

A LIMIT THEOREM FOR THE CONTOUR PROCESS OF CONDITIONED GALTON–WATSON TREES

BY THOMAS DUQUESNE

Ecole Normale Supérieure de Cachan, C.M.L.A.

In this work, we study asymptotics of the genealogy of Galton–Watson processes conditioned on the total progeny. We consider a fixed, aperiodic and critical offspring distribution such that the rescaled Galton–Watson processes converges to a continuous-state branching process (CSBP) with a stable branching mechanism of index $\alpha \in (1, 2]$. We code the genealogy by two different processes: the contour process and the height process that Le Gall and Le Jan recently introduced [21, 22]. We show that the rescaled height process of the corresponding Galton–Watson family tree, with one ancestor and conditioned on the total progeny, converges in a functional sense, to a new process: the normalized excursion of the continuous height process associated with the α -stable CSBP. We deduce from this convergence an analogous limit theorem for the contour process. In the Brownian case $\alpha = 2$, the limiting process is the normalized Brownian excursion that codes the continuum random tree: the result is due to Aldous who used a different method.

1. Introduction. The analogues in continuous time of the Galton–Watson branching processes (G–W processes) are the continuous-state branching processes (CSBP). This class of Markov processes was originally introduced by Jirina and Lamperti (see [15, 17]). These processes are the only possible weak limits that can be obtained from sequences of rescaled G–W processes (see [18] or [19]). The properties of CSBP have been extensively studied (see Grey [12] or Bingham [5]). Lamperti has shown that a general CSBP can be obtained from a Lévy process without negative jump by a random time change. The Laplace exponent ψ of the Lévy process is called the branching mechanism of the CSBP and it characterizes its law via a differential equation solved by the Laplace exponent of the process.

When one considers sequences of rescaled G–W processes with some fixed offspring distribution μ on \mathbb{N} , the possible limit processes are the CSBP with stable branching mechanism, that is, $\psi(\lambda) = c\lambda^\alpha$, for some positive c and α in $(0, 2]$ (see [19]). In the case $\psi(\lambda) = c\lambda^2$, the corresponding CSBP is the Feller diffusion.

In this work we use some recent results concerning the genealogical structure of CSBP that can be found in [21, 22, 11]. Our basic object is the G–W tree with offspring distribution μ that can be seen as the underlying family tree of the corresponding G–W process started with one ancestor; this random tree is chosen

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to be rooted and ordered (see Neveu [24] for a rigorous definition). If μ is critical or subcritical, the G–W tree is almost surely finite and it can be coded by two different discrete processes: the contour process and the height process that are both defined at the beginning of Section 2. These two processes are not Markovian in general but they can be written as a functional of a certain left-continuous random walk whose jump distribution depends on μ in a simple way.

If a sequence of rescaled G–W processes converges to a CSBP with branching mechanism ψ , then it has been shown in [11], Chapter 2, that the genealogical structure of the G–W processes converges too. More precisely, the corresponding rescaled sequences of contour processes and height processes, converge respectively to $(H_{t/2})_{t \geq 0}$ and $(H_t)_{t \geq 0}$, where the limit process $(H_t)_{t \geq 0}$ is the height process in continuous time that has been introduced by Le Gall and Le Jan in [21]. As in the discrete case, the height process is not Markovian in general but it can be written as a functional of the Lévy process without negative jump, with Laplace exponent ψ , that plays the role of the left-continuous random walk.

The case of a height process corresponding to a Lévy process with finite variation paths is treated in [21]. It has an interpretation in terms of queuing processes that has been used in some recent work of Limic (see [23]). In the present article, we are only dealing with the case of the α -stable branching mechanism, with α in $(1, 2]$. In that case the CSBP is conservative and becomes extinct almost surely. A general theorem implies that the corresponding height process is continuous (see Theorem 4.7. in [21] or [11], Chapter 1) and the convergence of the rescaled discrete height processes holds in a functional sense. Furthermore, as explained in Section 3, the height process has a scaling property of index $\alpha/(\alpha - 1)$. In the Brownian case $\alpha = 2$, the height process is proportional to a reflected standard Brownian motion.

In [1, 2], Aldous introduced the continuum random tree as the limit of rescaled G–W trees conditioned on the total progeny, in the case where the offspring distribution has finite variance. The continuum random tree is coded by a normalized Brownian excursion, in a way similar to our coding of discrete trees through the height process. In the present work, we aim to extend Aldous' result to G–W trees with possibly infinite variance offspring distribution. More precisely, we assume that the offspring distribution of the G–W tree belongs to the domain of attraction of a stable law with index α in $(1, 2]$. We then show that the (suitably rescaled) discrete height process of the G–W tree conditioned to have a large fixed progeny, converges in a functional sense to the normalized excursion of the height process associated with the α -stable CSBP. This is the main result of the present work and it is stated at the end of Section 3. We can think of our limiting process as the height process of an infinite tree: by analogy, we call it the α -stable continuum random tree. In the case $\alpha = 2$, it coincides with Aldous' continuum random tree. At the end of the Section 3, we also recall from [11], Chapter 3, the computation of finite-dimensional marginals of the α -stable continuum random tree.

The last section is devoted to the proof of the limit theorem. Our approach relies on an idea used by Kersting who introduced discrete bridges in [16] to study the convergence of rescaled G–W processes conditioned on the total progeny, in the case of an infinite variance offspring distribution. The limiting procedure is made easier in terms of discrete bridges thanks to their good properties of absolute continuity with respect to the law of the unconditioned random walk. In Section 4.1, we show that the height process of the G–W tree conditioned on its total progeny has the same law as a certain functional of the discrete bridge. In the next section, we state a similar result in the continuous setting. Then, we pass to the limit on functionals of discrete bridges. The identification of the limit process as the normalized excursion of the continuous height process involves several arguments that depend on continuity properties of the Vervaat transform (see [27]) and on certain path-decompositions of the α -stable Lévy bridge that are due to Chaumont.

2. The coding of discrete Galton–Watson trees. In this section, we introduce the contour process and the height process of a Galton–Watson tree with a critical or subcritical offspring distribution. Each of these processes provides a coding of the tree. The height process can be written as a simple functional of a left-continuous random walk. This observation explains the definition of the continuous height process, that is given in a forthcoming section. The results of this section are elementary and we refer to [21, 11] for details.

The trees considered in the present article are rooted ordered trees. Let us define them formally. We set $\mathbb{N}^* = \{1, 2, \dots\}$ and

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. U is the set of all possible words that can be written with the elements of \mathbb{N}^* . An element u of $(\mathbb{N}^*)^n$ is written $u = u_1 \dots u_n$, and we set $|u| = n$. If $u = u_1 \dots u_m$ and $v = v_1 \dots v_n$ belong to U , we write $uv = u_1 \dots u_m v_1 \dots v_n$ for the concatenation of u and v . In particular $u\emptyset = \emptyset u = u$. We write $u < v$ for the lexicographical order on U : $\emptyset < 1 < 11 < 12 < 121$ for example.

A rooted ordered tree τ is a subset of U such that:

- (i) $\emptyset \in \tau$.
- (ii) If $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}^*$, then $u \in \tau$.
- (iii) For every $u \in \tau$, there exists a number $k_u(\tau) \geq 0$ such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

We denote by \mathbf{T} the set of all trees. In the remainder, we see each vertex of a tree τ as an individual of some population whose τ is the family tree and we

shall often use a nonstandard “genealogical” terminology rather than the graph-theoretical one: for example, the individual \emptyset is called the ancestor of τ .

Let us set some notation. Let τ_1, \dots, τ_k be k trees, the concatenation of τ_1, \dots, τ_k , denoted by $[\tau_1, \dots, \tau_k]$, is defined in the following way: For $n \geq 1$, $u = u_1 u_2 \dots u_n$ belongs to $[\tau_1, \dots, \tau_k]$ if and only if $1 \leq u_1 \leq k$ and $u_2 \dots u_n$ belongs to τ_{u_1} .

A leaf of the tree τ is an individual u of τ that has no child, as-to-say $k_u(\tau) = 0$. We denote by \mathcal{L}_τ the set of all leaves of τ . If τ is a tree and $u \in \tau$, we define the shift of τ at u by $T_u \tau = \{v \in U, uv \in \tau\}$. Note that $T_u \tau \in \mathbf{T}$. We denote by $\zeta(\tau) = \text{Card}(\tau)$ the total progeny of τ . We write $u \preceq v$ if $v = uw$ for some w in U (\preceq is the “genealogical” order on τ). If $u \neq \emptyset$, we use the notation \overleftarrow{u} for the immediate predecessor of u with respect to \preceq , that can be seen as the “father” of u (thus $u = \overleftarrow{u} j$ for some positive integer j). We also denote by $u \wedge v$ the youngest common ancestor of u and v :

$$u \wedge v = \sup\{w \in \tau : w \preceq u \text{ and } w \preceq v\},$$

where the supremum is taken for the genealogical order.

