}$. If we denote by $a \cdot$ the action of the element a of $\mathcal{A}$ on $\mathcal{N}$, then the product on $\mathcal{G}$ is

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2,\ b_1\, a_1 \cdot b_2).$$

Our main assumption on $\mathcal{G}$ is that the action of $\mathcal{A}$ on $\mathcal{N}$ is contractive, that is, that there exists an element a in $\mathcal{A}$ such that

$$\forall b \in \mathcal{N}, \qquad a^k \cdot b \to 0 \qquad \text{when } k \to +\infty.$$

We have also an action of $\mathcal{G}$ on $\mathcal{N}$, that is, $g \cdot b = b(g)\, a(g) \cdot b$ where $a(g)$ and $b(g)$ are the projections of $g \in \mathcal{G}$ on $\mathcal{A}$ and $\mathcal{N}$, respectively.

We shall identify the group $\mathcal{A}$ with its Lie algebra $\mathfrak{A}$. We recall that since $\mathcal{N}$ is simply connected the exponential is a diffeomorphism between $\mathcal{N}$ and its Lie algebra $\mathfrak{N}$ and that for every $X \in \mathfrak{N}$,

$$a \cdot \exp(X) = \exp\bigl(\mathrm{Ad}(a)X\bigr)$$

where $\mathrm{Ad}(a)$ is, with an abuse of notation, the differential of the application $a\,\cdot$ at the unity of $\mathcal{N}$.

It has been shown [Elie (1982), Appendix A2] that it is possible to provide the algebra $\mathfrak{N}$ with a norm $\|\cdot\|$ such that if we identify $\mathcal{N}$ with $\mathfrak{N}$ and set

$$\|\exp(X)\| = \|X\|,$$

then there exists a polynomial function Q without constant term, such that for every $k \in \mathbb{N}$ and $b_1, \dots, b_k \in \mathcal{N}$,

$$\|b_1 \cdots b_k\| \le Q(\|b_1\| + \cdots + \|b_k\|).$$

Let Δ be the set of the roots for the action Ad of $\mathcal{A}$ on $\mathfrak{N}$. Then there is a decomposition

$$\mathfrak{N} = \bigoplus_{\alpha \in \Delta} \mathfrak{N}_\alpha,$$

where $\mathfrak{N}_\alpha = \{X \in \mathfrak{N} : \forall a \in \mathcal{A}\, \exists n \in \mathbb{N} (\mathrm{Ad}(a) - e^{\alpha(a)} I)^n X = 0\}$. For every $X \in \mathfrak{N}$ we will denote by X^α the projection of X on $\mathfrak{N}_\alpha$. If H is a vector of the Lie algebra $\mathfrak{A}$ of $\mathcal{A}$, we set, with an abuse of notation, $\alpha(H) = \alpha(\exp(H))$, so that α will indicate both the homomorphism of $\mathcal{A}$ and its correspondent linear form from the additive group of the Lie algebra on $\mathbb{R}$.

The fact that the action of $\mathcal{A}$ is contracting implies that all roots are non zero and that their real parts belong to an open half space of the dual of the Lie algebra. For simplicity, we shall suppose that all the roots are real valued and we shall denote by $\mathfrak{A}^-$ the negative Weyl chamber

$$\mathfrak{A}^- = \{H \in \mathfrak{A} : \alpha(H) < 0\ \forall \alpha \in \Delta\}.$$

An important example of such a group is given by the upper triangular matrices for d, with positive coefficients on the diagonal:

$$\mathcal{T}_+(d) = \left\{ \begin{bmatrix} x^{11} & \cdots & x^{1d} \\ 0 & \ddots & \vdots \\ 0 & 0 & x^{dd} \end{bmatrix} \in \mathrm{GL}(\mathbb{R}^d) \,\middle|\, x^{ii} > 0 \text{ for } i = 0, \dots, d \right\}.$$

