The Annals of Probability
1996, Vol. 24, No. 3, 1616–1637

A NOTE ON EXACT CONVERGENCE RATES IN SOME MARTINGALE CENTRAL LIMIT THEOREMS[1]

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Bolthausen established a bound of order $1/\sqrt{n}$ on the rate of convergence in the central limit theorem for martingale difference arrays having bounded conditional moments of order 4. In the present paper it is shown how much this moment condition can be relaxed while maintaining the same rate of convergence. An example shows that, unlike in the i.i.d. case, a moment condition of order 3 is not enough. Furthermore, exact rates of convergence are derived for moment conditions of order between 2 and 3.

1. Introduction. We consider sequences $X_1, \dots, X_n$ of real-valued r.v.'s which satisfy

$$E(X_i | X_1, \dots, X_{i-1}) = 0 \quad \text{a.s. for } 1 \le i \le n \tag{1.1}$$

and

$$E(X_i^2 | X_1, \dots, X_{i-1}) = 1 \quad \text{a.s. for } 1 \le i \le n. \tag{1.2}$$

In the fundamental paper by Bolthausen (1982), Theorem 4 yields as a special case

$$\delta(n) := \sup_{t \in \mathbb{R}} \left| P[S_n \le t] - \Phi(t) \right| = O(n^{-1/2}), \tag{1.3}$$

where the notation

$$S_i := \frac{1}{\sqrt{n}} \sum_{j=1}^{i} X_j, \qquad 0 \le i \le n, \tag{1.4}$$

is used and the following two conditions are additionally assumed:

$$E(X_i^3 | X_1, \dots, X_{i-1}) = \mu \quad \text{a.s. for } 1 \le i \le n, \tag{1.5}$$

$$E(X_i^4 | X_1, \dots, X_{i-1}) \le K \quad \text{a.s. for } 1 \le i \le n. \tag{1.6}$$

The aforementioned theorem does not include the classical i.i.d. Berry–Esseen theorem. Joos (1988) and Renz (1991) have independently shown that the moment condition (1.6) can be relaxed to

$$E\Big(|X_i|^3 \big(\ln(2 + |X_i|)\big)^{1+\varepsilon} | X_1, \dots, X_{i-1}\Big) \le K \quad \text{a.s. for } 1 \le i \le n, \tag{1.7}$$

Received January 1995; revised January 1996.

[1]Research supported by a grant from the Deutsche Forschungsgemeinschaft.

AMS 1991 *subject classifications*. Primary 60F05, 60G42.

Key words and phrases. Martingales, central limit theorem, rates of convergence.

$$E\big(|X_i|^{3+\varepsilon}|X_1,\dots,X_{i-1}\big) \le K \quad \text{a.s. for } 1 \le i \le n, \tag{1.8}$$

respectively, where in the second paper the more general situation of a stochastic approximation procedure is considered.

In Theorem 1 of this paper it is shown that a moment condition

$$E\big(|X_i|^3 h(|X_i|)|X_1,\dots,X_{i-1}\big) \le K \quad \text{a.s. for } 1 \le i \le n, \tag{1.9}$$

where $h\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a function such that

(1.10) h is nondecreasing on $\mathbb{R}^+$ and $h(\cdot)/\cdot$ is positive and nonincreasing on $\mathbb{R}^+$,

implies the order of convergence

$$O\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{ih(\sqrt{i}\,)}\right)$$

in the central limit theorem. Example 1 shows that this rate is exact. Hence the order $1/\sqrt{n}$ may not be attainable as soon as the series $\Sigma 1/(ih(\sqrt{i}\,))$ diverges. In particular, this answers a question posed by Bolthausen [Bolthausen (1982), page 674].

Theorem 2 and Example 2 treat the case where conditions (1.1), (1.2), (1.10) and

$$E\big(X_i^2 h(|X_i|)|X_1,\dots,X_{i-1}\big) \le K \quad \text{a.s. for } 1 \le i \le n \tag{1.11}$$

are assumed. In this case the order

$$O\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}\right)$$

is the exact rate of convergence in the martingale central limit theorem. In the i.i.d. case, according to Petrov (1975) (Theorem 5 in Chapter 5, Section 3) and Example 3 in Section 3 below, $1/h(\sqrt{n}\,)$ is the order of convergence. The two rates of convergence can be compared as follows. If $x \mapsto h(x)/x^{\varepsilon}$ is a nonincreasing function for some $\varepsilon \in (0,1)$, then the inequality

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)} \le \frac{1}{h(\sqrt{n}\,)}\frac{2}{1-\varepsilon} \tag{1.12}$$

holds, whereas the opposite inequality

$$2\left(1-\frac{1}{\sqrt{n}}\right)\frac{1}{h(\sqrt{n}\,)} \le \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}$$

holds for all functions h satisfying (1.10). Again, for a moment condition of order near 3, the difference from the i.i.d. case becomes apparent. For instance, $h(x) = x/\ln(e \vee x)$ leads to the order $O((\ln n)^2/\sqrt{n}\,)$ on the left-hand side of (1.12), while the right-hand side of (1.12) is of order $O(\ln n/\sqrt{n}\,)$.

While the positive results in Theorems 1 and 2 of Section 2 allow for slight generalizations toward a setting described in Bolthausen's Theorem 4, the main focus of this paper is on Section 3. Examples 1 and 2 in Section 3 show that the rates stated in Theorems 1 and 2 are sharp, even under the strong hypotheses of these theorems.

Methods of proof and construction of examples rely on modifications of ideas developed by Bolthausen (1982) and on Lemma 2 in the Appendix below, allowing us to treat moment conditions of noninteger order. Furthermore, in Lemmas 1 and 3 of the Appendix efforts are made to determine quantitative aspects of the method. These efforts lead to explicit constants in Theorems 1 and 2. We note that such constants are not necessary in the construction of the examples.

Kir'yanova and Rotar' (1991) recently presented a different approach to reach the order $1/\sqrt{n}$ in the martingale central limit theorem. They established a bound depending on distances between the conditional and the unconditional distribution functions of the r.v.'s X_i, instead of the higher conditional moments of these r.v.'s. It would be worthwhile to calculate the bounds derived in Kir'yanova and Rotar' (1991) for the examples given here.

2. Results and proofs. In this section we state and prove Theorems 1 and 2. The Appendix contains the lemmas needed for the proofs of these theorems.

THEOREM 1. *Let $n \in \mathbb{N}$. For $X_1, \dots, X_n$ and h: $\mathbb{R}_0^+ \to \mathbb{R}_0^+$, assume* (1.1), (1.2), (1.5), (1.9) *and* (1.10). *Then*

$$\delta(n) \le 100 \max\{h(1), K\} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})}.$$

In the following proof and throughout the paper we employ the notation

$$H_i(u,v) := \Phi(u-v) - \Phi(u) - \sum_{j=1}^{i-1} \frac{(-v)^j}{j!} \varphi^{(j-1)}(u), \qquad i = 2,3,4, \tag{2.1}$$

which was introduced in Bolthausen (1982).

PROOF. Let $\tilde{h}(x) := h(x)/h(1)$. Then $\tilde{h}$ obeys condition (1.10) and $\tilde{h}(1) = 1$. Now the proof will be finished by showing $\delta(n) \le 100\tilde{K}n^{-1/2}\sum_{j=1}^{n} 1/(j\tilde{h}(\sqrt{j}))$, where $\tilde{K} := K/h(1)$ satisfies w.l.o.g. the additional assumption $1 \le \tilde{K}$. Therefore we will assume $h(1) = 1$ and $1 \le K$.

We define new random variables $\xi, Z_1, \dots, Z_n$, where $Z_1, \dots, Z_n$ are identically $N(0,1)$-distributed, ξ is $N(0, \kappa^2)$-distributed ($\kappa \ge 1$ to be defined below) and $(X_1, \dots, X_n)$, $\xi, Z_1, \dots, Z_n$ are independent. In the sequel we will employ the following notation:

$$\kappa := 20K \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})},$$

$$W_i := \frac{1}{\sqrt{n}} \left(\xi + \sum_{j=i+1}^{n} Z_j \right), \tag{2.2}$$

$$\lambda_i := \sqrt{\frac{n - i + \kappa^2}{n}}, \qquad 0 \le i \le n.$$

The choice of κ implies $\kappa \ge 20K \ge 20$. For $n \le 10{,}000$, or more generally in the case $\sqrt{n} \le 5\kappa$, it follows that $\delta(n) \le 1 \le 100Kn^{-1/2}\sum_{j=1}^{n} 1/(jh(\sqrt{j}))$, and nothing remains to be proven. Now we proceed by induction and consider the remaining case $5\kappa < \sqrt{n}$.