We now introduce the height process associated with a finite tree τ . Let us denote by $u(0) = \emptyset < u(1) < u(2) < \dots < u(\zeta(\tau) - 1)$ the individuals of τ listed in lexicographical order. The height process $H(\tau) = (H_n(\tau); 0 \leq n < \zeta(\tau))$ is defined by

$$H_n(\tau) = |u(n)|, \quad 0 \leq n < \zeta(\tau).$$

The height process is thus the sequence of generations of the individuals of τ visited in lexicographical order. It is easy to check that $H(\tau)$ fully characterizes the tree.

We also define the contour process associated with a tree τ . We see τ embedded in the oriented half-plane. We suppose that the edges of τ have length one. Let us think of a particle visiting continuously each edge of τ at speed one, from the left to the right: after having reached $u(n)$, the particle goes to the individual $u(n + 1)$, taking the shortest way that consists first to move backward on the line of descent from $u(n)$ to $u(n) \wedge u(n + 1)$ and then, to move forward along the single edge between $u(n) \wedge u(n + 1)$ to $u(n + 1)$. The value $C_t(\tau)$ of the contour process at time t is the distance from the root to the position of the particle at time t . See Figure 1 for an example.

More formally, we denote by $l_1 < l_2 < \dots < l_p$ the p leaves of τ listed in lexicographical order. The contour process $(C_t(\tau); t \in \mathbb{R}_+)$ is the piecewise linear continuous path with slope equal to $+1$ or -1 , that takes successive local extremes with values: $0, |l_1|, |l_1 \wedge l_2|, |l_2|, \dots, |l_{p-1} \wedge l_p|, |l_p|$ and 0 . Observe that the contour process visits each edge of τ exactly two times. The contour process can be recovered from the height process through the following transform. First,

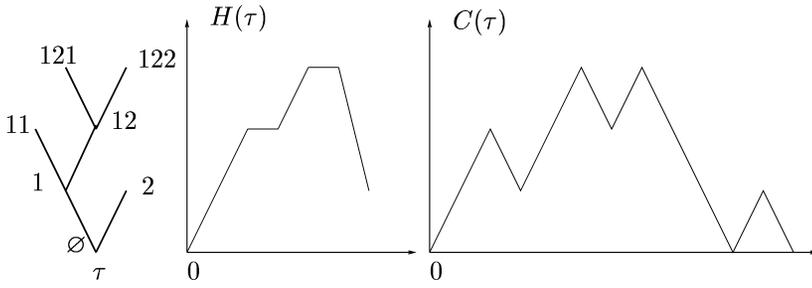


FIG. 1. Height process and contour process.

set $b_n = 2n - H_n(\tau)$, for $0 \leq n < \zeta(\tau)$ and $b_{\zeta(\tau)} = 2(\zeta(\tau) - 1)$. Then, observe that

$$0 = b_0 < b_1 < \dots < b_{\zeta(\tau)-1} < b_{\zeta(\tau)} = 2(\zeta(\tau) - 1).$$

For $n < \zeta(\tau) - 1$ and t in $[b_n, b_{n+1})$,

$$(1) \quad C_t(\tau) = \begin{cases} H_n(\tau) - (t - b_n), & \text{if } t \in [b_n, b_{n+1} - 1), \\ t - b_{n+1} + H_{n+1}(\tau), & \text{if } t \in [b_{n+1} - 1, b_{n+1}), \end{cases}$$

and

$$C_t(\tau) = H_{\zeta(\tau)-1}(\tau) - (t - b_{\zeta(\tau)-1}) \quad \text{if } t \in [b_{\zeta(\tau)-1}, b_{\zeta(\tau)}).$$

We can consider still another function coding τ , which is denoted by $(W_n(\tau); 0 \leq n < \zeta(\tau))$ and defined by $W_0(\tau) = 0$ and

$$W_{n+1}(\tau) - W_n(\tau) = k_{u(n)}(\tau) - 1, \quad 0 \leq n < \zeta(\tau).$$

Observe that the jumps of $W(\tau)$ are not smaller than -1 . The height process can be deduced from $W(\tau)$ by the following formula (see Corollary 2.2. of [21]):

$$(2) \quad H_n(\tau) = \text{Card} \left\{ 0 \leq j < n : W_j(\tau) = \inf_{j \leq k \leq n} W_k(\tau) \right\}, \quad 0 \leq n < \zeta(\tau).$$

As we will see in the next section, the continuous height process is defined by analogy with this formula.

We now extend the definition of the height process, the contour process and the path W to a forest (i.e., a sequence) of finite trees: let $\varphi = (\tau_p)_{p \geq 1}$ be such a forest and set $n_p = \zeta(\tau_1) + \dots + \zeta(\tau_p)$ with $n_0 = 0$. For any $p \geq 1$, we define

$$\begin{aligned} H_{n_p+k}(\varphi) &= H_k(\tau_{p+1}), & 0 \leq k < \zeta(\tau_{p+1}), \\ W_{n_p+k}(\varphi) &= W_k(\tau_{p+1}) - p, & 0 \leq k < \zeta(\tau_{p+1}), \end{aligned}$$

and

$$C_{t+2n_p-2p}(\varphi) = C_t(\tau_{p+1}), \quad t \in [0, 2(\zeta(\tau_{p+1}) - 1)).$$

Observe that $\{n_p, p \geq 0\}$ is the set of integers k such that $H_k(\varphi) = 0$ or equivalently $W_k(\varphi) < \inf_{0 \leq j < k} W_j(\varphi)$. Consequently, the excursions of $(H_n(\varphi); n \geq 0)$ above 0 [resp. the excursions of $(W_n(\varphi); n \geq 0)$ between the successive times of decrease of its infimum] are the $(H_{n_p+k}(\varphi); 0 \leq k < \zeta(\tau_{p+1}))$ [resp. the $(W_{n_p+k}(\varphi) - p; 0 \leq k < \zeta(\tau_{p+1}))$]. To the p th tree of φ corresponds the p th excursion of above level zero of $H(\varphi)$ and this excursion coincides with its height process.

REMARK 2.1. In particular, this implies that (2) still holds when $H(\tau)$ and $W(\tau)$ are replaced by $H(\varphi)$ and $W(\varphi)$.

Let μ be a probability measure on \mathbb{N} . We assume that μ is critical or subcritical:

$$\sum_{k=1}^{\infty} k\mu(k) \leq 1$$

and in order to avoid trivialities, we assume $\mu(1) < 1$. The law of the Galton–Watson tree with offspring distribution μ is the unique probability measure P_μ on \mathbf{T} such that:

- (i) $P_\mu(k_\emptyset = j) = \mu(j), j \in \mathbb{N}$.
- (ii) For every $j \geq 1$ with $\mu(j) > 0$, the shifted trees $T_1\tau, \dots, T_j\tau$ are independent under the conditional probability $P_\mu(\cdot | k_\emptyset = j)$ and their conditional distribution is P_μ .

Let $\varphi = (\tau_p)_{p \geq 0}$ be an i.i.d. sequence of G–W trees with offspring distribution μ . In general, neither $H(\varphi)$ nor $C(\varphi)$ are Markovian. But it is easy to see that $W(\varphi)$ is a random walk started at zero; its jump distribution is $\nu(k) = \mu(k + 1), k \in \{-1, 0, 1, 2, \dots\}$. This property and (2) imply the following proposition.

PROPOSITION 2.1. *Let μ be a critical or subcritical offspring distribution. Let $(W_n; n \geq 0)$ be a random walk started at 0 with jump distribution $\nu(k) = \mu(k + 1), k \in \{-1, 0, 1, \dots\}$, defined under the probability measure \mathbf{P} . Let us set $\zeta = \inf\{n \geq 0: W_n = -1\}$. Define the process $(H_n; n \geq 0)$ by*

$$H_n = \text{Card} \left\{ 0 \leq j < n : W_j = \inf_{j \leq l \leq n} W_l \right\}.$$

Let $n \geq 1$ be such that $\mathbf{P}(\zeta = n) > 0$. The law of the process $(H_n; 0 \leq n < \zeta)$ under $\mathbf{P}(\cdot | \zeta = n)$ is the same as the law of $H(\tau)$ under $P_\mu(\cdot | \zeta(\tau) = n)$.

REMARK 2.2. The law of the G–W tree with a geometric offspring distribution conditioned to have its total progeny equal to n , is the uniform probability measure on the set of all ordered rooted trees with n vertices.