Every matrix of this group may be written as

$$
\begin{bmatrix} x^{11} & \cdots & x^{1d} \\ 0 & \ddots & \vdots \\ 0 & 0 & x^{dd} \end{bmatrix} = \begin{bmatrix} 1 \; x^{12}/x^{22} \cdots & & x^{1d}/x^{dd} \\ & & \vdots \\ 0 & \ddots & x^{d-1d}/x^{dd} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x^{dd} \end{bmatrix} \tag{4.1}
$$

so that we may decompose $\mathcal{T}_+(d)$ as the semidirect product of the Abelian group of diagonal matrices with positive coefficients

$$
\mathcal{D}_+(d) = \left\{ \begin{bmatrix} a^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a^d \end{bmatrix} \in \mathrm{GL}(\mathbb{R}^d) \,\middle|\, a^i > 0 \text{ for } i = 0, \dots, d \right\}
$$

acting by conjugation on the nilpotent group of triangular matrices with only ones as diagonal elements,

$$
\mathcal{N}(d) = \left\{ \begin{bmatrix} 1 & x^{12} \cdots & x^{1d} \\ & & \vdots \\ 0 & \ddots & x^{d-1d} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(\mathbb{R}^d) \right\}.
$$

The Lie algebra of this group is the nilpotent algebra of triangular matrices with zeros on the diagonal,

$$
\mathfrak{N}(d) = \left\{ \begin{bmatrix} 0 & x^{12} \cdots & x^{1d} \\ & & \vdots \\ 0 & \ddots & x^{d-1d} \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}(\mathbb{R}^d) \right\}.
$$

To suppose that all the roots are real is equivalent to restrict to matrices with positive coefficient on the diagonal; in this case, the roots are the homomorphisms

$$
\alpha_{ij} \left(\begin{bmatrix} a^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a^d \end{bmatrix} \right) = \log(a^i) - \log(a^j) \qquad \text{for } i < j \tag{4.2}
$$

and $\mathfrak{N}_{\alpha_{ij}}$ is the one-dimensional subspace of $\mathfrak{N}(d)$ of the matrices where all the coefficients except the one in the ith line and jth column are zero.

Let $g_n = (a_n, b_n)$ be a sequence of $\mathcal{G}$-valued independent and identically distributed random variables, $L_n = g_n \cdots g_1$ the left random walk and

$$
Y_n^y = L_n \cdot y = b\bigl(L_n(1, y)\bigr) \tag{4.3}
$$

the associated Markov chain obtained by the projection b of $\mathcal{G}$ on $\mathcal{N}$. We observe that, as for an autoregressive model, this process can be defined recursively by the equation

$$
Y_n^y = b_n \, a_n \cdot Y_{n-1}^y, \qquad Y_0^y = y.
$$

The goal of this section is to show that if the action of $\mathcal{A}$ on $\mathfrak{N}$ is centered in one "direction" and contractive on the others the Markov chain Y_n^y is a locally contractive process. Although the definition of this process is formally very close to the process on a vector space we have treated in the previous section, we have to face the problem that $\mathcal{N}$ is not an Abelian group. However, we will see that, for our purpose, the process on the nilpotent group does not differ too much from a convenient autoregressive model defined on the Lie algebra.

Since the exponential is surjective we are able to rise the random variables b_i on $\mathcal{N}$ in a sequence of random vectors B_i on the Lie algebra $\mathfrak{N}$ such that $b_i = \exp(B_i)$. Let now consider the autoregressive process on $\mathfrak{N}$ defined by

$$\widetilde{Y}_n^Y = \mathrm{Ad}(a_n)\widetilde{Y}_{n-1}^Y + B_n, \qquad \widetilde{Y}_0^Y = Y.$$

We observe that the sequence of i.i.d. random transformations $(\mathrm{Ad}(a_n), B_n)$ are of the type we studied in the previous section. Although the exponential is not a homomorphism and we cannot directly deduce the behavior of the process Y_n^y on $\mathcal{N}$ from $\widetilde{Y}_n^Y$, we know that there is a sequence of random variables Z_n in $[\mathfrak{N}, \mathfrak{N}]$ such that if $y = \exp(Y)$,

$$\begin{aligned} Y_n^y &= b_n\, a_n \cdot b_{n-1} \cdots (a_n \cdots a_1) \cdot y \\ &= \exp(B_n) \exp\big(\mathrm{Ad}(a_n) B_{n-1}\big) \cdots \exp\big(\mathrm{Ad}(a_n \cdots a_1) Y\big) \\ &= \exp\big(B_n + \mathrm{Ad}(a_n) B_{n-1} + \cdots + \mathrm{Ad}(a_n) \cdots \mathrm{Ad}(a_1) Y + Z_n\big) \\ &= \exp\big(\widetilde{Y}_n^Y + Z_n\big). \end{aligned}$$