Notice that W_i is an $N(0, \lambda_i^2)$-distributed r.v. which is independent of (S_{i-1}, X_i) and of (S_{i-1}, Z_i). Since

$$\begin{aligned} &\sup_{x \in \mathbb{R}} |\Phi(x/r) - \Phi(x)| \le 0.75|r - 1|, \qquad r > 0, \\ &|r - 1| \le |r^2 - 1|, \qquad r \ge 0, \end{aligned} \tag{2.3}$$

we obtain

$$\sup_{t \in \mathbb{R}} |\Phi(t) - P[W_0 \le t]| \le \frac{3}{4} \frac{\kappa^2}{n}. \tag{2.4}$$

By Lemma 1 in Bolthausen (1982), (2.4) and Lemma 3, we obtain

$$\begin{aligned} \delta(n) &\le 2 \sup_{t \in \mathbb{R}} \left| P\left[\frac{1}{\sqrt{n}} \xi + S_n \le t \right] - P[W_0 \le t] \right| + \frac{5}{\sqrt{2\pi}} \frac{\kappa}{\sqrt{n}} + \frac{3}{2} \frac{\kappa^2}{n} \\ &= 2 \sup_{t \in \mathbb{R}} \left| \sum_{i=1}^{n} \left(P\left[S_{i-1} + \frac{X_i}{\sqrt{n}} + W_i \le t \right] - P\left[S_{i-1} + \frac{Z_i}{\sqrt{n}} + W_i \le t \right] \right) \right| \\ &\quad + \frac{5}{\sqrt{2\pi}} \frac{\kappa}{\sqrt{n}} + \frac{3}{2} \frac{\kappa^2}{n} \\ &= 2 \sup_{t \in \mathbb{R}} \left| \sum_{i=1}^{n} \left(E\Phi\left(\frac{t - S_{i-1}}{\lambda_i} - \frac{X_i}{\lambda_i \sqrt{n}} \right) - E\Phi\left(\frac{t - S_{i-1}}{\lambda_i} - \frac{Z_i}{\lambda_i \sqrt{n}} \right) \right) \right| \\ &\quad + \frac{5}{\sqrt{2\pi}} \frac{\kappa}{\sqrt{n}} + \frac{3}{2} \frac{\kappa^2}{n} \\ &= 2 \sup_{t \in \mathbb{R}} \left| \sum_{i=1}^{n} \left(EH_4\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i \sqrt{n}} \right) - EH_4\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{Z_i}{\lambda_i \sqrt{n}} \right) \right. \right. \\ &\qquad \left. \left. - \frac{E(X_i^3 | S_{i-1})}{6\lambda_i^3 n^{3/2}} E\varphi''\left(\frac{t - S_{i-1}}{\lambda_i} \right) \right) \right| \end{aligned}$$

$$
\begin{aligned}
&+\frac{5}{\sqrt{2\pi}}\frac{\kappa}{\sqrt{n}}+\frac{3}{2}\frac{\kappa^2}{n}\\
&\le 2\sup_{t\in\mathbb{R}}\Bigg\{\sum_{i=1}^{\lceil n/2\rceil}\left(\frac{0.134K}{\lambda_i^3 n^{3/2}h(\lambda_i\sqrt{n})}+\frac{0.069}{\lambda_i^4 n^2}+\frac{0.067|\mu|}{\lambda_i^3 n^{3/2}}\right)\\
&\qquad+\sum_{i=\lceil n/2\rceil+1}^{n}\Bigg(\frac{K(2.096\delta(n)+2.896\lambda_i)}{\lambda_i^3 n^{3/2}h(\lambda_i\sqrt{n})}\\
&\qquad\qquad+\frac{0.906\delta(n)+1.443\lambda_i}{\lambda_i^4 n^2}\\
&\qquad\qquad+\frac{|\mu|\,|E\varphi''((t-S_{i-1})/\lambda_i)|}{6\lambda_i^3 n^{3/2}}\Bigg)\Bigg\}\\
&\quad+\frac{5}{\sqrt{2\pi}}\frac{\kappa}{\sqrt{n}}+\frac{3}{2}\frac{\kappa^2}{n}.
\end{aligned}
$$

According to Lemma 2 in Bolthausen (1982), we obtain, from (A.5) and (A.8),

$$
\begin{aligned}
\left|E\varphi''\left(\frac{t-S_{i-1}}{\lambda_i}\right)\right| &\le \|\varphi''\|_V\sup_{t\in\mathbb{R}}|P[S_{i-1}\le t]-\Phi(t)|+\|\varphi\|_1\|\varphi''\|_\infty\lambda_i^3\\
&\le 1.52\sup_{t\in\mathbb{R}}|P[S_{i-1}\le t]-\Phi(t)|+0.4\lambda_i^3.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}|P[S_{i-1}\le t]-\Phi(t)|\\
&\quad\le \sup_{t\in\mathbb{R}}\left|P\left[\frac{1}{\sqrt{i-1}}\sum_{j=1}^{i-1}X_j\le t\sqrt{\frac{n}{i-1}}\right]-\Phi\left(t\sqrt{\frac{n}{i-1}}\right)\right|\\
&\qquad+\sup_{t\in\mathbb{R}}\left|\Phi\left(t\sqrt{\frac{n}{i-1}}\right)-\Phi(t)\right|\\
&\quad\le \sup_{t\in\mathbb{R}}\left|P\left[\frac{1}{\sqrt{i-1}}\sum_{j=1}^{i-1}X_j\le t\right]-\Phi(t)\right|+\frac{3}{4}\lambda_i^2,\qquad \lceil n/2\rceil+1\le i\le n,
\end{aligned}
$$

where the last inequality is obtained from (2.3). The induction hypothesis yields

$$
\begin{aligned}
\sup_{t\in\mathbb{R}}\left|P\left[\frac{1}{\sqrt{i-1}}\sum_{j=1}^{i-1}X_j\le t\right]-\Phi(t)\right| &\le 100K\frac{1}{\sqrt{i-1}}\sum_{j=1}^{i-1}\frac{1}{jh(\sqrt{j})}\\
&\le \frac{\sqrt{2}\,100K}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{jh(\sqrt{j})},
\end{aligned}
$$

and therefore

$$\left|E\varphi''\left(\frac{t-S_{i-1}}{\lambda_i}\right)\right| \le 215K\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{jh\left(\sqrt{j}\right)} + 1.14\lambda_i^2 + 0.4\lambda_i^3,$$

$$\lceil n/2\rceil + 1 \le i \le n.$$

For $|\mu|$ we find the bound

$$\begin{aligned}|\mu| &\le E\left(|X_j|^3|X_1,\dots,X_{j-1}\right)\\ &\le E\left(X_j^2 1_{[|X_j|\le 1]}|X_1,\dots,X_{j-1}\right) + E\left(|X_j|^3 h\left(|X_j|\right)1_{[|X_j|>1]}|X_1,\dots,X_{j-1}\right)\\ &\le 1 + K \le 2K.\end{aligned}$$

This yields

$$\begin{aligned}\delta(n) \le{} & \sum_{i=1}^{\lceil n/2\rceil}\left(\frac{0.268K}{\lambda_i^3 n^{3/2} h\left(\lambda_i\sqrt{n}\right)} + \frac{0.138}{\lambda_i^4 n^2} + \frac{0.268K}{\lambda_i^3 n^{3/2}}\right) + 2\frac{\kappa}{\sqrt{n}} + 1.5\frac{\kappa^2}{n}\\ & + \sum_{i=\lceil n/2\rceil+1}^{n}\left(\frac{K(4.192\,\delta(n) + 5.792\lambda_i)}{\lambda_i^3 n^{3/2} h\left(\lambda_i\sqrt{n}\right)} + \frac{1.812\,\delta(n) + 2.886\lambda_i}{\lambda_i^4 n^2}\right.\\ &\left. + \frac{K\left(144Kn^{-1/2}\sum_{j=1}^{n}1/\left(jh\left(\sqrt{j}\right)\right) + 0.76\lambda_i^2 + 0.27\lambda_i^3\right)}{\lambda_i^3 n^{3/2}}\right).\end{aligned}$$

Next we make the following observations:

$$\begin{aligned}\sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lambda_i^3 n^{3/2} h\left(\lambda_i\sqrt{n}\right)} &\le \sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lambda_i^3 n^{3/2}} = \sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\left(n-i+\kappa^2\right)^{3/2}}\\ &\le \sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lceil n/2\rceil^{3/2}} \le \frac{\sqrt{2}}{\sqrt{n}},\\ \sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lambda_i^4 n^2} &\le \sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\left(n-i+\kappa^2\right)^2} \le \sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lceil n/2\rceil^2} \le \frac{2}{n},\\ \sum_{i=\lceil n/2\rceil+1}^{n}\frac{1}{\lambda_i^3 n^{3/2}} &= \sum_{i=\lceil n/2\rceil+1}^{n}\frac{1}{\left(n-i+\kappa^2\right)^{3/2}}\\ &\le \frac{1}{\kappa^3} + \int_{\kappa^2}^{\infty}\frac{1}{x^{3/2}}\,dx \le \frac{1}{\kappa^3} + \frac{2}{\kappa} \le \frac{3}{\kappa}, \qquad (2.5)\\ \sum_{i=\lceil n/2\rceil+1}^{n}\frac{1}{\lambda_i^4 n^2} &= \sum_{i=\lceil n/2\rceil+1}^{n}\frac{1}{\left(n-i+\kappa^2\right)^2}\\ &\le \frac{1}{\kappa^4} + \int_{\kappa^2}^{\infty}\frac{1}{x^2}\,dx \le \frac{2}{\kappa^2},\end{aligned}$$

$$\sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i\sqrt{n}} = \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\sqrt{n-i+\kappa^2}}$$
$$\le \frac{1}{\kappa} + \int_{\kappa^2}^{n/2+\kappa^2} \frac{1}{\sqrt{x}}\,dx \le 1.5\sqrt{n}\,,$$
$$\sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i^3 n^{3/2} h(\lambda_i\sqrt{n})} \le \frac{1}{\kappa} \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i^2 n h(\lambda_i\sqrt{n})}$$
$$\le \frac{1}{\kappa} \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})} = \frac{1}{20K}.$$