3. The α -stable continuum random tree.

3.1. *The height process.* We define the height process in continuous time by analogy with (2). The role of the left-continuous random walk is played by a stable Lévy process without negative jump. In this section, we use several results about stable Lévy processes and we refer to [4], Chapter VIII, or to the original work of Chaumont [7, 8] for further details.

Let us denote by $(\Omega, \mathcal{F}, \mathbf{P})$ the underlying probability space. Let X be a process with paths in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, the space of right-continuous with left limit (càdlàg) real-valued functions, endowed with the Skorokhod topology. We denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by X and augmented with the \mathbf{P} -null sets. We assume that X is a stable Lévy process without negative jump with index $\alpha \in (1, 2]$. Then we have

$$\mathbf{E}[\exp(-\lambda X_t)] = \exp(-c\lambda^\alpha), \quad \lambda > 0,$$

for some positive constant c . The process $(k^{-1/\alpha} X_{kt}; t \geq 0)$ has the same law as $(X_t; t \geq 0)$. Thanks to this scaling property, we can take $c = 1$, without loss of generality in our purpose. When $\alpha = 2$, the process X is $1/\sqrt{2}$ times the standard Brownian motion on the line. When $1 < \alpha < 2$, the Lévy measure of X is

$$\pi(dr) = \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} r^{-\alpha-1} dr.$$

We use the following notation: for any $s < t$, we set

$$I_{s,t} = \inf_{[s,t]} X, \quad I_t = \inf_{[0,t]} X \quad \text{and} \quad S_t = \sup_{[0,t]} X.$$

Let us fix $t > 0$. By analogy with the discrete case, we want to define the height H_t as the “measure” of the set

$$(3) \quad \left\{ s \leq t, X_s = \inf_{s \leq r \leq t} X_r \right\}.$$

To give a meaning to the word “measure,” we use a time-reversal argument. Let $\widehat{X}^{(t)}$ be the time-reversed process

$$\begin{aligned} \widehat{X}_s^{(t)} &= X_t - X_{(t-s)-}, & \text{if } 0 \leq s < t, \\ \widehat{X}_t^{(t)} &= X_t. \end{aligned}$$

It is easy to check that $(\widehat{X}_s^{(t)}, 0 \leq s \leq t) \stackrel{\text{law}}{=} (X_s, 0 \leq s \leq t)$, that is referred to as the “duality property.” We set $\widehat{S}_s^{(t)} = \sup_{r \in [0,s]} \widehat{X}_r^{(t)}$. Under the transformation $s \rightarrow t - s$, the set (3) corresponds to

$$\{s \leq t, \widehat{X}_s^{(t)} = \widehat{S}_s^{(t)}\},$$

that is the zero set of the process $\widehat{S}^{(t)} - \widehat{X}^{(t)}$ over $[0, t]$. Note that the process $\widehat{S}^{(t)} - \widehat{X}^{(t)}$ has the same distribution as $S - X$. However, the process $S - X$ is a

Markov process. As $\alpha \in (1, 2]$, the point $\{0\}$ is regular for itself with respect to this Markov process. Hence, we can define the local time at 0 of $S - X$ and denote it by $L = (L_t; t \geq 0)$. Note that L is only defined up to a multiplicative constant. Let us specify this normalization: Let L^{-1} denote the right-continuous inverse of L ,

$$L^{-1}(t) = \inf\{s > 0 : L_s > t\}.$$

Both processes $(L^{-1}(t), t \geq 0)$ and $(S_{L^{-1}(t)}, t \geq 0)$ are subordinators, called respectively the ladder time process and the ladder height process. The Laplace exponent of the ladder height process is given by

$$E[\exp(-\lambda S_{L^{-1}(t)})] = \exp(-k\lambda^{\alpha-1}),$$

where the positive constant k depends on the normalization of L (see [4], Theorem VII-4). We fix it by choosing $k = 1$.

REMARK 3.1. Observe that in the Brownian case $\alpha = 2$, we have $L = S$.

If $1 < \alpha < 2$, we recall from [21] the following approximation of L . Let us denote by (g_j, d_j) , $j \in J$, the excursion intervals of $S - X$ above 0. A classical argument of fluctuation theory shows that the point measure

$$\sum_{j \in J} \delta_{(L_{g_j}, \Delta S_{d_j}, \Delta X_{d_j})}(dl dr dx)$$

is a Poisson measure with intensity $dl \pi(dx) \mathbf{1}_{[0,x]}(r) dr$ (see [26] or [4], Chapter VI). Set

$$\beta_\varepsilon = \int_{(\varepsilon, +\infty)} x \pi(dx) = \frac{\alpha}{\Gamma(2 - \alpha) \varepsilon^{\alpha-1}}.$$

By standard arguments, we see that **P**-a.s. for every $t \geq 0$,

$$(4) \quad L_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card}\{s \in [0, t] : S_{s-} < X_s; \Delta X_s > \varepsilon\}.$$

Thanks to this approximation, we can view L_t as a function of $(X_s; 0 \leq s \leq t)$. Then we define the *height process in continuous time*, denoted by H_t , by the formula $H_t = L_t(\widehat{X}^{(t)})$. In the Brownian case, the height process is $H_t = \widehat{S}_t^{(t)} = X_t - I_t$ and obviously has continuous paths. If $1 < \alpha < 2$, general Theorem 4.7 of [21] shows that H admits a continuous modification. Using the Fubini theorem, we deduce from (4) and from the duality property that the limit

$$(5) \quad H_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card}\{u \in [0, t] : X_{u-} < I_{u,t}; \Delta X_u > \varepsilon\}$$

holds **P**-a.s. on a set of values of t of full Lebesgue measure. We deduce from the scaling property of X and from the previous approximation formula that H has a scaling property of index $\frac{\alpha}{\alpha-1}$: For any $k > 0$,

$$(k^{1/\alpha-1} H_{kt}; t \geq 0) \stackrel{\text{law}}{=} (H_t; t \geq 0).$$

3.2. *The normalized excursion of the height process.* Recall that $X - I$ is a strong Markov process and that 0 is regular for $X - I$. We may and will choose $-I$ as the local time of $X - I$ at level 0. Let $(g_i, d_i), i \in \mathcal{I}$, be the excursion intervals of $X - I$ above 0. Let us set

$$\omega_s^i = X_{g_i+s} - X_{g_i}, \quad 0 \leq s \leq \zeta_i = d_i - g_i.$$

The point measure

$$\mathcal{N}(dt d\omega) = \sum_{i \in \mathcal{I}} \delta_{(-I_{g_i}, \omega^i)}$$

is a Poisson measure with intensity $dt N(d\omega)$. Here $N(d\omega)$ is a σ -finite measure on the set of finite paths $(\omega(s); 0 \leq s \leq \zeta(\omega))$. Thanks to (5), we see that H_t only depends on the excursion of $X - I$ straddling t . Thus we can use excursion theory arguments in order to define the height process under the excursion measure N . We can also deduce from (5) that the excursions of H above 0 are almost surely equal to the $H(\omega^i), i \in \mathcal{I}$, with an evident functional notation (see [11], Chapter 1).

We first have to define the normalized excursion of the α -stable Lévy process. Let us simply denote by $\zeta = \zeta(\omega)$ the lifetime of ω under $N(d\omega)$. A standard result of fluctuation theory says that $N(1 - e^{-\lambda\zeta}) = \lambda^{1/\alpha}$ (see [4]). Thus we have

$$N(\zeta > t) = \frac{1}{\Gamma(1 - 1/\alpha)} t^{-1/\alpha}.$$

Define for any $\lambda > 0$ the functional $S^{(\lambda)}$ by

$$S^{(\lambda)}(\omega) = (\lambda^{1/\alpha} \omega(s/\lambda); 0 \leq s \leq \lambda\zeta(\omega)).$$

Thanks to the scaling property of X , one can show that the image of $N(\cdot|\zeta > t)$ under $S^{(1/\zeta)}$ is the same for every $t > 0$. This law, defined on the càdlàg paths with unit lifetime, is the law of the *normalized excursion of X* denoted by \mathbf{P}^{exc} . Informally \mathbf{P}^{exc} can be seen as $N(\cdot|\zeta = 1)$ (see [4], Chapter VIII). We may assume that there exists a process X^{exc} defined on $(\Omega, \mathcal{F}, \mathbf{P})$ that takes values in $\mathbb{D}([0, 1], \mathbb{R}_+)$ and whose law under \mathbf{P} is \mathbf{P}^{exc} .