We will suppose that the autoregressive process $\widetilde{Y}_n$ satisfies the condition of the previous section and in particular that only one of the logarithms of the eigenvalues of the matrix $\mathrm{Ad}(a_1)$ is centered while the others are negative in mean. Thus, we assume

$$\text{(4.4)} \quad \exists \alpha_0 \in \Delta, \qquad \mathbb{E}[\alpha_0(a_1)] = 0 \quad \text{and} \quad \mathbb{E}[\alpha(a_1)] < 0 \qquad \text{for all } \alpha \neq \alpha_0$$

so that the vector $\mathbb{E}[\log a_1]$ belong to a wall of the Weyl chamber $\mathfrak{A}^-$.

As it appears from the next lemma, α_0 has to be taken among the extremal roots.

LEMMA 4.1. *Under hypothesis* (4.4),

$$[\mathfrak{N}, \mathfrak{N}] \subseteq \bigoplus_{\alpha \neq \alpha_0} \mathfrak{N}_\alpha.$$

PROOF. Let $X = \sum_{\alpha \in \Delta} X^\alpha$ and $Y = \sum_{\alpha \in \Delta} Y^\alpha$ be any two vectors of $\mathfrak{N}$ then

$$[X, Y] = \sum_{\alpha, \beta \in \Delta} [X^\alpha, Y^\beta].$$

As it is known that $[X^\alpha, Y^\beta] \in \mathfrak{N}_{\alpha+\beta}$, we just need to show that $\alpha + \beta \neq \alpha_0$. First of all, as α_0 is not null, $2\alpha_0 \neq \alpha_0$. On the other hand, if either α or β are different from α_0, we cannot have $\alpha + \beta = \alpha_0$ since, by hypothesis (4.4),

$$\mathbb{E}[\alpha(a_1) + \beta(a_1)] < 0 = \mathbb{E}[\alpha_0(a_1)]. \qquad \square$$

For our purpose this lemma has the crucial consequence that the projection of the variable Z_n on the subspace $\mathfrak{N}_{\alpha_0}$ is zero, so that Y_n does not differ from its "commutative part," $\widetilde{Y}_n$, along the centered direction.

In order to prove the recurrence of the system we need also the following lemma that translates in the case of a nilpotent group a classical criterion of convergence for real series.

LEMMA 4.2. *Let $\{x_n\}_n$ a sequence in $\mathcal{N}$ such that*

$$\limsup_{n\to\infty} \frac{\log \|x_n\|}{n} < 0,$$

then the product $x_1 \cdots x_n$ converges to an element of $\mathcal{N}$.

PROOF. Set $y_n = x_1 \cdots x_n$ and denote by $\log : \mathcal{N} \to \mathfrak{N}$ the continuous inverse of the exponential, we will then prove that $\{\log y_n\}_n$ converges, or equivalently that it is a Cauchy sequence in $(\mathfrak{N}, \|\cdot\|)$. We first observe that $\{y_n\}_n$ (and therefore $\{\log y_n\}_n$) is contained in a compact set. Indeed,

$$\|y_n\| \le Q(\|x_1\| + \cdots + \|x_n\|) \le C_1 \left(\sum_{n=1}^{\infty} \|x_n\| \right)^D + C_2 < +\infty$$

for some positive constants C_1, C_2 and D.

As the logarithm is uniformly continuous on every compact set K, for every $\varepsilon > 0$, it exists $\delta > 0$ such that for all $y \in K$,

$$\|v\| \le \delta \quad \Longrightarrow \quad \|\log y - \log yv\| \le \varepsilon.$$

Since there is $N \in \mathbb{N}$ such that for all $n > m \ge N$,

$$\|y_m^{-1} y_n\| = \|x_{m+1} \cdots x_n\| \le Q\left(\sum_{k=m+1}^{n} \|x_k\| \right) \le \delta,$$

then for all $n > m \ge N$, $\|\log y_n - \log y_m\| \le \varepsilon$, that is, $\{\log y_n\}_n$ is a Cauchy sequence. $\square$

We are now able to prove the following.