Combining these results, we have

$$\delta(n) \le \frac{0.761K}{\sqrt{n}} + \delta(n)\left(0.210 + \frac{3.624}{\kappa^2}\right) + \frac{K}{\sqrt{n}}\{0.135 + 1.14\} + \frac{8.658}{\sqrt{n}\,\kappa}$$
$$+ \frac{1}{\sqrt{n}} K \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})} \left\{5.792 + \frac{288K}{\kappa} + \frac{144K}{\kappa^3} + 40 + 30\frac{\kappa}{\sqrt{n}}\right\}.$$

Therefore,

$$\delta(n)(1 - 0.210 - 0.010)$$
$$\le \frac{1}{\sqrt{n}} K \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})}\{5.792 + 14.4 + 0.018 + 40 + 6\}$$
$$+ \frac{2.036K}{\sqrt{n}} + \frac{0.433}{\sqrt{n}}$$
$$\le \frac{1}{\sqrt{n}} K \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})} 68.679.$$

This implies $\delta(n) \le 100Kn^{-1/2}\sum_{j=1}^{n} 1/(jh(\sqrt{j}))$. $\square$

THEOREM 2. *Let* $n \in \mathbb{N}$. *For* $X_1, \dots, X_n$ *and* $h\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$, *assume* (1.1), (1.2), (1.10) *and* (1.11). *Then*

$$\delta(n) \le 300 \max\{h(1), K\} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}\,h(\sqrt{j})}.$$

PROOF. We use the same notation as in the proof of Theorem 1, except that we set $\kappa := 60K\sum_{j=1}^{n} 1/(\sqrt{j}\,h(\sqrt{j}))$. Again it is enough to prove the assertion under the additional assumption $h(1) = 1 \le K$. The choice of κ implies $\kappa \ge 60K \ge 60$. For $5\kappa \ge \sqrt{n}$ we obtain $\delta(n) \le 1 \le 300Kn^{-1/2}\sum_{j=1}^{n} 1/$

$(\sqrt{j}\,h(\sqrt{j}))$. In the case $5\kappa < \sqrt{n}$ we obtain as in the previous proof

$$\begin{aligned}
\delta(n) \le{}& 2\sup_{t\in\mathbb{R}}\left|\sum_{i=1}^{n}\left(EH_3\left(\frac{t-S_{i-1}}{\lambda_i},\frac{X_i}{\lambda_i\sqrt{n}}\right)-EH_4\left(\frac{t-S_{i-1}}{\lambda_i},\frac{Z_i}{\lambda_i\sqrt{n}}\right)\right)\right|\\
&+\frac{2\kappa}{\sqrt{n}}+\frac{1.5\kappa^2}{n}\\
\le{}& 2\left\{\sum_{i=1}^{\lceil n/2\rceil}\left(\frac{0.242K}{\lambda_i^2 nh(\lambda_i\sqrt{n})}+\frac{0.069}{\lambda_i^4 n^2}\right)\right.\\
&\left.+\sum_{i=\lceil n/2\rceil+1}^{n}\left(\frac{K(12.150\delta(n)+14.843\lambda_i)}{\lambda_i^2 nh(\lambda_i\sqrt{n})}+\frac{0.906\delta(n)+1.443\lambda_i}{\lambda_i^4 n^2}\right)\right\}\\
&+\frac{2\kappa}{\sqrt{n}}+\frac{1.5\kappa^2}{n}.
\end{aligned}$$

In addition to (2.5), we will need the following auxiliary facts:

$$\begin{aligned}
\sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lambda_i^2 nh(\lambda_i\sqrt{n})} &\le \frac{\sqrt{2}}{\sqrt{n}}\sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\lambda_i\sqrt{n}\,h(\lambda_i\sqrt{n})}\\
&\le \frac{\sqrt{2}}{\sqrt{n}}\sum_{i=1}^{\lceil n/2\rceil}\frac{1}{\sqrt{i}\,h(\sqrt{i})} \le \frac{\sqrt{2}}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})},\\
\sum_{i=\lceil n/2\rceil+1}^{n}\frac{1}{\lambda_i^2 nh(\lambda_i\sqrt{n})} &\le \frac{1}{\kappa}\sum_{i=\lceil n/2\rceil+1}^{n}\frac{1}{\lambda_i\sqrt{n}\,h(\lambda_i\sqrt{n})}\\
&\le \frac{1}{\kappa}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})} = \frac{1}{60K}.
\end{aligned}$$

We obtain

$$\begin{aligned}
\delta(n) \le{}& \frac{0.685K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})}+\frac{0.276}{n}+\delta(n)\left\{0.405+\frac{3.624}{\kappa^2}\right\}\\
&+\frac{29.686K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})}+\frac{8.658}{\sqrt{n}\,\kappa}+\frac{120K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})}\\
&+\frac{90K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})}\frac{\kappa}{\sqrt{n}}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\delta(n)\left(1-0.405-\frac{3.624}{\kappa^2}\right)\\
&\quad\le \frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i})}\left\{0.685+29.686+120+90\frac{\kappa}{\sqrt{n}}+\frac{0.276}{\sqrt{n}}+\frac{8.658}{\kappa}\right\}.
\end{aligned}$$

This yields

$$\delta(n)(1-0.407) \le \frac{K}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}\,h(\sqrt{i})}\{0.685+29.686+120+18+0.001+0.145\}$$

and the assertion is proved. □

3. Examples. In this section we give two examples showing that the rates of convergence stated in Theorems 1 and 2 cannot be improved under the assumptions made in these theorems. For the sake of convenient reference, we sketch a third example showing that the rate of convergence in the theorem of Petrov (1975) (Theorem 5 in Chapter 5, Section 3) is sharp.

EXAMPLE 1. Suppose that h satisfies (1.10) and the following two additional properties: $\sum_{j=1}^{\infty} 1/(jh(\sqrt{j})) = \infty$ and $h(x) \nearrow \infty$ for $x \to \infty$. Furthermore, let $\max\{1, 5.2h(2)\} \le K$. We consider for $\varrho \ge 1$ the following probability measures on $(\mathbb{R}, \mathscr{B})$:

$$R_\varrho := \frac{2(\varrho-1)(\varrho+1)}{(\varrho+2)(2\varrho-1)}\left(\frac{1}{5}\delta_{-2}+\frac{4}{5}\delta_{1/2}\right) + \frac{3\varrho}{(\varrho+2)(2\varrho-1)}\left(\frac{\varrho^2}{1+\varrho^2}\delta_{-1/\varrho}+\frac{1}{1+\varrho^2}\delta_\varrho\right)$$

and

$$P_\varrho := (1-\gamma_{\varrho,K})R_1 + \gamma_{\varrho,K}R_\varrho \quad \text{where } \gamma_{\varrho,K} := \min\left\{1, \frac{K}{3h(\varrho)}\right\}.$$

The following properties hold: $\int xP_\varrho(dx) = 0$, $\int x^2P_\varrho(dx) = 1$, $\int x^3P_\varrho(dx) = 0$ and

$$\begin{aligned}\int |x|^3 h(|x|)P_\varrho(dx) &\le (1-\gamma_{\varrho,K})h(1) + \gamma_{\varrho,K}\{1.6h(2)+0.1h(1/2)+0.5h(1)+1.5h(\varrho)\}\\ &\le h(1)+1.6h(2)+1.5\gamma_{\varrho,K}h(\varrho) \le K.\end{aligned}$$

Let $n \in \mathbb{N}$ and $1 \le \kappa^2 \le n/2$, where the constant κ will be specified later. We define r.v.'s $X_1, \dots, X_n$ and $Z_1, \dots, Z_n$ on a suitable probability space as follows: $X_i = Z_i$, $i \le \lceil n/2 \rceil$, are independent identically Bernoulli-distributed r.v.'s (± 1 w.p. $1/2$),

$$P_{X_i|(X_1,\dots,X_{i-1})}((x_1,\dots,x_{i-1}),A) := 1_{J_i^c}(x_1+\cdots+x_{i-1})P_1(A) + 1_{J_i}(x_1+\cdots+x_{i-1})P_{\varrho_i}(A),$$

with $\lceil n/2 \rceil < i \le n$, $\varrho_i := \sqrt{n-i+\kappa^2}$, $J_i := [\varrho_i/4, \varrho_i/2]$ and $Z_{\lceil n/2\rceil+1}, \dots, Z_n, \xi$ are normal-distributed r.v.'s, where $(X_1,\dots,X_n), Z_{\lceil n/2\rceil+1},\dots,Z_n,\xi$

are independent, ξ is $N(0,\kappa^2)$-distributed and $Z_{\lceil n/2\rceil+1},\dots,Z_n$ are identically $N(0,1)$-distributed. This yields

$$\begin{aligned}&X_1,\dots,X_n \text{ is an m.d.s. with } E(X_i^2|X_1,\dots,X_{i-1})=1,\\ &E(X_i^3|X_1,\dots,X_{i-1})=0 \text{ and } E(|X_i|^3h(|X_i|)|X_1,\dots,X_{i-1})\le K \text{ a.s.,}\\ &i=1,\dots,n.\end{aligned}\tag{3.1}$$