We recall Chaumont's path-construction of the normalized excursion of X (see [7, 8] or [4], Chapter VIII): let $(\underline{g}_1, \underline{d}_1)$ be the excursion interval of $X - I$ straddling 1:

$$\underline{g}_1 = \sup\{s \leq 1 : X_s = I_s\},$$

$$\underline{d}_1 = \inf\{s > 1 : X_s = I_s\}.$$

We define $\zeta_1 = \underline{d}_1 - \underline{g}_1$, the length of this excursion and we set

$$X^* = \left(\zeta_1^{-1/\alpha} (X_{\underline{g}_1 + \zeta_1 s} - X_{\underline{g}_1}); 0 \leq s \leq 1 \right).$$

Then, we have

$$(6) \quad X^{\text{exc}} \stackrel{\text{law}}{=} X^*.$$

Now, let us define the *normalized excursion of the height process*. In the Brownian case $\alpha = 2$, this is the normalized excursion of X . In the case $1 < \alpha < 2$, the approximation (5) and the identity (6) imply that the limit

$$\zeta_1^{1/\alpha-1} H_{\underline{g}_1 + \zeta_1 t} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card} \left\{ u \in [0, t] : X_{u-}^* < \inf_{[u,t]} X^*; \Delta X_u^* > \varepsilon \right\}$$

holds \mathbf{P} -a.s. for a set of values of t of full Lebesgue measure on $[0, 1]$. So there exists a continuous process $(H_t^{\text{exc}}; 0 \leq t \leq 1)$ such that the limit

$$(7) \quad H_t^{\text{exc}} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card} \left\{ u \in [0, t] : X_{u-}^{\text{exc}} < \inf_{[u,t]} X^{\text{exc}}; \Delta X_u^{\text{exc}} > \varepsilon \right\}$$

holds \mathbf{P} -a.s. for a set of values of t of full Lebesgue measure in $[0, 1]$. The process H^{exc} is called the normalized excursion of the height process. Moreover, we have

$$(8) \quad H^{\text{exc}} \stackrel{\text{law}}{=} (\zeta_1^{1/\alpha-1} H_{\underline{g}_1 + \zeta_1 s}; 0 \leq s \leq 1).$$

This result also holds in the Brownian case.

3.3. The limit theorem. In this section, we state a limit theorem for the rescaled discrete contour process and the rescaled discrete height process of a G–W tree conditioned on its total progeny. First we need to introduce some notation and to recall some results that are proved in [11], Chapter 2.

Let μ be a critical or subcritical offspring distribution such that $\mu(1) < 1$ and let $(Z_n^p; n \geq 0)$ be a G–W process with offspring distribution μ , starting with p ancestors: $Z_0^p = p$. We let $\varphi = (\tau_p)_{p \geq 1}$ be a sequence of i.i.d. G–W trees with offspring distribution μ . By convenience, we denote by $(H_n; n \geq 0)$, $(C_t; t \geq 0)$ and $(W_n; n \geq 0)$ the corresponding height process, contour process and random walk associated with φ . As was observed in Section 2, W is a left-continuous random walk with jump distribution ν defined by $\nu(k) = \mu(k + 1)$, $k \in \{-1, 0, 1 \dots\}$.

We assume that ν is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$. The condition $\nu((-\infty, -1)) = 0$, implies that the limit law is spectrally positive. Thus, there exists an increasing sequence of positive real numbers $(a_p)_{p \geq 0}$ such that $a_p \rightarrow \infty$ and

$$(H) \quad \frac{1}{a_p} W_p \xrightarrow{d} X_1$$

where X is a stable Lévy process without negative jump with Laplace exponent $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2]$. Note that we have automatically $a_p/p \rightarrow 0$. Grimvall has shown in [13] that (H) is equivalent to

$$(9) \quad \left(\frac{1}{a_p} Z_{\lfloor (pt/a_p) \rfloor}^p; t \geq 0 \right) \rightarrow (Z_t; t \geq 0),$$

where $(Z_t; t \geq 0)$ is a CSBP with branching mechanism $\psi(\lambda) = \lambda^\alpha$. Here and later, *the convergence in distribution of processes always holds in the functional sense*, that is in the sense of the weak convergence of the probability distributions of the processes in the Skorokhod space where they have their paths (which is meant by the symbol \xrightarrow{d}). We will use the notation \xrightarrow{fd} to indicate weak convergence of finite-dimensional marginals.

Our starting points are Theorems 2.3.2 and 2.4.1 in [11], that we recall in our particular setting: under assumption (H), the following convergences hold:

$$(10) \quad \begin{aligned} \left(\frac{a_p}{p} H_{[pt]}; t \geq 0\right) &\xrightarrow{d} (H_t; t \geq 0), \\ \left(\frac{a_p}{p} C_{pt}; t \geq 0\right) &\xrightarrow{d} (H_{t/2}; t \geq 0), \end{aligned}$$

where $(H_t; t \geq 0)$ stands for the continuous height process associated with X .

As in Section 2, we let τ be a G–W tree with offspring distribution μ , under the probability measure P_μ . To simplify notation, we denote by ζ the total progeny of τ . If μ is assumed to be aperiodic [i.e., $\gcd(k \in \{1, 2, \dots\}; \mu(k) > 0) = 1$], the conditional probability $P_\mu(\cdot | \zeta = p)$ is well defined for $p \geq 1$ sufficiently large. Let $(H_n^{\text{exc},p}; 0 \leq n \leq p)$, $(C_t^{\text{exc},p}; 0 \leq t \leq 2p)$ and $(W_n^{\text{exc},p}; 0 \leq n \leq p)$ be three processes defined on $(\Omega, \mathcal{F}, \mathbf{P})$ such that

$$((H_n^{\text{exc},p}; 0 \leq n < p), (C_t^{\text{exc},p}; 0 \leq t < 2p - 2), (W_n^{\text{exc},p}; 0 \leq n < p))$$

has the same law as $(H(\tau), C(\tau), W(\tau))$ under $P_\mu(\cdot | \zeta = p)$ and such that $H_p^{\text{exc},p} = 0$, $C_t^{\text{exc},p} = 0$ for $t \in [2p - 2, 2p]$ and $W_p^{\text{exc},p} = -1$. Let also H^{exc} be the normalized excursion of the height process H defined in the previous section. The main goal of the present work is to prove the following limit theorem:

THEOREM 3.1. *Assume (H) and that μ is aperiodic. Then, we have*

$$\left(\frac{a_p}{p} H_{[pt]}^{\text{exc},p}; 0 \leq t \leq 1\right) \xrightarrow{d} (H_t^{\text{exc}}; 0 \leq t \leq 1)$$

and

$$\left(\frac{a_p}{p} C_{pt}^{\text{exc},p}; 0 \leq t \leq 2\right) \xrightarrow{d} (H_{t/2}^{\text{exc}}; 0 \leq t \leq 2).$$

REMARK 3.2. Thanks to (1), the second convergence of the theorem follows from the first one: Set $b_n = 2n - H_n^{\text{exc},p}$, $0 \leq n < p$, and $b_p = 2p - 2$. We deduce from (1) that, for $0 \leq n < p$,

$$(11) \quad \sup_{b_n \leq t < b_{n+1}} |C_t^{\text{exc},p} - H_n^{\text{exc},p}| \leq |H_{n+1}^{\text{exc},p} - H_n^{\text{exc},p}| + 1.$$

Define the random function $g_p : [0, 2p] \rightarrow \mathbb{N}$ by setting $g_p(t) = n$, if $t \in [b_n, b_{n+1})$ and $n < p$, and $g_p(t) = p$, if $t \in [2p - 2, 2p]$. The definition of b_n implies

$$\sup_{0 \leq t \leq 2p} \left| g_p(t) - \frac{t}{2} \right| \leq \frac{1}{2} \sup_{0 \leq k \leq p} H_k^{\text{exc}, p} + 1.$$

Set $f_p(t) = g_p(pt)/p$. By (11), we have

$$\sup_{t \in [0, 2]} \frac{a_p}{p} |C_{pt}^{\text{exc}, p} - H_{pf_p(t)}^{\text{exc}, p}| \leq \frac{a_p}{p} + \frac{a_p}{p} \sup_{t \in [0, 1]} |H_{[pt]+1}^{\text{exc}, p} - H_{[pt]}^{\text{exc}, p}|$$

and

$$\sup_{t \in [0, 2]} \left| f_p(t) - \frac{t}{2} \right| \leq \frac{1}{p} + \frac{1}{2a_p} \sup_{t \in [0, 1]} \frac{a_p}{p} H_{[pt]}^{\text{exc}, p}.$$

Assuming that the first convergence of the theorem holds, we have

$$\frac{a_p}{p} + \frac{a_p}{p} \sup_{t \in [0, 1]} |H_{[pt]+1}^{\text{exc}, p} - H_{[pt]}^{\text{exc}, p}| \rightarrow 0$$

and

$$\frac{1}{p} + \frac{1}{2a_p} \sup_{t \in [0, 1]} \frac{a_p}{p} H_{[pt]}^{\text{exc}, p} \rightarrow 0$$

in probability. Thus, the preceding bounds imply

$$\left(\frac{a_p}{p} C_{pt}^{\text{exc}, p}; 0 \leq t \leq 2 \right) \xrightarrow{d} (H_{t/2}^{\text{exc}}; 0 \leq t \leq 2).$$

REMARK 3.3. We denote by $Z^{\text{exc}, p}$ the G–W process started with one ancestor conditioned on having a total progeny equal to p . Under the same assumptions as Theorem 3.1, Kersting has proved in [16] that $(p^{-1} Z_{[pt/a_p]}^{\text{exc}, p}; t \geq 0)$ converges in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ to a process Z^{exc} that is obtained from X^{exc} by the Lamperti time change. Theorem 3.1 can be used to simplify Kersting’s proof. More precisely, it implies Lemma 9 in [16], that is the key-argument showing that the laws of $(p^{-1} Z_{[pt/a_p]}^{\text{exc}, p}; t \geq 0)$ are tight.