THEOREM 4.3. *Suppose that* $(\mathrm{Ad}(a_1), B_1)$ *satisfies the moment and irreducibility hypotheses of Theorem* 3.2 *and that* (4.4) *is fulfilled. Then for every compact set* K *of* $\mathcal{N}$ *and every* $\varepsilon > 0$,

$$\limsup_{\substack{n\to+\infty\\ Y_n^y\in K}} \frac{\log\|(Y_n^x)^{-1}Y_n^y\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} = \limsup_{\substack{n\to+\infty\\ Y_n^y\in K}} \frac{\log\|\mathrm{Ad}(a_1\cdots a_n)\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0,$$

$\mathbb{P}$*-almost surely. Furthermore the set of accumulation points of* Y_n^y *is almost surely not empty, so that it is a locally contractive system.*

PROOF. As exp is a diffeomorphism between $\mathcal{N}$ and $\mathfrak{N}$, for every compact set K the projection, $\mathcal{K}^{\alpha_0}$, of $\exp(K)$ on $\mathfrak{N}_{\alpha_0}$ is still compact. On the other side,

$$[Y_n\in K]\subseteq\big[(\widetilde{Y}_n+Z_n)^{\alpha_0}\in\mathcal{K}^{\alpha_0}\big]=\big[(\widetilde{Y}_n)^{\alpha_0}\in\mathcal{K}^{\alpha_0}\big],$$

because Z_n is contained in $\bigoplus_{\alpha\neq\alpha_0}\mathfrak{N}_\alpha$. As the process $\widetilde{Y}_n$ satisfies the hypothesis of Theorem 3.2, it has a local contraction properties when $(\widetilde{Y}_n)^{\alpha_0}\in\mathcal{K}^{\alpha_0}$ (it is clear from the proof of this theorem [see (3.8)] that the only component that we need to have in a compact set is the one along the direction that in not globally contracted) so that

$$\limsup_{\substack{n\to+\infty\\ Y_n^y\in K}} \frac{\log\|\mathrm{Ad}(a_1\cdots a_n)\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} \le \limsup_{\substack{n\to+\infty\\ \widetilde{Y}_n^Y\in \mathcal{K}^{\alpha_0}}} \frac{\log\|\mathrm{Ad}(a_1\cdots a_n)\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0.$$

The proof of the recurrence property follows the same schema of the real case. Let l_n be the ladder times of $\alpha_0(a_1\cdots a_n)$ and consider the increments $L_{l_{n+1}}^{l_n} := g_{l_{n+1}}\cdots g_{l_n+1}$ of the random walk L_{l_n} on $\mathcal{G}$.

We have seen (3.10) that $\log^+\|\mathrm{Ad}(a(L_{l_1}^{l_0}\cdots L_{l_n}^{l_{n-1}}))\|$ is integrable and that

$$\limsup_{n\to\infty}\frac{\log\|\mathrm{Ad}(a(L_{l_1}^{l_0}\cdots L_{l_n}^{l_{n-1}}))\|}{n}<0.$$

Furthermore, using (3.9) and Lemma 3.3, we show that

$$\mathbb{E}\big[\log^+\|b(L_{l_1}^{l_0})\|\big]\le\mathbb{E}\left[\log^+ Q\left(\sum_{j=1}^{l_1}\|\mathrm{Ad}(a_{l_1}\cdots a_{j+1})\|\,\|B_j\|\right)\right]<+\infty.$$

Then

$$\begin{aligned}\limsup_{n\to\infty}&\frac{\log\|a(L_{l_1}^{l_0}\cdots L_{l_n}^{l_{n-1}})\cdot b(L_{l_n}^{l_{n-1}})\|}{n}\\ &\le\limsup_{n\to\infty}\frac{\log\|\mathrm{Ad}(a(L_{l_1}^{l_0}\cdots L_{l_n}^{l_{n-1}}))\|}{n}+\frac{\log\|b(L_{l_n}^{l_{n-1}})\|}{n}<0\end{aligned}$$

and, by the convergence criterion given in Lemma 4.2, the sequence

$$L_{l_1}^{l_0}\cdots L_{l_n}^{l_{n-1}}\cdot 1 = b\big(L_{l_1}^{l_0}\big)\big(a\big(L_{l_1}^{l_0}\big)\cdot b\big(L_{l_2}^{l_1}\big)\big)\cdots\big(a\big(L_{l_1}^{l_0}\cdots L_{l_n}^{l_{n-1}}\big)\cdot b\big(L_{l_n}^{l_{n-1}}\big)\big)$$

converges to a random element whose law is the unique invariant measure of $Y_{l_n}^y$. Then the sets of accumulation points of $Y_{l_n}^y$ and therefore of Y_n^y are not empty. □