With the notation S_i, W_i and λ_i as introduced in the proof of Theorem 1, we obtain that W_i is $N(0,\lambda_i^2)$-distributed and independent of (S_{i-1},X_i) and (S_{i-1},Z_i), $i>\lceil n/2\rceil$. Furthermore, we obtain $P[W_0\le 0]=1/2$ because W_0 is symmetrically distributed and has a density. This yields

$$\begin{aligned}-\delta(n)&\le P[S_n\le 0]-\frac{1}{2}\\ &=P\left[\frac{\xi}{\sqrt{n}}+S_n\le 0\right]-P[W_0\le 0]+P[S_n\le 0]-P\left[\frac{\xi}{\sqrt{n}}+S_n\le 0\right]\\ &\le P\left[\frac{\xi}{\sqrt{n}}+S_n\le 0\right]-P[W_0\le 0]+2\delta(n)+\frac{2\kappa}{\sqrt{2\pi}\sqrt{n}}\\ &=\sum_{i=1}^{n}\left(P\left[S_{i-1}+\frac{X_i}{\sqrt{n}}+W_i\le 0\right]-P\left[S_{i-1}+\frac{Z_i}{\sqrt{n}}+W_i\le 0\right]\right)\\ &\quad+2\delta(n)+\frac{2\kappa}{\sqrt{2\pi}\sqrt{n}}\\ &=\sum_{i=\lceil n/2\rceil+1}^{n}\left(EH_4\left(-\frac{S_{i-1}}{\lambda_i},\frac{X_i}{\lambda_i\sqrt{n}}\right)-EH_4\left(-\frac{S_{i-1}}{\lambda_i},\frac{Z_i}{\lambda_i\sqrt{n}}\right)\right)\\ &\quad+2\delta(n)+\frac{2\kappa}{\sqrt{2\pi}\sqrt{n}},\end{aligned}\tag{3.2}$$

where we have used (3.1), the independence of $(X_1,\dots,X_{i-1})$ and Z_i and the fact that the r.v.'s Z_i are standard normal ($i>\lceil n/2\rceil$). By the construction of the sequence (X_i), we have

$$P_{X_i|S_{i-1}}(x,A)=1_{[\lambda_i/4,\lambda_i/2]^c}(x)P_1(A)+1_{[\lambda_i/4,\lambda_i/2]}(x)P_{\varrho_i}(A),\qquad i>\lceil n/2\rceil,$$

and therefore, for $i>\lceil n/2\rceil$,

$$\begin{aligned}&EH_4\left(-\frac{S_{i-1}}{\lambda_i},\frac{X_i}{\lambda_i\sqrt{n}}\right)\\ &\quad=\iint H_4\left(-\frac{x}{\lambda_i},\frac{y}{\lambda_i\sqrt{n}}\right)P_{X_i|S_{i-1}}(x,dy)P_{S_{i-1}}(dx)\\ &\quad=\int_{[\lambda_i/4,\lambda_i/2]^c}\int H_4\left(-\frac{x}{\lambda_i},\frac{y}{\lambda_i\sqrt{n}}\right)P_1(dy)P_{S_{i-1}}(dx)\\ &\qquad+\int_{[\lambda_i/4,\lambda_i/2]}\int H_4\left(-\frac{x}{\lambda_i},\frac{y}{\lambda_i\sqrt{n}}\right)P_{\varrho_i}(dy)P_{S_{i-1}}(dx)\end{aligned}$$

$$= \int_{[\lambda_i/4,\,\lambda_i/2]^c} \frac{1}{2}\left(H_4\left(-\frac{x}{\lambda_i}, \frac{1}{\lambda_i\sqrt{n}}\right) + H_4\left(-\frac{x}{\lambda_i}, \frac{-1}{\lambda_i\sqrt{n}}\right)\right) P_{S_{i-1}}(dx)$$

$$+ \int_{[\lambda_i/4,\,\lambda_i/2]} \Bigg\{ (1-\gamma_{\varrho_i,K}) \frac{1}{2}\left(H_4\left(-\frac{x}{\lambda_i}, \frac{1}{\lambda_i\sqrt{n}}\right) + H_4\left(-\frac{x}{\lambda_i}, \frac{-1}{\lambda_i\sqrt{n}}\right)\right)$$

$$+ \gamma_{\varrho_i,K} \frac{2(\varrho_i-1)(\varrho_i+1)}{(\varrho_i+2)(2\varrho_i-1)} \left(\frac{1}{5} H_4\left(-\frac{x}{\lambda_i}, \frac{-2}{\lambda_i\sqrt{n}}\right) + \frac{4}{5} H_4\left(-\frac{x}{\lambda_i}, \frac{1}{2\lambda_i\sqrt{n}}\right)\right)$$

$$+ \gamma_{\varrho_i,K} \frac{3\varrho_i}{(\varrho_i+2)(2\varrho_i-1)} \left(\frac{\varrho_i^2}{1+\varrho_i^2} H_4\left(-\frac{x}{\lambda_i}, \frac{-1}{\lambda_i^2 n}\right) w\left(+\frac{1}{1+\varrho_i^2} H_4\left(-\frac{x}{\lambda_i}, 1\right)\right)\right)\Bigg\} P_{S_{i-1}}(dx).$$

Using $3\varrho_i/((\varrho_i+2)(2\varrho_i-1)(1+\varrho_i^2)) \geq 1/(2\varrho_i^3)$, we obtain, by Lemma 1(c),

$$EH_4\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right)$$

$$\leq \frac{1}{2}\left(EH_4\left(-\frac{S_{i-1}}{\lambda_i}, \frac{1}{\lambda_i\sqrt{n}}\right) + EH_4\left(-\frac{S_{i-1}}{\lambda_i}, \frac{-1}{\lambda_i\sqrt{n}}\right)\right)$$

$$+ 0.023\gamma_{\varrho_i,K}\left(1+\frac{13}{4}\right)\frac{1}{\lambda_i^4 n^2} P\left[S_{i-1} \in \left[\frac{1}{4}\lambda_i, \frac{1}{2}\lambda_i\right]\right]$$

$$+ 0.0208\gamma_{\varrho_i,K} \frac{1}{\lambda_i^3 n^{3/2}} \max_{t\in[-3/2,-1/4]} \varphi'''(t) P\left[S_{i-1} \in \left[\frac{1}{4}\lambda_i, \frac{1}{2}\lambda_i\right]\right].$$

Lemma 3(a) together with

$$\max_{t\in[-3/2,-1/4]} \varphi'''(t) = \max\left\{\varphi'''\left(-\tfrac{3}{2}\right), \varphi'''\left(-\tfrac{1}{4}\right)\right\} = \varphi'''\left(-\tfrac{3}{2}\right) \leq -0.145$$

and $\lambda_i \leq 1$ yields

$$EH_4\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right)$$

$$\leq \frac{1}{\lambda_i^4 n^2}(0.302\,\delta(n) + 0.481\lambda_i)$$

$$+ \frac{0.098}{\lambda_i^4 n^2}\left(2\sup_{t\in\mathbb{R}}|P[S_{i-1}\leq t] - \Phi(t)| + 0.1\lambda_i\right)$$

$$- \frac{0.003}{\lambda_i^3 n^{3/2}}\gamma_{\varrho_i,K}\left(0.088\lambda_i - 2\sup_{t\in\mathbb{R}}|P[S_{i-1}\leq t]-\Phi(t)|\right).$$

Lemma 1 in Bolthausen (1982) implies

$$\sup_{t \in \mathbb{R}} \left| P[S_{i-1} \le t] - \Phi(t) \right| \le 2\delta(n) + \frac{5}{\sqrt{2\pi}} \lambda_i.$$

We obtain, by Theorem 1 and (2.3),

$$\begin{aligned}\sup_{t \in \mathbb{R}} \left| P[S_{i-1} \le t] - \Phi(t) \right| &\le \delta(i-1) + \frac{3}{4}\lambda_i^2 \\ &\le \sqrt{2}\, 100 K \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})} + \frac{3}{4}\lambda_i^2.\end{aligned}$$

Putting the last three statements together leads to

$$\begin{aligned}&EH_4\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right) \\ &\quad \le \frac{0.694}{\lambda_i^4 n^2}\delta(n) + \frac{0.882}{\lambda_i^3 n^2} + \frac{0.849}{\lambda_i^3 n^{3/2}} \frac{K}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})} \\ &\qquad + \frac{0.0045}{\lambda_i n^{3/2}} - 0.000264 \frac{\gamma_{\varrho_i, K}}{\lambda_i^2 n^{3/2}}.\end{aligned} \tag{3.3}$$