REMARK 3.4. If the offspring distribution has a finite variance, then, $\alpha = 2$, H^{exc} is proportional to the normalized Brownian excursion and Theorem 3.1 is due to Aldous with a very different proof (see [2]). Let us mention that Bennies and Kersting proved a weaker version of Aldous’ theorem using a method closer to our (see [3]).

The convergence of Theorem 3.1 suggests that H^{exc} is the height process of a “continuous tree.” By analogy with Aldous’ continuum random tree, we call the limiting tree the α -stable continuum random tree that can be defined as a random

compact metric space in the following way: Each $s \in [0, 1]$ corresponds to a vertex at height H_s^{exc} in the α -stable continuum random tree. Let $t \in [0, 1]$. The distance in α -stable continuum random tree between the two vertices corresponding to s and t must be equal to

$$d(s, t) = H_t^{\text{exc}} + H_s^{\text{exc}} - 2 \inf_{u \in [\min(s,t), \max(s,t)]} H_u^{\text{exc}}.$$

Then, we say the s and t are equivalent if and only if $d(s, t) = 0$ and we denote it by $s \sim t$. We set $\mathcal{T} = [0, 1] / \sim$ and we define the α -stable continuum random tree as the (random) compact metric space (\mathcal{T}, d) . For a general theory, we refer to [10, 9].

For any $s \in [0, 1]$ we denote by \tilde{s} the corresponding vertex in \mathcal{T} ; by analogy with the discrete case, we call $\tilde{0}$ the root. The order on \mathcal{T} induced by the order on $[0, 1]$ is the continuous analogue of the lexicographical order on discrete ordered trees. We can also define a ‘‘genealogical’’ order \preceq on \mathcal{T} : Let $\sigma, \sigma' \in \mathcal{T}$. Then we say that

$$\sigma \preceq \sigma' \quad \text{iff } d(\sigma, \sigma') = d(\tilde{0}, \sigma') - d(\tilde{0}, \sigma).$$

The set of leaves is the set of vertices that are maximal with respect to \preceq . We denote it by \mathcal{L} . Here we give some properties of \mathcal{T} without proof (more general results are to be given in a forthcoming paper):

- **P**-a.s. the Lebesgue measure of $\{s \in [0, 1] : \tilde{s} \in \mathcal{L}\}$ is 1;
- **P**-a.s. the Hausdorff and packing dimensions of (\mathcal{T}, d) are both equal to $\frac{\alpha}{\alpha-1}$;
- **P**-a.s. the Hausdorff and packing dimensions of $\mathcal{T} \setminus \mathcal{L}$ are both equal to 1.

Following Aldous [1, 2], we can define the finite-dimensional marginals of \mathcal{T} . Let us say a word about it: Aldous’ first construction of the (2-stable) continuum random tree was based on explicit formulas for the finite-dimensional marginals of this random tree. Later, Aldous identified the continuum random tree as the tree coded by a normalized Brownian excursion, in the sense of [2]. Le Gall [20] provided a derivation of the finite-dimensional marginals from properties of Brownian excursions. A similar approach has been used in [11] to get the finite-dimensional marginals of the α -stable continuum random tree. For sake of completeness let us explain how Theorem 3.1 provides asymptotics for the finite-dimensional marginals of the G–W tree conditioned on its total progeny.

Let τ be distributed under $P_\mu(\cdot | \zeta(\tau) = p)$ and fix $k \leq p$. Let (v_1, \dots, v_k) be a k -uple of distinct vertices of τ . Aldous has defined (Section 2 of [2]) the k th marginal of τ as the reduced subtree at $\{v_1, \dots, v_k\}$ that is the (graph-theoretical) tree whose set of vertices V is $\{v_i \wedge v_j; 0 \leq i \leq j \leq k\} \cup \{\emptyset\}$ and whose edges are all (u, v) , for u and v distinct in V such that $u \preceq w \preceq v$ or $v \preceq w \preceq u$ occurs for $w \in V$ iff $w = u$ or $w = v$; furthermore, the length of the edge (u, v) is $||u| - |v||$. Let us explain how the k th marginal can be recovered from the height process of τ .

First we need to define what is a marked tree is: A marked tree is a pair $\theta = (\tau, \{h_v, v \in \tau\})$, where $\tau \in \mathbf{T}$ and $h_v \geq 0$ for every $v \in \tau$. The number h_v is interpreted as the lifetime of individual v and τ is called the skeleton of θ . Let $\theta_1 = (\tau_1, \{h_v^1, v \in \tau_1\}), \dots, \theta_k = (\tau_k, \{h_v^k, v \in \tau_k\})$ be k marked trees and $h \geq 0$. The concatenation of $[\theta_1, \dots, \theta_k]_h$ is the marked tree whose skeleton is $[\tau_1, \dots, \tau_k]$ and such that the lifetimes of vertices in $\tau_i, 1 \leq i \leq k$ become the lifetimes of the corresponding vertices in $[\tau_1, \dots, \tau_k]$, and finally the lifetime of \emptyset in $[\tau_1, \dots, \tau_k]$ is h .

Assume that $k < p$ and let us explain how we deduce the k th marginals of τ under $P_\mu(\cdot | \zeta(\tau) = p)$ from $H^{\text{exc}, p}$. Let $\omega : [a, b] \rightarrow [0, +\infty)$ be a càdlàg function defined on the subinterval $[a, b]$ of $[0, +\infty)$. Let $t_1, t_2, \dots, t_k \in [0, +\infty)$ be such that $a \leq t_1 \leq \dots \leq t_k \leq b$. We first give the definition of the marked tree associated to ω and t_1, \dots, t_k . For every $a \leq u \leq v \leq b$, we set

$$m(u, v) = \inf_{u \leq t \leq v} \omega(t).$$

We will now construct a marked tree

$$\theta(\omega, t_1, \dots, t_k) = (\tau(\omega, t_1, \dots, t_k), \{h_v(\omega, t_1, \dots, t_k), v \in \tau\})$$

associated with the function ω and the instants t_1, \dots, t_k . We proceed by induction on k . If $k = 1$, $\tau(\omega, t_1) = \{\emptyset\}$ and $h_{\emptyset}(\omega, t_1) = \omega(t_1)$.

Let $k \geq 2$ and suppose that the tree has been constructed up to order $k - 1$. Then there exists an integer $l \in \{1, \dots, k - 1\}$ and l integers $1 \leq i_1 < i_2 < \dots < i_l \leq k - 1$ such that $m(t_i, t_{i+1}) = m(t_1, t_k)$ if and only if $i \in \{i_1, i_2, \dots, i_l\}$. For every $q \in \{0, 1, \dots, l\}$, define ω^q by the formulas

$$\begin{aligned} \omega^0(t) &= \omega(t) - m(t_1, t_k), & t \in [a, t_{i_1}], \\ \omega^q(t) &= \omega(t) - m(t_1, t_k), & t \in [t_{i_q+1}, t_{i_{q+1}}], \\ \omega^l(t) &= \omega(t) - m(t_1, t_k), & t \in [t_{i_l+1}, b]. \end{aligned}$$

We then set

$$\begin{aligned} &\theta(\omega, t_1, \dots, t_k) \\ &= [\theta(\omega^0, t_1, \dots, t_{i_1}), \theta(\omega^1, t_{i_1+1}, \dots, t_{i_2}), \dots, \theta(\omega^l, t_{i_l+1}, \dots, t_k)]_{m(t_1, t_k)}. \end{aligned}$$

This completes the construction of the tree by induction. Note that $l + 1$ is the number of children of \emptyset in $\theta(\omega, t_1, \dots, t_k)$ and $m(t_1, t_k)$ is its lifetime. Figure 2 gives an example of a tree $\theta(\omega, t_1, \dots, t_k)$ when $k = 4$ and $[a, b] = [0, 1]$.