We conclude this section by translating in the context of the upper triangular matrices the results of the last theorem. Let

$$T_n = \begin{bmatrix} t_n^{11} & \cdots & t_n^{1d} \\ 0 & \ddots & \vdots \\ 0 & 0 & t_n^{dd} \end{bmatrix}$$

be a sequence of i.i.d. random matrices in $\mathcal{T}_+(d)$ and, for every $y \in \mathcal{N}(d)$ and $a \in \mathcal{D}_+(d)$, let Y_n^y and D_n the Markov chain in $\mathcal{N}(d)$ and in $\mathcal{D}_+(d)$ such that, according to the decomposition (4.1),

$$T_n \cdots T_1 y a = Y_n^y D_n a.$$

We observe that D_n is just the diagonal part of the product $T_n \cdots T_1$ and that $Y_n^y = Y_n^1 D_n y D_n^{-1}$.

If the closed group generated by support of T_1 is the whole group $\mathcal{T}_+(d)$, then all the homomorphisms (4.2) are involved and the condition (4.4) is equivalent to

$$\mathbb{E}[\log(t_1^{11})] < \mathbb{E}[\log(t_1^{22})] < \cdots < \mathbb{E}[\log(t_1^{d-1d-1})] = \mathbb{E}[\log(t_1^{dd})]. \tag{4.5}$$

In the case where the support of T_1 charges a smaller subgroup we may have other possibilities. In this context, the irreducibility condition (3.3) becomes

$$\forall y \in \mathbb{R}, \qquad \begin{aligned} &\mathbb{P}[t_1^{d-1d-1} y + t_1^{d-1d} = t_1^{dd} y] < 1, \\ &\mathbb{P}[t_1^{d-1d-1} = t_1^{dd}] < 1. \end{aligned} \tag{4.6}$$

COROLLARY 4.4. *Under the hypotheses* (4.5) *and* (4.6), *and the moment condition* (3.7), Y_n^y *is recurrent and*

$$\limsup_{\substack{n\to+\infty \\ Y_n^y \in K}} \frac{\log \|Y_n^y - Y_n^x\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} < 0, \qquad \mathbb{P}\text{-almost surely}$$

for every compact set K of $\mathcal{N}(d)$ and every $x, y \in \mathcal{N}(d)$.

PROOF. Since $I - y^{-1}x$ is a nilpotent matrix and, by Theorem 4.3, the norm on the adjoint action of D_n on $\mathfrak{N}(d)$ (i.e., in this case, the action by conjugation) goes to 0 with a convenient speed, one has

$$\begin{aligned}\limsup_{\substack{n\to+\infty\\ Y_n^y\in K}} \frac{\log\|Y_n^y - Y_n^x\|}{\sqrt{n}(\log n)^{-1-\varepsilon}} &\le \limsup_{\substack{n\to+\infty\\ Y_n^y\in K}} \frac{\log C\|I-(Y_n^y)^{-1}Y_n^x\|}{\sqrt{n}(\log n)^{-1-\varepsilon}}\\ &= \limsup_{\substack{n\to+\infty\\ Y_n^y\in K}} \frac{\log\|D_n(I-y^{-1}x)D_n^{-1}\|}{\sqrt{n}(\log n)^{-1-\varepsilon}}\\ &< 0,\end{aligned}$$

where $C = \max_{u\in K}\|u\|$. $\square$

APPENDIX: DECREASING RANDOM WALK

In this Appendix we will come back to the case where the linear part of the autoregressive model is a real number and study in more detail the random walks on the affine group that are strongly contractive. We will give a proof of Lemma 2.3 in the following proposition.