Furthermore, we obtain, as a consequence of Lemma 3(a),

$$E\left| H_4\left(-\frac{S_{i-1}}{\lambda_i}, \frac{Z_i}{\lambda_i\sqrt{n}}\right)\right| \le \frac{0.906}{\lambda_i^4 n^2}\delta(n) + \frac{1.443}{\lambda_i^3 n^2}. \tag{3.4}$$

By inserting (3.3) and (3.4) in (3.2), we obtain

$$\begin{aligned}&0.000264 \sum_{i=\lceil n/2\rceil+1}^{n} \frac{\gamma_{\varrho_i, K}}{\lambda_i^2 n^{3/2}} \\ &\quad \le \delta(n)\left(3 + 1.6 \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i^4 n^2}\right) + \frac{0.798\kappa}{\sqrt{n}} \\ &\qquad + 2.325 \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i^3 n^2} + 0.0045 \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i n^{3/2}} \\ &\qquad + 0.849 \frac{K}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{jh(\sqrt{j})} \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{\lambda_i^3 n^{3/2}}.\end{aligned}$$

Choose κ large enough that $\kappa \ge (0.849 \times 3 \times 2 \times 12)/0.000264$ and $h(\varrho_i) = h(\sqrt{n - i + \kappa^2}) \ge h(\kappa) \ge K/3$, that is, $\gamma_{\varrho_i, K} = K/(3h(\varrho_i))$. For n large enough we obtain $\sum_{i=1}^{n} 1/(ih(\sqrt{i})) \geq 4\sum_{i=1}^{\lfloor \kappa^2 \rfloor} 1/(ih(\sqrt{i}))$ and therefore

$$\begin{aligned}
\sum_{i=\lceil n/2\rceil+1}^{n} \frac{\gamma_{\varrho_i, K}}{\lambda_i^2 n^{3/2}} &= \frac{K}{3}\frac{1}{\sqrt{n}} \sum_{i=\lceil n/2\rceil+1}^{n} \frac{1}{(n-i+\kappa^2)h\left(\sqrt{n-i+\kappa^2}\right)} \\
&\geq \frac{K}{3}\frac{1}{\sqrt{n}} \sum_{i=\lfloor \kappa^2\rfloor+1}^{\lceil n/2\rceil} \frac{1}{ih(\sqrt{i})} \\
&\geq \frac{K}{6}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{ih(\sqrt{i})} - \frac{K}{3}\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor \kappa^2\rfloor} \frac{1}{ih(\sqrt{i})} \\
&\geq \frac{K}{12}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{ih(\sqrt{i})}.
\end{aligned}$$

With these observations we conclude

$$0.00001\frac{K}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{ih(\sqrt{i})} \leq 3\delta(n) + 0.8\frac{\kappa}{\sqrt{n}}.$$

It follows that $Kn^{-1/2}\sum_{i=1}^{n} 1/(ih(\sqrt{i}))$ is the exact order of magnitude for $\delta(n)$ in this example.

EXAMPLE 2. Suppose that h satisfies (1.10) and the following two additional properties: $h(x) \nearrow \infty$ and $h(x)/x \searrow 0$ for $x \to \infty$. Furthermore, let $\max\{1, 2h(1)\} \leq K$. We consider for $\varrho \geq 1$ the following probability measures on $(\mathbb{R}, \mathscr{B})$:

$$\begin{aligned}
R_\varrho &:= \left(\frac{\varrho^2}{1+\varrho^2}\delta_{1/\varrho} + \frac{1}{1+\varrho^2}\delta_{-\varrho}\right), \\
P_\varrho &:= (1-\gamma_{\varrho, K})R_1 + \gamma_{\varrho, K}R_\varrho,
\end{aligned} \tag{3.5}$$

where $\gamma_{\varrho, K} := \min\{1, K/(2h(4\varrho))\}$. The following properties hold: $\int xP_\varrho(dx) = 0$, $\int x^2 P_\varrho(dx) = 1$ and

$$\int x^2 h(|x|)P_\varrho(dx) \leq (1-\gamma_{\varrho, K})h(1) + \gamma_{\varrho, K}\{0.5h(1) + h(\varrho)\} \leq K.$$

Let $\varrho_i := \sqrt{n-i}/4$ and $\kappa := c^{-1}\sum_{i=1}^{n} 1/(\sqrt{i}\,h(\sqrt{i}))$, where $c := 0.0000625/(6 \times 2.30 \times K)$. The assumptions on h lead to the following properties:

$$\frac{\kappa}{\sqrt{n}} = \frac{1}{c}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}\,h(\sqrt{i})} \to 0, \qquad n \to \infty,$$

in particular,

$$\frac{\kappa}{\sqrt{n}} \leq \frac{1}{2} \quad \text{for } n \geq n_0,$$

$$\kappa \geq \frac{1}{c}\sum_{i=1}^{n} \frac{1}{i}\frac{\sqrt{i}}{h(\sqrt{i})} \geq \frac{\ln n}{ch(1)} \to \infty, \qquad n \to \infty$$

and $\kappa \geq 4$ for $n \geq n_1$, $K \leq 2h(\kappa)$ for $n \geq n_2$ and

$$\gamma_{\varrho_i, K} = \frac{K}{2h(\sqrt{n-i})}$$

for $i \leq \lfloor n - \kappa^2 \rfloor$ and $n \geq \max\{n_1, n_2\}$,

$$\frac{1}{\kappa}\sum_{i=1}^{n} \frac{1}{\sqrt{i}\,h(\sqrt{i})} = c \geq 8\frac{1}{\kappa}\sum_{i=1}^{\lfloor \kappa^2 \rfloor} \frac{1}{\sqrt{i}\,h(\sqrt{i})} (\to 0 \text{ as } \kappa \to \infty) \quad \text{for } n \geq n_3$$

and

$$\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{i=\lceil n/2 \rceil + 1}^{\lfloor n-\kappa^2 \rfloor} \frac{1}{\sqrt{n-i}\,h(\sqrt{n-i})} &\geq \frac{1}{\sqrt{n}} \sum_{i=\lceil \kappa^2 \rceil}^{\lfloor n/2 \rfloor - 1} \frac{1}{\sqrt{i}\,h(\sqrt{i})} \\
&\geq \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n/2 \rfloor - 1} \frac{1}{\sqrt{i}\,h(\sqrt{i})} - \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor \kappa^2 \rfloor} \frac{1}{\sqrt{i}\,h(\sqrt{i})} \\
&\geq \frac{1}{4}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}\,h(\sqrt{i})} - \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor \kappa^2 \rfloor} \frac{1}{\sqrt{i}\,h(\sqrt{i})} \\
&\geq \frac{1}{8}\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}\,h(\sqrt{i})} \quad \text{for } n \geq \max\{n_3, 6\}.
\end{aligned}$$

It will be assumed that $n \geq \max\{n_0, \dots, n_3, 6\}$. We define r.v.'s $X_1, \dots, X_n$ and $Z_1, \dots, Z_n$ on a suitable probability space as follows: X_i, $i \leq \lceil n/2 \rceil$, are independent identically $N(0,1)$-distributed r.v.'s,

$$\begin{aligned}
&P_{X_i|(X_1,\dots,X_{i-1})}((x_1,\dots,x_{i-1}), A) \\
&\quad := 1_{J_i^c}(x_1 + \cdots + x_{i-1}) P_1(A) + 1_{J_i}(x_1 + \cdots + x_{i-1}) P_{\varrho_i}(A),
\end{aligned}$$

with $\lceil n/2 \rceil < i \leq \lfloor n - \kappa^2 \rfloor$, $J_i := [-\varrho_i, \varrho_i]$ and X_i, $\lfloor n - \kappa^2 \rfloor < i \leq n$, are identically $N(0,1)$-distributed r.v.'s such that $(X_1, \dots, X_{\lfloor n-\kappa^2 \rfloor}), X_{\lfloor n-\kappa^2 \rfloor + 1}, \dots, X_n$ are independent and $Z_1, \dots, Z_n$ are i.i.d. $N(0,1)$-distributed r.v.'s such that $(X_1, \dots, X_n), Z_1, \dots, Z_n$ are independent. This yields

(3.6) $X_1, \dots, X_n$ is an m.d.s. with $E(X_i^2 | X_1, \dots, X_{i-1}) = 1$ and $E(X_i^2 h(|X_i|) | X_1, \dots, X_{i-1}) \leq K$ a.s., $i = 1, \dots, n$.