Let $(U_1^p, U_2^p, \dots, U_k^p)$ be independent of $H^{\text{exc}, p}$ and uniformly distributed on the set of all (n_1, n_2, \dots, n_k) with $0 \leq n_1 < n_2 < \dots < n_k \leq p - 1$. From our construction of the height process, it should be clear that the k th marginal under $P_\mu(\cdot | \zeta = p)$ is close to the tree $\theta((H_{[pt]}^{\text{exc}, p})_{0 \leq t \leq 1}, U_1^p, \dots, U_k^p)$ (in a sense that we do not make precise, but the reader can easily convince himself that both trees have the same scaling limits when $p \rightarrow \infty$). On the other hand, Theorem 3.1 implies

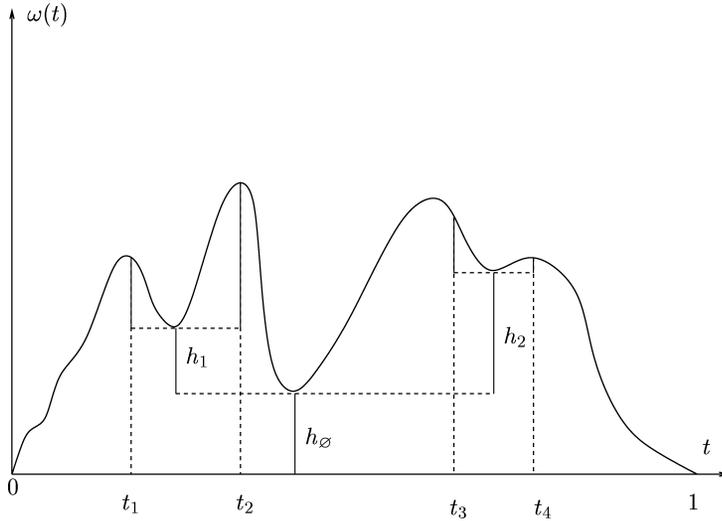


FIG. 2. *Reduced tree.*

that the rescaled trees $\theta((\frac{ap}{p} H_{[pt]}^{\text{exc}, p})_{0 \leq t \leq 1}, \frac{U_1^p}{p}, \dots, \frac{U_k^p}{p})$ converges in distribution to $\theta(H^{\text{exc}}, U_1, \dots, U_k)$, where the k -tuple (U_1, \dots, U_k) is independent of H^{exc} and distributed according to the measure

$$k! \mathbf{1}_{\{0 \leq u_1 < u_2 < \dots < u_k \leq 1\}} du_1 \cdots du_k.$$

We define $\theta(H^{\text{exc}}, U_1, \dots, U_k)$ as the k th marginal of \mathcal{T} . The following theorem gives the law $\theta(H^{\text{exc}}, U_1, \dots, U_k)$:

THEOREM 3.2 (Theorem 3.3.3 of [11]). *The law of $\theta(H^{\text{exc}}, U_1, \dots, U_k)$ is characterized by the following properties:*

(i) *The probability of a given skeleton $\tau \in \{\theta \in \mathbf{T} : |\mathcal{L}_\theta| = k \text{ and } k_u(\theta) \neq 1, u \in \theta\}$ is*

$$\frac{k!}{\prod_{v \in \tau \setminus \mathcal{L}_\tau} k_v(\tau)!} \frac{\prod_{v \in \tau \setminus \mathcal{L}_\tau} |(1 - \alpha)(2 - \alpha)(3 - \alpha) \cdots (k_v(\tau) - 1 - \alpha)|}{|(\alpha - 1)(2\alpha - 1) \cdots ((k - 1)\alpha - 1)|}.$$

(ii) *Conditionally on the skeleton τ , the marks $(h_v)_{v \in \tau}$ have a density with respect to the Lebesgue measure on \mathbb{R}_+^τ given by*

$$\frac{\Gamma(k - \frac{1}{\alpha})}{\Gamma(\delta_\tau)} \alpha^{|\tau|} \int_0^1 du u^{\delta_\tau - 1} q\left(\alpha \sum_{v \in \tau} h_v, 1 - u\right),$$

where $\delta_\tau = k - (1 - \frac{1}{\alpha})|\tau| - \frac{1}{\alpha} > 0$, and $q(s, u)$ is the continuous density at time s of the stable subordinator with index $1 - \frac{1}{\alpha}$, that is characterized by

$$\int_0^{+\infty} du e^{-\lambda s} q(s, u) = \exp(-s \lambda^{1 - (1/\alpha)}).$$

REMARK 3.5. In particular the skeleton of $\theta(H^{\text{exc}}, U_1, U_2, U_3)$ is equal to the discrete tree $\{\emptyset, 1, 2, 3\}$ with probability $\frac{2-\alpha}{2\alpha-1}$. Consequently, \mathcal{T} has branching points of order greater than 2 if $\alpha < 2$. General arguments (see [11], Chapter 1) imply that \mathcal{T} has an infinite number of infinitely branching vertices.

4. Proof of the main theorem. The proof of Theorem 3.1 use Chaumont’s result on the Vervaat transform of the bridge of a α -stable Lévy process (see [8] or [4], Chapter VIII). In this section, we explain how the normalized excursion of the height process is connected (through the Vervaat transform) to the height process associated with the bridge of the Lévy process. Before that, we need to establish some properties in the discrete setting. This is the purpose of the following subsection.

4.1. *The discrete bridge.* Let us start with some notation. We denote by Ω_0 the set of all discrete-time finite paths in \mathbb{Z} :

$$\Omega_0 = \bigcup_{n \geq 0} \mathbb{Z}^{\{0,1,\dots,n\}}.$$

If w is in $\mathbb{Z}^{\{0,1,\dots,n\}}$, we denote by $z(w) = n$ its lifetime. Let w be such that $z(w) \geq n$. We denote by $w^{(n)}$ and \widehat{w}^n , respectively, the shifted path and the time and space reversed path at time n :

$$w^{(n)}(k) = w(k + n) - w(n), \quad 0 \leq k \leq z(w) - n,$$

and

$$\widehat{w}^n(k) = w(n) - w(n - k), \quad 0 \leq k \leq n.$$

We set

$$L_n(w) = \text{Card} \left\{ 0 < j \leq n : w(j) = \sup_{0 \leq k \leq j} w(k) \right\}.$$

We also define

$$H_n(w) = L_n(\widehat{w}^n) = \text{Card} \left\{ 0 \leq j < n : w(j) = \inf_{j \leq k \leq n} w(k) \right\}.$$

For any integer a , we define $t(a, w)$ by

$$t(a, w) = \inf \{ k \in [0, z(w)] : w(k) \geq a \}$$

(with the convention $\inf \emptyset = +\infty$). A careful counting leads to the following formulas, valid for any $0 \leq m \leq z(w) - n$:

$$(12) \quad \begin{aligned} H_{n+m}(w) - \inf_{n \leq k \leq n+m} H_k(w) &= H_m(w^{(n)}), \\ \inf_{n \leq k \leq n+m} H_k(w) &= L_n(\widehat{w}^n) - L_{\beta(n,m)}(\widehat{w}^n), \end{aligned}$$

where

$$\beta(n, m) = \begin{cases} 0, & \text{if } \inf_{0 \leq k \leq m} w^{(n)}(k) \geq 0, \\ t\left(-\inf_{0 \leq k \leq m} w^{(n)}(k), \widehat{w}^n\right) - 1, & \text{if } -\sup_{0 \leq k \leq n} \widehat{w}^n(k) \leq \inf_{0 \leq k \leq m} w^{(n)}(k) < 0, \\ n, & \text{if } \inf_{0 \leq k \leq m} w^{(n)}(k) < -\sup_{0 \leq k \leq n} \widehat{w}^n(k). \end{cases}$$

Shortly written, we have

$$\beta(n, m) = n \wedge \left(t\left(-\inf_{0 \leq k \leq m} w^{(n)}(k), \widehat{w}^n\right) - 1 \right)_+.$$

We also set

$$G(w) = \inf \left\{ 0 \leq k \leq z(w) : w(k) = \inf_{0 \leq j \leq z(w)} w(j) \right\}.$$

We now define the Vervaat transform $V_0 : \Omega_0 \rightarrow \Omega_0$ by

$$V_0(w)(k) = \begin{cases} w(k + G(w)) - \inf w, & \text{if } 0 \leq k \leq z(w) - G(w), \\ w(k + G(w) - z(w)) + w(z(w)) - \inf w - w(0), & \text{if } z(w) - G(w) \leq k \leq z(w). \end{cases}$$

Observe that the path $V_0(w)$ starts at 0 and that its lifetime is $z(w)$.