PROPOSITION A.1. *Let* $\{(a_n, B_n)\}_n \subset \mathrm{Aff}(\mathbb{R}^d)$ *be a sequence of independent identically distributed random variables such that the irreducibility hypothesis* (2.1) *holds and that* $\log^+|B_1|$ *and* $|\log a_1|$ *are integrable. Furthermore suppose that*

$$\mathbb{E}[\log a_1] < 0.$$

Let $Y_n^y = a_n Y_{n-1}^y + B_n$ *be the induced process on* $\mathbb{R}^d$ *and* $\bar\nu$ *its invariant probability measure. Let* K *be a subset of* $\mathbb{R}^d$ *such that* $\bar\nu(K) > 0$ *and denote by* $\bar\nu_K$ *the normalized restriction of* $\bar\nu$ *to* K.

If $t_n^K = t_n^{y,K}$ *is the sequence of entrance times of* Y_n^y *in* K *then*:

(1) *If the starting point* y *is distributed as* $\bar\nu_K$,

$$\mathbb{E}_{\bar\nu_K}[t_1^K] = \frac{1}{\bar\nu(K)} \quad \textit{and} \quad \mathbb{E}_{\bar\nu_K}[S_{t_1^K}] = \frac{\mathbb{E}[\log a_1]}{\bar\nu(K)}.$$

(2) *There exists a compact set* K_0 *such that for every* $K \supseteq K_0$ *and every compact set* C *there is a positive constant* M *such that for all* $x \in C$,

$$\mathbb{E}_x[t_1^K] < M \quad \textit{and} \quad \mathbb{E}_x[S_{t_1^K}] > M\mathbb{E}[\log a_1].$$

PROOF. (1) The Markov chain Y_n^y has a unique invariant probability measure $\bar\nu$ and it is ergodic. Then by the ergodic theorem,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n \mathbb{1}_K(Y_n^{y_0})}{n} = \bar\nu(K), \qquad \mathbb{P}_{\bar\nu}\text{-almost surely.}$$

Since $\sum_{k=1}^{t_n^K} \mathbb{1}_K(Y_n^y) = n$, we have

$$\lim_{n\to\infty} \frac{t_n^K}{n} = \frac{1}{\bar{\nu}(K)}, \qquad \mathbb{P}_{\bar{\nu}}\text{-almost surely.} \tag{A.1}$$

On the other hand, $Y_{t_n^K}^y$ is a Markov chain on K with invariant probability measure $\bar{\nu}_K$, so that under the distribution $\mathbb{P}_{\bar{\nu}_K}$ the sequence of positive real random variables $\{t_n^K - t_{n-1}^K\}_n$ is stationary. Then applying once more the ergodic theorem we have

$$\lim_{n\to\infty} \frac{t_n^K}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^n t_k^K - t_{k-1}^K}{n} = \mathbb{E}_{\bar{\nu}_K}[t_1^K \,|\, \mathcal{I}], \qquad \mathbb{P}_{\bar{\nu}_K}\text{-almost surely}$$

where $\mathcal{I}$ is the invariant σ-algebra for the shift on the sequence $\{t_n^K - t_{n-1}^K\}_n$. As $\mathbb{P}_{\bar{\nu}_K}$ is absolutely continuous with respect to $\mathbb{P}_{\bar{\nu}}$ the last two equations imply that $\mathbb{E}_{\bar{\nu}_K}[t_1^K \,|\, \mathcal{I}]$ is $\mathbb{P}_{\bar{\nu}_K}$-almost surely constant and

$$\mathbb{E}_{\bar{\nu}_K}[t_1^K] = \frac{1}{\bar{\nu}(K)}.$$

To obtain the mean of $S_{t_1^K}$ we observe that it is the sum of the real random variables $\log a_n$ and that t_1^K is a stopping time with respect to the filtration generated by the random variables (a_n, B_n). Using Wald's identity [see Doob (1953), page 350, for a proof], we conclude that

$$\mathbb{E}_{\bar{\nu}_K}\big[S_{t_1^K}\big] = \mathbb{E}_{\bar{\nu}_K}[t_1^K]\mathbb{E}[\log a_1] = \frac{\mathbb{E}[\log a_1]}{\bar{\nu}(K)}.$$

(2) To show the second result we extract from Y_n^y a subchain that is even more contractive and we compare the trajectories of this subchain with the trajectories starting from a generic point x.