Let $\lambda_i := \sqrt{(n-i)/n}$. We obtain similarly as in Example 1

$$\begin{aligned}
0 \leq \sum_{i=\lceil n/2 \rceil + 1}^{\lfloor n-\kappa^2 \rfloor} &\left(EH_3\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right) - EH_3\left(-\frac{S_{i-1}}{\lambda_i}, \frac{Z_i}{\lambda_i\sqrt{n}}\right)\right) \qquad (3.7)\\
&+ \delta(n),
\end{aligned}$$

$$P_{X_i|S_{i-1}}(x, A) = 1_{[-\lambda_i/4, \lambda_i/4]^c}(x) P_1(A) + 1_{[-\lambda_i/4, \lambda_i/4]}(x) P_{\varrho_i}(A)$$

and

$$
\begin{aligned}
EH_3&\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right)\\
&= \int_{[-\lambda_i/4,\,\lambda_i/4]^c} \frac{1}{2}\left(H_3\left(-\frac{x}{\lambda_i}, \frac{1}{\lambda_i\sqrt{n}}\right) + H_3\left(-\frac{x}{\lambda_i}, \frac{-1}{\lambda_i\sqrt{n}}\right)\right) P_{S_{i-1}}(dx)\\
&\quad + \int_{[-\lambda_i/4,\,\lambda_i/4]} \Bigg\{(1-\gamma_{\varrho_i,K})\frac{1}{2}\left(H_3\left(-\frac{x}{\lambda_i}, \frac{1}{\lambda_i\sqrt{n}}\right) + H_3\left(-\frac{x}{\lambda_i}, \frac{-1}{\lambda_i\sqrt{n}}\right)\right)\\
&\qquad\qquad + \gamma_{\varrho_i,K}\Bigg(\frac{\varrho_i^2}{1+\varrho_i^2} H_3\left(-\frac{x}{\lambda_i}, \frac{4}{\lambda_i^2 n}\right)\\
&\qquad\qquad\qquad + \frac{1}{1+\varrho_i^2} H_3\left(-\frac{x}{\lambda_i}, -\frac{1}{4}\right)\Bigg)\Bigg\} P_{S_{i-1}}(dx)
\end{aligned}
$$

for $i > \lceil n/2 \rceil$. Using $1/(1+\varrho_i^2) \geq 1/(2\varrho_i^2)$, we obtain, by Lemma 1(c),

$$
\begin{aligned}
EH_3&\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right)\\
&\leq \frac{1}{2}\left(EH_3\left(-\frac{S_{i-1}}{\lambda_i}, \frac{1}{\lambda_i\sqrt{n}}\right) + EH_3\left(-\frac{S_{i-1}}{\lambda_i}, \frac{-1}{\lambda_i\sqrt{n}}\right)\right)\\
&\quad + 0.067\gamma_{\varrho_i,K}\left(1+\frac{1}{2}\right)\frac{1}{\lambda_i^3 n^{3/2}} P\left[S_{i-1} \in \left[-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_i\right]\right]\\
&\quad + 0.0208\gamma_{\varrho_i,K}\frac{1}{\lambda_i^2 n} \max_{t\in[-1/4,1/2]} \varphi''(t) P\left[S_{i-1} \in \left[-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_i\right]\right].
\end{aligned}
$$

Lemma 1(a), Lemma 2 in Bolthausen (1982) and Theorem 2 together with

$$
\max_{t\in[-1/4,1/2]} \varphi''(t) = \max\left\{\varphi''\left(-\frac{1}{4}\right), \varphi''\left(\frac{1}{2}\right)\right\} = \varphi''\left(\frac{1}{2}\right) \leq -0.264,
$$

$$
\begin{aligned}
P&\left[S_{i-1} \in \left[-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_i\right]\right]\\
&= P\left[\frac{1}{\sqrt{i-1}}\sum_{j=1}^{i-1} X_j \in \left[-\frac{1}{4}\sqrt{\frac{n-i}{i-1}}, \frac{1}{4}\sqrt{\frac{n-i}{i-1}}\right]\right]\\
&\geq \frac{1}{2}\sqrt{\frac{n-i}{i-1}}\,\varphi\left(\frac{1}{4}\sqrt{\frac{n-i}{i-1}}\right) - 2\delta(i-1) \geq 0.19\lambda_i - 2\delta(i-1)
\end{aligned}
$$

and

$$\sqrt{\frac{n-i}{i-1}} \le \sqrt{2}\,\lambda_i$$

yield

$$\begin{aligned}
&EH_3\left(-\frac{S_{i-1}}{\lambda_i}, \frac{X_i}{\lambda_i\sqrt{n}}\right) + E\left|H_3\left(-\frac{S_{i-1}}{\lambda_i}, \frac{Z_i}{\lambda_i\sqrt{n}}\right)\right| \\
&\quad\le \frac{1}{\lambda_i^3 n^{3/2}}\left(0.643\sqrt{2}\,300K\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{\sqrt{j}\,h\left(\sqrt{j}\right)}\right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{1.407}{\sqrt{2\pi}}\sqrt{\frac{n-i}{i-1}}\right)\left(1 + \frac{4}{\sqrt{2\pi}}\right) \\
&\qquad + \frac{0.101}{\lambda_i^3 n^{3/2}}\left(2\sqrt{2}\,300K\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{\sqrt{j}\,h\left(\sqrt{j}\right)} + \frac{1}{2\sqrt{2\pi}}\sqrt{\frac{n-i}{i-1}}\right) \\
&\qquad - \frac{0.0054}{\lambda_i^2 n}\gamma_{\varrho_i, K}\left(0.19\lambda_i - 2\sqrt{2}\,300K\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{\sqrt{j}\,h\left(\sqrt{j}\right)}\right) \\
&\quad\le \frac{794}{\lambda_i^3 n^{3/2}}\frac{K}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{\sqrt{j}\,h\left(\sqrt{j}\right)} + \frac{2.09}{\lambda_i^2 n^{3/2}} \\
&\qquad + \frac{2.30K}{\lambda_i^2 nh\left(\lambda_i\sqrt{n}\right)}\frac{K}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{\sqrt{j}\,h\left(\sqrt{j}\right)} - \frac{0.0005K}{\lambda_i nh\left(\lambda_i\sqrt{n}\right)}.
\end{aligned} \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned}
&0.0000625\frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)} \\
&\quad\le \delta(n) + 794\frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}\sum_{j=\lceil\kappa^2\rceil}^{\lfloor n/2\rfloor}\frac{1}{j^{3/2}} \\
&\qquad + 2.30\frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}\sum_{j=\lceil\kappa^2\rceil}^{\lfloor n/2\rfloor}\frac{K}{jh\left(\sqrt{j}\right)} + 2.09\frac{1}{\sqrt{n}}\sum_{j=\lceil\kappa^2\rceil}^{\lfloor n/2\rfloor}\frac{1}{j} \\
&\quad\le \delta(n) + 794\frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}\frac{3}{\kappa} + 2.30\frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}cK \\
&\qquad + 2.09\frac{K}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{i}\,h(\sqrt{i}\,)}\frac{h(\kappa)}{\kappa K}.
\end{aligned}$$

This shows $\delta(n) \ge 0.00003Kn^{-1/2}\sum_{i=1}^{n}1/(\sqrt{i}\,h(\sqrt{i}\,))$ for n large enough.

EXAMPLE 3. Let h be a function as specified in Example 2. For $n \geq 2$ with $h(\sqrt{n}) \geq 8$ we consider a sequence $X_1, \dots, X_n$ of real-valued i.i.d. r.v.'s with $P_{X_1} := (1 - \gamma(n))R_1 + \gamma(n)R_{\sqrt{n}}$, where R_ϱ is defined according to (3.5) and $\gamma(n) := 1/h(\sqrt{n})$. Let $X_{ni} := X_i/\sqrt{n}$ for $1 \leq i \leq n$. The following properties hold:

$$E(X_1) = 0, \qquad E\big(X_1^2\big) = 1, \qquad E\big(X_1^2 h(|X_1|)\big) \leq h(1) + 1,$$

$$E\Big(X_{n1}^2 1_{(|X_{n1}| > 1)}\Big) = 0, \qquad E\Big(X_{n1}^2 1_{(|X_{n1}| > 1/\sqrt{n})}\Big) = \frac{\gamma(n)}{1+n},$$

$$E\Big(X_{n1}^4 1_{(|X_{n1}| \leq 1)}\Big) = \frac{1}{n^2} + \gamma(n)\frac{(n-1)^2}{n^3}$$

and

$$E\Big(X_{n1}^3 1_{(|X_{n1}| \leq 1)}\Big) = \gamma(n)\frac{1-n}{n^2}.$$

Thus, for n large enough, Theorems 2.2 and 2.7 in Hall (1982) yield

$$C_0\gamma(n) \leq C_1\gamma(n) - C_2\left(\frac{1}{\sqrt{n}} + \gamma(n)^2\right) \leq \left| P\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \leq 1\right] - \Phi(1)\right|,$$

where the first inequality holds because $(1/\sqrt{n} + \gamma(n)^2)/\gamma(n) \to 0$.

APPENDIX

In this section we state and prove Lemmas 1, 2 and 3.

LEMMA 1. (a) $|H_i(u,v)| \leq |v|^i\psi_i(u)$, $i = 2,3,4$, *with* $\|\psi_2\|_1 \leq 5.023$, $\|\psi_2\|_V \leq 4.947$, $\|\psi_3\|_1 \leq 1.407$, $\|\psi_3\|_V \leq 0.643$, $\|\psi_4\|_1 \leq 0.449$ *and* $\|\psi_4\|_V \leq 0.151$.

(b) $|H_i(u,v)| \leq \min\{|v|^{i-1}, |v|^i\}\tilde{\psi}_i(u)$, $i = 3,4$, *with* $\|\tilde{\psi}_3\|_1 \leq 6.830$, $\|\tilde{\psi}_3\|_V \leq 6.075$, $\|\tilde{\psi}_4\|_1 \leq 2.018$ *and* $\|\tilde{\psi}_4\|_V \leq 1.048$.