Let us consider the random walk W whose jump distribution is given by $\nu(k) = \mu(k + 1)$, $k \in \{-1, 0, 1, \dots\}$. Recall from Section 2 that $\zeta = \inf\{k \geq 0 : W_k = -1\}$. Set for any positive integer p , $G_p = G((W_k)_{0 \leq k \leq p})$. A well-known result states that for any positive integer p , $p\mathbf{P}(\zeta = p) = \mathbf{P}(W_p = -1)$ (see [25]). In the remainder, we assume that $\mathbf{P}(W_p = -1) > 0$. Then, $\mathbf{P}(\zeta = p) > 0$. We recall the classical result on random walk that is due to Vervaat (see [27]):

$$(13) \quad V_0(W) \text{ under } \mathbf{P}(\cdot | W_p = -1) \stackrel{\text{law}}{=} W \text{ under } \mathbf{P}(\cdot | \zeta = p).$$

This identity connects the discrete bridge of length p with the excursion conditioned to last p . We want to establish a similar identity for the height process. To this end, we introduce the process $M = (M_k)_{0 \leq k \leq p}$ that is defined by the formula

$$M_k = L_p(\widehat{W}^p) - L_{\gamma_p(k)}(\widehat{W}^p),$$

where we have set $\gamma_p(k) = p \wedge (t(-\inf_{0 \leq i \leq k} W_i, \widehat{W}^p) - 1)_+$.

Let us explain the intuition behind M_k : Consider τ under $P(\cdot | \zeta = p)$. Let $\emptyset = u_0 < u_1 < \dots < u_{p-1}$ be the vertices of τ lexicographically ordered. Pick N

at random in $\{0, 1, \dots, p - 1\}$ and assume that N is independent of τ . We denote by $N(k)$ the integer of $\{0, 1, \dots, p - 1\}$ equal to $N + k$ modulo p . Then M_k has the same law as $|u_N \wedge u_{N(k)}|$ that is the height of the common ancestor of u_N and $u_{N(k)}$. We have the following proposition.

PROPOSITION 4.1. *The law of the process $(V_0(W), V_0(H(W) + M))$ under $\mathbf{P}(\cdot | W_p = -1)$ is the same as that of $(W, H(W))$ under $\mathbf{P}(\cdot | \zeta = p)$.*

PROOF. Set $W' = V_0(W)$. Thanks to (13), it is sufficient to prove that $\mathbf{P}(\cdot | W_p = -1)$ -a.s. $H(W') = V_0(H(W) + M)$. Let $0 \leq l \leq G_p$. Applying (12) with $w = (W'_k; 0 \leq k \leq p)$, $n = p - G_p$ and $m = l$, we get

$$H_{p-G_p+l}(W') = H_l(W'^{(p-G_p)}) + L_{p-G_p}(\widehat{W}'^{p-G_p}) - L_{\beta'(p-G_p,l)}(\widehat{W}'^{p-G_p})$$

where $\beta'(p - G_p, l) = \beta(p - G_p, l)(W')$. However,

$$W'^{(p-G_p)} = (W_k)_{0 \leq k \leq G_p} \quad \text{and} \quad \widehat{W}'^{p-G_p} = (\widehat{W}_k^p)_{0 \leq k \leq p-G_p}.$$

Then, $H_l(W'^{(p-G_p)}) = H_l(W)$ and it is easily verified that

$$L_p(\widehat{W}^p) = L_{p-G_p}(\widehat{W}'^{p-G_p}) \quad \text{and} \quad L_{\gamma_p(l)}(\widehat{W}^p) = L_{\beta'(p-G_p,l)}(\widehat{W}'^{p-G_p}),$$

so that

$$L_{p-G_p}(\widehat{W}'^{p-G_p}) - L_{\beta'(p-G_p,l)}(\widehat{W}'^{p-G_p}) = M_l.$$

So we have

$$(14) \quad H_{p-G_p+l}(W') = M_l + H_l(W), \quad 0 \leq l \leq G_p.$$

Let us consider now $0 \leq l \leq p - G_p$. We then have

$$W'_l = W_{l+G_p} - W_{G_p} = W_{l+G_p} - \inf_{0 \leq k \leq p} W_k.$$

It easily follows that $H_l(W') = H_{l+G_p}(W)$. But $M_{l+G_p} = 0$ because $\gamma_p(l + G_p) = p$ [note that $-\inf_{0 \leq k \leq p} W_k = \sup_{0 \leq k \leq p} \widehat{W}_k^p + 1$, $\mathbf{P}(\cdot | W_p = -1)$ -a.s.]. We conclude that

$$(15) \quad H_l(W') = H_{l+G_p}(W) + M_{l+G_p}, \quad 0 \leq l \leq p - G_p.$$

Thanks to (14) and (15), we see that it only remains to prove that $G(M + H(W)) = G_p$: First note that if $G_p = p$, we have $M_l = 0$ for every $l \in [0, p]$ and $H(W) = V_0(H(W))$ in a trivial way. We can therefore suppose $0 < G_p < p$. Then it is easily seen that $\mathbf{P}(\cdot | W_p = -1)$ -a.s., for every $l \in [0, p]$,

$$M_l = \text{Card} \left\{ 0 \leq j < p : W_j = \inf_{j \leq k \leq p} W_k \text{ and } W_j \leq -1 + \inf_{0 \leq k \leq l} W_k \right\}.$$

If $0 \leq l < G_p$, then $\inf_{0 \leq k \leq l} W_k > \inf_{0 \leq k \leq p} W_k$ and thus $M_l > 0$ because we can take $j = G_p$ in the previous formula. On the other hand, $H_{G_p}(W) + M_{G_p} = 0$. This proves $G(M + H(W)) = G_p$. \square

4.2. *Auxiliary processes.* In this section we introduce the Lévy bridge X^{br} that can be seen informally as the path $(X_t; 0 \leq t \leq 1)$ conditioned to be at level 0 at time 1. Standard arguments make this singular conditioning rigorous and we refer to the original work of Chaumont [7, 8] or to [4], Chapter VIII, for the proofs. We also define the height process associated with the bridge, denoted by H^{br} and the process M^{br} that will play the role of M in continuous time.

We denote by p_t the continuous density of the law of X_t ; it is characterized by

$$\int_{\mathbb{R}} \exp(-\lambda x) p_t(x) dx = \exp(-t\lambda^\alpha).$$

For $0 < t < 1$, the law of $(X_s^{\text{br}}; 0 \leq s \leq t)$ is absolutely continuous with respect to the law of $(X_s; 0 \leq s \leq t)$. More precisely, for any bounded continuous functional F defined on $\mathbb{D}([0, t], \mathbb{R})$, we have

$$(16) \quad \mathbf{E}[F(X_s^{\text{br}}; 0 \leq s \leq t)] = \mathbf{E}\left[F(X_s; 0 \leq s \leq t) \frac{p_{1-t}(-X_t)}{p_1(0)}\right].$$

It follows that

$$(17) \quad \widehat{X}^{\text{br}} \stackrel{\text{law}}{=} X^{\text{br}}$$

where, for convenience, we denote by \widehat{X}^{br} the process X^{br} reversed at time 1 ($\widehat{X}_t^{\text{br}} = -X_{1-t}^{\text{br}}$, $t \in [0, 1]$). Chaumont provides in [8] a path-construction for X^{br} : set $G = \sup\{t \in [0, 1] : X_t = 0\}$, the last passage time at the origin on $[0, 1]$ of the unconditioned process X . Let us set $\widetilde{X} = (G^{-1/\alpha} X_{Gt})_{0 \leq t \leq 1}$. Chaumont has shown that

$$(18) \quad X^{\text{br}} \stackrel{\text{law}}{=} \widetilde{X}.$$

In the Brownian case $\alpha = 2$, we define the two processes H^{br} and $L(X^{\text{br}})$ by setting

$$H_t^{\text{br}} = X_t^{\text{br}} - I_t^{\text{br}}, \quad L_t(X^{\text{br}}) = S_t^{\text{br}}, \quad t \in [0, 1],$$

with an evident notation for I^{br} and S^{br} .