Fix $\rho > 0$ and consider the sequence of stopping times

$$r_n = \inf\{k > r_{n-1} : S_k - S_{r_{n-1}} < -\rho\}, \qquad r_0 = 0.$$

We shall first show that the process $Y_{r_n}^y$ is recurrent. Observe that it is an autoregressive process on $\mathbb{R}^d$ induced by the left random walk L_{r_n} on $\mathrm{Aff}(\mathbb{R})$. Thus it suffices to show that its step law, that is, the distribution of L_{r_1}, satisfies the moment conditions

$$\mathbb{E}\big[|\log a(L_{r_1})|\big] < +\infty \quad \text{and} \quad \mathbb{E}\big[\log^+ |b(L_{r_1})|\big] < +\infty.$$

Since $\mathbb{E}[S_1] < 0$ we have $\mathbb{E}[r_1] < +\infty$ and then, using Wald's identity,

$$\mathbb{E}\big[|\log a(L_{r_1})|\big] = \mathbb{E}[-S_{r_1}] < +\infty.$$

Furthermore,

$$\begin{aligned}|b(L_{r_1})| &= \left|\sum_{k=1}^{r_1} A_{r_1}\cdots A_{k+1}B_k\right| \\ &= \left|\sum_{k=1}^{r_1} e^{S_{r_1}-S_k}B_k\right| \\ &\le \sum_{k=1}^{r_1}|B_k| \qquad \text{since } S_k \ge -\rho \text{ for } k<r_1 \text{ and } S_{r_1}<-\rho.\end{aligned}$$

Thus, by Wald's identity, we deduce that $\mathbb{E}[\log^+|b(L_{r_1})|]$ is finite, because it is smaller than $\log r_1 + \sum_1^{r_1}\log^+|B_k|$. Thus the sub-chain $Y_{r_n}^y = L_{r_n}\cdot y$ of Y_n^y is recurrent and satisfies the property

$$|Y_{r_n}^y - Y_{r_n}^x| = e^{S_{r_n}}|y-x| \le e^{-n\rho}|y-x|.$$

Let $\bar{\nu}^r$ be the unique invariant probability measure for $Y_{r_n}^y$. We now fix a compact set K' such that $\bar{\nu}^r(K')>0$ and consider a compact set K such that $K\supseteq\{x\in\mathbb{R}^d : d(x,K')\le\varepsilon\}$ for some $\varepsilon\ge 0$.

We want now to show that for every fixed compact set C there exists a finite constant M such that $\mathbb{E}[t_1^{x,K}]\le M$ for all starting point x in C. Let N be an integer number such that $e^{-\delta N}\sup_{x\in C,y\in K'}|x-y|<\varepsilon$, then for every $n\ge N$, $x\in C$ and $y\in K'$,

$$|Y_{r_n}^x - Y_{r_n}^y| \le e^{-n\rho}|x-y| \le \varepsilon.$$

Let $\bar{t}^{y,K'}$ be the Nth entrance time of $Y_{r_n}^y$ in K'. We observe that, if y is distributed as $\bar{\nu}_{K'}^r$ then $Y^x_{r_{\bar{t}^{y,K'}}}$ is in K because $\bar{t}^{y,K'}\ge N$ and $Y^y_{r_{\bar{t}^{y,K'}}}\in K'$; thus $t_1^{x,K}\le r_{\bar{t}^{y,K'}}$. Applying the first part of this lemma to the Markov chain $Y_{r_n}^y$ leads to

$$\mathbb{E}_{\bar{\nu}_{K'}^r}[\bar{t}^{y,K'}] = \frac{N}{\bar{\nu}^r(K')} < +\infty.$$

Finally using once more Wald's identity on the sequence of random variables $\{(r_{n+1}-r_n)\}_n$ and to the filtration $\mathcal{F}_{r_n}=\sigma((a_i,B_i), i\le r_n)$, we conclude that

$$\mathbb{E}[t_1^{x,K}] \le \mathbb{E}_{\nu_{K'}^r}[r_{\bar{t}^{y,K'}}] = \mathbb{E}[r_1]\mathbb{E}_{\nu_{K'}^r}[\bar{t}^{y,K'}] = \frac{N\mathbb{E}[r_1]}{\nu^r(K')} < +\infty.$$

The uniform integrability of $S_{t_1^{x,K}}$ follows from this result and Wald's identity. □

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