(c) $|H_i(u,v)| \leq C_i|v|^i$, $i = 3,4$, *with* $C_3 = 0.067$ *and* $C_4 = 0.023$.

(d) $|H_i(u,v)| \leq D_i \min\{|v|^{i-1}, |v|^i\}$, $i = 3,4$, *with* $D_3 = 0.242$ *and* $D_4 = 0.134$.

PROOF. With $\vartheta, \tilde{\vartheta} \in [0,1]$ depending on u, v, we obtain in (2.1), for $i = 2,3,4$,

$$\begin{aligned} H_i(u,v) &= \frac{(-v)^i}{i!}\varphi^{(i-1)}(u - \vartheta v) \\ &= \frac{(-v)^{i-2}}{(i-2)!}\big(\Phi^{(i-2)}(u - \tilde{\vartheta} v) - \Phi^{(i-2)}(u)\big) \qquad \text{(A.1)} \\ &\quad - \frac{(-v)^{i-1}}{(i-1)!}\varphi^{(i-2)}(u). \end{aligned}$$

Let $a_2 := 2$, $a_3 := 4$ and $a_4 := 5$. For $|v| \le |u|/2$ and $|u| > a_i$ it follows that $|u - \vartheta v| \ge |u|/2$ and therefore

$$|H_i(u,v)| \le \frac{|v|^i}{i!} \max_{|t| \ge |u|/2} |\varphi^{(i-1)}(t)| = \frac{|v|^i}{i!} \left| \varphi^{(i-1)}\left(\frac{1}{2}u\right)\right|, \qquad i = 2,3,4. \tag{A.2}$$

Using (A.1) and (A.2), we get

$$\begin{aligned}
|H_i(u,v)| \le{} & \frac{|v|^i}{i!}\|\varphi^{(i-1)}\|_\infty 1_{\{x \mid |x| \le a_i\}}(u) + \frac{|v|^i}{i!}\left|\varphi^{(i-1)}\left(\frac{1}{2}u\right)\right| 1_{\{x \mid |x| > a_i\}}(u) \\
& + \Bigg[\frac{|v|^{i-2}}{(i-2)!} \sup_{s \in \mathbb{R},\, |t| > a_i} |\Phi^{(i-2)}(s) - \Phi^{(i-2)}(t)| \\
& \qquad + \frac{|v|^{i-1}}{(i-1)!}|\varphi^{(i-2)}(u)| \Bigg] 1_{\{(x,y) \mid |x| > a_i,\, |x|/2 < |y|\}}(u,v) \qquad \text{(A.3)} \\
\le{} & |v|^i \left\{ \frac{\|\varphi^{(i-1)}\|_\infty}{i!} f_i(u) + \frac{1}{i!} g_i(u) \right. \\
& \left. + \frac{4b_i}{(i-2)!} h_i(u) + \frac{2}{a_i(i-1)!} k_i(u) \right\} =: |v|^i \psi_i(u),
\end{aligned}$$

where

$$b_i := \sup_{s \in \mathbb{R},\, |t| > a_i} |\Phi^{(i-2)}(s) - \Phi^{(i-2)}(t)|, \qquad f_i(u) := 1_{\{x \mid |x| \le a_i\}}(u),$$

$$g_i(u) := \left|\varphi^{(i-1)}\left(\frac{1}{2}u\right)\right| 1_{\{x \mid |x| > a_i\}}(u), \qquad h_i(u) := u^{-2} 1_{\{x \mid |x| > a_i\}}(u),$$

$$k_i(u) := |\varphi^{(i-2)}(u)| 1_{\{x \mid |x| > a_i\}}(u).$$

Exploiting the local monotonicity of the functions involved, we obtain $b_2 \le 1$, $b_3 \le 0.4$, $b_4 \le 0.25$, $\|f_i\|_1 = 2a_i$, $\|f_i\|_V = 2$, $\|g_i\|_1 = 4|\varphi^{(i-2)}(a_i/2)|$, $\|g_i\|_V = 4|\varphi^{(i-1)}(a_i/2)|$, $\|h_i\|_1 = 2/a_i$, $\|h_i\|_V = 4/a_i^2$, $\|k_2\|_1 = 2(1 - \Phi(2))$, $\|k_i\|_1 = 2|\varphi^{(i-3)}(a_i)|$, $i = 3,4$, and $\|k_i\|_V = 4|\varphi^{(i-2)}(a_i)|$. In particular, we obtain

$$\|f_2\|_1 = 4, \qquad \|f_3\|_1 = 8, \qquad \|f_4\|_1 = 10, \qquad \|f_i\|_V = 2,$$

$$\|g_2\|_1 = \frac{4}{\sqrt{2\pi}} e^{-1/2} \le 0.97, \qquad \|g_3\|_1 = \frac{8}{\sqrt{2\pi}} e^{-2} \le 0.44,$$

$$\|g_4\|_1 = \frac{21}{\sqrt{2\pi}} e^{-25/8} \le 0.37, \qquad \|g_2\|_V = \frac{4}{\sqrt{2\pi}} e^{-1/2} \le 0.97,$$

$$\|g_3\|_V = \frac{12}{\sqrt{2\pi}} e^{-2} \le 0.65, \qquad \|g_4\|_V = \frac{65}{2\sqrt{2\pi}} e^{-25/8} \le 0.57,$$

$$\|h_2\|_1 = 1, \qquad \|h_3\|_1 = 0.5, \qquad \|h_4\|_1 = 0.4,$$

$$\text{(A.4)}\quad \|h_2\|_V = 1, \qquad \|h_3\|_V = 0.25, \qquad \|h_4\|_V = 0.16,$$
$$\|k_2\|_1 = 2(1-\Phi(2)) \le \frac{1}{\sqrt{2\pi}}e^{-2} \le 0.054,$$
$$\|k_3\|_1 = \frac{2}{\sqrt{2\pi}}e^{-8} \le 0.0003, \qquad \|k_4\|_1 = \frac{10}{\sqrt{2\pi}}e^{-25/2} \le 0.00002,$$
$$\|k_2\|_V = \frac{4}{\sqrt{2\pi}}e^{-2} \le 0.22, \qquad \|k_3\|_V = \frac{16}{\sqrt{2\pi}}e^{-8} \le 0.0022,$$
$$\|k_4\|_V = \frac{96}{\sqrt{2\pi}}e^{-25/2} \le 0.0002.$$

Furthermore, we have

$$\text{(A.5)}\quad \begin{aligned} &\|\varphi'\|_\infty = \frac{1}{\sqrt{2\pi e}} \le 0.242, \ \|\varphi''\|_\infty = \frac{1}{\sqrt{2\pi}} \le 0.4,\\ &\|\varphi'''\|_\infty = \frac{\sqrt{3-\sqrt{6}}\,\sqrt{6}}{\sqrt{2\pi}}\exp\left(-\frac{3-\sqrt{6}}{2}\right) \le 0.551. \end{aligned}$$

From (A.4), (A.5) and the definition of the functions ψ_i, $i = 2,3,4$, we get

$$\text{(A.6)}\quad \begin{aligned} \|\psi_2\|_1 &\le \frac{\|\varphi'\|_\infty}{2}\|f_2\|_1 + \frac{1}{2}\|g_2\|_1 + 4b_2\|h_2\|_1 + \frac{2}{a_2}\|k_2\|_1 \le 5.023,\\ \|\psi_2\|_V &\le \frac{\|\varphi'\|_\infty}{2}\|f_2\|_V + \frac{1}{2}\|g_2\|_V + 4b_2\|h_2\|_V + \frac{2}{a_2}\|k_2\|_V \le 4.947,\\ \|\psi_3\|_1 &\le \frac{\|\varphi''\|_\infty}{6}\|f_3\|_1 + \frac{1}{6}\|g_3\|_1 + 4b_3\|h_3\|_1 + \frac{1}{a_3}\|k_3\|_1 \le 1.407,\\ \|\psi_3\|_V &\le \frac{\|\varphi''\|_\infty}{6}\|f_3\|_V + \frac{1}{6}\|g_3\|_V + 4b_3\|h_3\|_V + \frac{1}{a_3}\|k_3\|_V \le 0.643,\\ \|\psi_4\|_1 &\le \frac{\|\varphi'''\|_\infty}{24}\|f_4\|_1 + \frac{1}{24}\|g_4\|_1 + 2b_4\|h_4\|_1 + \frac{1}{3a_4}\|k_4\|_1 \le 0.449,\\ \|\psi_4\|_V &\le \frac{\|\varphi'''\|_\infty}{24}\|f_4\|_V + \frac{1}{24}\|g_4\|_V + 2b_4\|h_4\|_V + \frac{1}{3a_4}\|k_4\|_V \le 0.151. \end{aligned}$$

Inequality (A.3) together with (A.6) implies assertion (a). According to (a), we obtain

$$\begin{aligned} |H_i(u,v)| &\le |H_{i-1}(u,v)| + \frac{|v|^{i-1}}{(i-1)!}|\varphi^{(i-2)}(u)|\\ &\le |v|^{i-1}\left\{\psi_{i-1}(u) + \frac{1}{(i-1)!}|\varphi^{(i-2)}(u)|\right\}\\ &=: |v|^{i-1}\overline{\psi}_i(u), \qquad i = 3,4. \end{aligned}$$

This implies

$$|H_i(u,v)| \le \min\{|v|^{i-1},|v|^i\}\big(\psi_i(u)+\bar{\psi}_i(u)\big) =: \min\{|v|^{i-1},|v|^i\}\tilde{\psi}_i(u). \tag{A.7}$$

Next we observe that

$$\begin{aligned}
\|\varphi'\|_1 &= 2\varphi(0) = \frac{2}{\sqrt{2\pi}} \le 0.8,\\
\|\varphi'\|_V &= 4|\varphi'(1)| = \frac{4}{\sqrt{2\pi}}e^{-1/2} \le 0.97,\\
\|\varphi''\|_1 &= 2(\varphi'(0)-2\varphi'(1)) = -4\varphi'(1) = \frac{4}{\sqrt{2\pi}}e^{-1/2} \le 0.97,\\
\|\varphi''\|_V &= 4\varphi''(\sqrt{3}) - 2\varphi''(0) = \frac{2}{\sqrt{2\pi}}(4e^{-3/2}+1) \le 1.52.
\end{aligned} \tag{A.8}$$

Recalling the definition of the functions $\tilde{\psi}_i$, $i = 3, 4$, we can infer from (a) and (A.8) that

$$\begin{aligned}
\|\tilde{\psi}_3\|_1 &\le \|\psi_3\|_1 + \|\psi_2\|_1 + \tfrac{1}{2}\|\varphi'\|_1 \le 6.830,\\
\|\tilde{\psi}_3\|_V &\le \|\psi_3\|_V + \|\psi_2\|_V + \tfrac{1}{2}\|\varphi'\|_V \le 6.075,\\
\|\tilde{\psi}_4\|_1 &\le \|\psi_4\|_1 + \|\psi_3\|_1 + \tfrac{1}{6}\|\varphi''\|_1 \le 2.018,\\
\|\tilde{\psi}_4\|_V &\le \|\psi_4\|_V + \|\psi_3\|_V + \tfrac{1}{6}\|\varphi''\|_V \le 1.048.
\end{aligned}$$

This completes the proof of assertion (b). Concerning parts (c) and (d), we observe that (A.1) together with (A.5) implies

$$|H_i(u,v)| \le \frac{\|\varphi^{(i-1)}\|_\infty}{i!}|v|^i \le C_i|v|^i$$

and

$$|H_i(u,v)| \le |H_{i-1}(u,v)| + \frac{|\varphi^{(i-2)}(u)|}{(i-1)!}|v|^{i-1} \le \frac{2\|\varphi^{(i-2)}\|_\infty}{(i-1)!}|v|^{i-1} \le D_i|v|^{i-1}.$$

Therefore we obtain

$$|H_i(u,v)| \le \min\{|v|^{i-1},|v|^i\}\max\{D_i,C_i\} = \min\{|v|^{i-1},|v|^i\}D_i. \qquad \square$$

LEMMA 2. *Suppose that* $h\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *satisfies* (1.10). *Then, for* $x > 0$ *and* $y \ge 0$,

$$\min\left\{\left(\frac{y}{x}\right)^{i-1}, \left(\frac{y}{x}\right)^i\right\} \le \frac{y^{i-1}h(y)}{x^{i-1}h(x)}, \qquad i \ge 1.$$

PROOF. In the case $y \ge x$ we obtain $\min\{(y/x)^{i-1}, (y/x)^i\} = y^{i-1}/x^{i-1}$. Thus the assertion is implied by $0 < h(x) \le h(y)$. For $y < x$ we have $\min\{(y/x)^{i-1}, (y/x)^i\} = y^i/x^i$. The assertion is true for $y = 0$ and for $y > 0$ it follows from $0 < h(x)/x \le h(y)/y$. $\square$

LEMMA 3. *Let $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $\kappa \ge 1$. For $X_1, \dots, X_n$ and $h: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, assume* (1.1), (1.2) *and* (1.10). *Suppose that Y is a real-valued r.v. Then, for $i = 1, \dots, n$,*

$$\text{(a)} \quad E\left|H_j\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{Y}{\lambda_i\sqrt{n}}\right)\right| \le \frac{\left\|E\left(|Y|^j | S_{i-1}\right)\right\|_\infty}{\left(\lambda_i\sqrt{n}\right)^j} \times \begin{cases} \left(1.286\,\delta(n) + 1.844\lambda_i\right), & \text{for } j = 3, \\ \left(0.302\,\delta(n) + 0.481\lambda_i\right), & \text{for } j = 4, \end{cases}$$

$$\text{(b)} \quad E\left|H_j\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{Y}{\lambda_i\sqrt{n}}\right)\right| \le \frac{\left\|E\left(|Y|^{j-1} h(|Y|) | S_{i-1}\right)\right\|_\infty}{\left(\lambda_i\sqrt{n}\right)^{j-1} h\left(\lambda_i\sqrt{n}\right)} \times \begin{cases} \left(12.150\,\delta(n) + 14.843\lambda_i\right), & \text{for } j = 3, \\ \left(2.096\,\delta(n) + 2.896\lambda_i\right), & \text{for } j = 4, \end{cases}$$

$$\text{(c)} \quad E\left|H_j\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{Y}{\lambda_i\sqrt{n}}\right)\right| \le \frac{E\left(|Y|^j\right)}{\left(\lambda_i\sqrt{n}\right)^j} \times \begin{cases} 0.067, & \text{for } j = 3, \\ 0.023, & \text{for } j = 4, \end{cases}$$

$$\text{(d)} \quad E\left|H_j\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{Y}{\lambda_i\sqrt{n}}\right)\right| \le \frac{E\left(|Y|^{j-1} h(|Y|)\right)}{\left(\lambda_i\sqrt{n}\right)^{j-1} h\left(\lambda_i\sqrt{n}\right)} \times \begin{cases} 0.242, & \text{for } j = 3, \\ 0.134, & \text{for } j = 4, \end{cases}$$

where $\delta(n)$, S_{i-1} and λ_i have been defined in (1.3), (1.4) *and* (2.2), *respectively.*

PROOF. By using Lemmas 1 and 2 in Bolthausen (1982) and Lemma 1, we obtain

$$\begin{aligned} E&\left|H_j\left(\frac{t - S_{i-1}}{\lambda_i}, \frac{Y}{\lambda_i\sqrt{n}}\right)\right| \\ &\le \frac{\left\|E\left(|Y|^j | S_{i-1}\right)\right\|_\infty}{\left(\lambda_i\sqrt{n}\right)^j} E\psi_j\left(\frac{t - S_{i-1}}{\lambda_i}\right) \\ &\le \frac{\left\|E\left(|Y|^j | S_{i-1}\right)\right\|_\infty}{\left(\lambda_i\sqrt{n}\right)^j} \left\{\|\psi_j\|_V \sup_{t \in \mathbb{R}} \left|P[S_{i-1} \le t] - \Phi(t)\right| + \|\psi_j\|_1 \|\varphi\|_\infty \lambda_i\right\} \end{aligned}$$

$$\le \frac{\left\| E\left(|Y|^j|S_{i-1}\right)\right\|_\infty}{\left(\lambda_i\sqrt{n}\right)^j}\left\{2\|\psi_j\|_V\delta(n)+\left(\frac{5}{\sqrt{2\pi}}\|\psi_j\|_V+\frac{1}{\sqrt{2\pi}}\|\psi_j\|_1\right)\lambda_i\right\}.$$

With the additional use of Lemma 2, one can similarly prove part (b). The assertions in parts (c) and (d) are immediate consequences of Lemma 1(c) and (d). □

Acknowledgments. The author would like to thank Professor Erich Haeusler for drawing his attention to the papers by Joos (1988) and by Kir'yanova and Rotar' (1991). Part of this work was done while the author was visiting the Department of Statistics and Probability, Michigan State University, East Lansing, and the Department of Statistics, University of Illinois at Urbana-Champaign. He thanks the members of both departments for their hospitality. Finally, the author wishes to thank the referee for comments leading to a clearer and more consistent exposition throughout the paper.

REFERENCES

BOLTHAUSEN, E. (1982). Exact convergence rates in some martingale central limit theorems. *Ann. Probab.* **10** 672–688.

HALL, P. (1982). *Rates of Convergence in the Central Limit Theorem. Research Notes in Mathematics* **62**. Pitman, Boston.

JOOS, K. (1988). Abschätzungen der konvergenzgeschwindigkeit in asymptotischen verteilungsaussagen für martingale. Ph.D. dissertation, Univ. Munich.

KIR'YANOVA, L. V. and ROTAR', V. I. (1991). Estimates for the rate of convergence in the central limit theorem for martingales. *Theory Probab. Appl.* **36** 289–302.

PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer, Berlin.

RENZ, J. (1991). Konvergenzgeschwindigkeit und asymptotische konfidenzintervalle in der stochastischen approximation. Ph.D. dissertation, Univ. Stuttgart.

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