If $1 < \alpha < 2$, we define H^{br} and $L(X^{\text{br}})$ by use of the approximation formula (5): By (16) and a continuity argument it is easy to check that $\mathbf{P}(\exists \delta > 0 : S_{1-\delta}^{\text{br}} = S_t^{\text{br}}, t \in [1-\delta, 1]) = 1$. Then, by (4) and (16), it follows that we may define a continuous increasing process $L(X^{\text{br}})$ by setting \mathbf{P} -a.s. for every t in $[0, 1]$,

$$(19) \quad L_t(X^{\text{br}}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card}\{s \in [0, t] : S_{s-}^{\text{br}} < X_s^{\text{br}}; \Delta X_s^{\text{br}} > \varepsilon\}.$$

Next, by (5), it follows that the limit

$$G^{1/\alpha-1} H_{Gt} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card}\left\{s \in [0, t] : \widetilde{X}_{s-} < \inf_{[s,t]} \widetilde{X}; \Delta \widetilde{X}_s > \varepsilon\right\}$$

holds **P**-a.s. for a set of values of t of full Lebesgue measure in $[0, 1]$. Then, thanks to Chaumont’s identity (18) we can show that there exists a continuous process $(H_t^{\text{br}}; 0 \leq t \leq 1)$ such that the limit

$$(20) \quad H_t^{\text{br}} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card} \left\{ s \in [0, t] : X_{s-}^{\text{br}} < \inf_{[s,t]} X^{\text{br}}; \Delta X_s^{\text{br}} > \varepsilon \right\}$$

holds **P**-a.s. for a set of values of t of full Lebesgue measure in $[0, 1]$. We also have

$$(21) \quad H^{\text{br}} \stackrel{\text{law}}{=} (G^{1/\alpha-1} H_{Gt})_{0 \leq t \leq 1}.$$

And by (16), it follows that, for $0 < t < 1$,

$$(22) \quad \mathbf{E}[F(H_s^{\text{br}}; 0 \leq s \leq t)] = \mathbf{E} \left[F(H_s; 0 \leq s \leq t) \frac{p_{1-t}(-X_t)}{p_1(0)} \right].$$

REMARK 4.1. Equations (22) and (21) both hold in the Brownian case.

We now define the Vervaat transform in continuous time, denoted by $V : \mathbb{D}([0, 1], \mathbb{R}) \rightarrow \mathbb{D}([0, 1], \mathbb{R})$: For any ω in $\mathbb{D}([0, 1], \mathbb{R})$, we set $g_1(\omega) = \inf\{t \in [0, 1] : \omega(t-) \wedge \omega(t) = \inf_{[0,1]} \omega\}$. Then, we define V by

$$V(\omega)(t) = \begin{cases} \omega(t + g_1(\omega)) - \inf_{[0,1]} \omega, & \text{if } t + g_1(\omega) \leq 1, \\ \omega(t + g_1(\omega) - 1) + \omega(1) - \inf_{[0,1]} \omega - \omega(0), & \text{if } t + g_1(\omega) \geq 1. \end{cases}$$

Thanks to (16), it is easy to see that the bridge X^{br} reaches its infimum almost surely at a unique random time (that must be $g_1(X^{\text{br}})$ and that is uniformly distributed in $[0, 1]$). The bridge is connected to the normalized excursion X^{exc} through the Vervaat transform

$$(23) \quad V(X^{\text{br}}) \stackrel{\text{law}}{=} X^{\text{exc}}.$$

(For a proof, see Chaumont [8] or Bertoin [4], Chapter VIII.) Next, we define the analogue of M in continuous time: For any ω in $\mathbb{D}([0, 1], \mathbb{R})$ and any positive real number x , let us denote by $T_x(\omega)$, the first passage time above x :

$$T_x(\omega) = \inf\{t \geq 0 : \omega(t) \geq x\},$$

(with the convention: $\inf \emptyset = +\infty$). For any $1 < \alpha \leq 2$, $L_t(\widehat{X}^{\text{br}})$ is well defined thanks to (17) and (19). So we can set

$$B_x = L_1(\widehat{X}^{\text{br}}) - L_{1 \wedge T_x}(\widehat{X}^{\text{br}}), \quad x \geq 0,$$

and we define M^{br} by

$$M_t^{\text{br}} = B_{-\inf_{[0,t]} X^{\text{br}}}, \quad 0 \leq t \leq 1.$$

The following proposition is an analogue in continuous time of Proposition 4.1.

PROPOSITION 4.2. *The processes H^{br} , M^{br} and B have the following properties:*

(i) **P**-a.s. $(B_x; x \geq 0)$ is a nonnegative and nonincreasing continuous process. Furthermore we have $B_x = 0$ if and only if $x \geq -\inf_{[0,1]} X^{\text{br}}$.

(ii) **P**-a.s. $(M_t^{\text{br}} + H_t^{\text{br}}; 0 \leq t \leq 1)$ is a nonnegative continuous process that attains its minimal value 0 at a unique instant.

(iii) $V(M^{\text{br}} + H^{\text{br}}) \stackrel{\text{law}}{=} H^{\text{exc}}$.

PROOF. Thanks to Chaumont’s result (23), we can assume that X^{exc} and X^{br} are related in the following way:

$$(24) \quad \begin{aligned} X_s^{\text{exc}} &= X_{g_1+s}^{\text{br}} - X_{g_1}^{\text{br}}, & 0 \leq s \leq 1 - g_1, \\ X_s^{\text{exc}} &= X_{s+g_1-1}^{\text{br}} - X_{g_1}^{\text{br}}, & 1 - g_1 \leq s \leq 1, \end{aligned}$$

where we have set $g_1 = g_1(X^{\text{br}})$.

In the Brownian case $\alpha = 2$, we have $L_t(\widehat{X}^{\text{br}}) = \sup_{s \leq t} \widehat{X}_s^{\text{br}}$. It easily follows that $B_x = (-x - I_1^{\text{br}})_+$, $M_t^{\text{br}} = I_t^{\text{br}} - I_1^{\text{br}}$ and $M_t^{\text{br}} + H_t^{\text{br}} = X_t^{\text{br}} - I_1^{\text{br}}$, for $x \geq 0$ and $0 \leq t \leq 1$. Assertions (i) and (ii) follow immediately and (iii) is a direct consequence of (24).

From now on, we assume that $1 < \alpha < 2$. Let us prove (i) first. Recall that $(S_{L_t^{-1}}; t \geq 0)$ is a stable subordinator with index $\alpha - 1$. Hence, its right-continuous inverse $(L_{T_x(X)}; x \geq 0)$ is **P**-a.s. continuous. If $x \geq S_1$, then $T_x(X) \geq 1$ and $L_1 = L_{1 \wedge T_x(X)}$. However, for any positive rational q , the Markov property for X implies that $T_q(X)$ is an increase time for L . Then

$$L_{T_q(X)} < L_1 \quad \text{on } \{q < S_1\}.$$

Hence, $(L_1 - L_{1 \wedge T_x(X)}; x \geq 0)$ is **P**-a.s. a nonincreasing and nonnegative continuous process that vanishes if and only if $x \geq S_1$.

Let $t < 1$. We can use property (16) to show that **P**-a.s. the process $(L_{t \wedge T_x(X^{\text{br}})}(X^{\text{br}}); x \geq 0)$ is continuous and $L_{T_x(X^{\text{br}})}(X^{\text{br}}) < L_t(X^{\text{br}})$ if and only if $x < \sup_{[0,t]} X^{\text{br}}$. But **P**-a.s. there exists $t \in [0, 1)$ such that $\sup_{[0,t]} X^{\text{br}} = \sup_{[0,1]} X^{\text{br}}$ and so

$$L_{t \wedge T_x(X^{\text{br}})}(X^{\text{br}}) = L_{1 \wedge T_x(X^{\text{br}})}(X^{\text{br}}), \quad x \geq 0.$$

Hence, we have proved that **P**-a.s. the process $(L_{1 \wedge T_x(X^{\text{br}})}(X^{\text{br}}); x \geq 0)$ is continuous and $L_{T_x(X^{\text{br}})}(X^{\text{br}}) < L_1(X^{\text{br}})$ if and only if $x < \sup_{[0,1]} X^{\text{br}}$. Then, (i) follows from the duality property (17). Then, the continuity of M^{br} follows from the continuity of I^{br} .

Recall that X^{exc} and X^{br} are related by (24). We now establish the a.s. identity

$$(25) \quad H^{\text{exc}} = V(M^{\text{br}} + H^{\text{br}}).$$

First, observe that if $t > g_1$, then the conditions $X_{s-}^{\text{br}} < \inf_{[s,t]} X^{\text{br}}$ and $s \in [0, t]$ imply that $s \geq g_1$. Thanks to the approximations (7) and (20), and the continuity of the processes H^{exc} and H^{br} , we easily verify that, **P**-a.s.,

$$(26) \quad H_t^{\text{exc}} = H_{g_1+t}^{\text{br}}, \quad 0 \leq t \leq 1 - g_1.$$

However (i) and the definition of M^{br} imply that **P**-a.s. $M_t^{\text{br}} = 0$ for any t in $[g_1, 1]$. Then, by (26), it follows that

$$(27) \quad H_t^{\text{exc}} = H_{g_1+t}^{\text{br}} + M_{g_1+t}^{\text{br}}, \quad 0 \leq t \leq 1 - g_1.$$

Next we have to prove

$$(28) \quad H_t^{\text{exc}} = H_{g_1+t-1}^{\text{br}} + M_{g_1+t-1}^{\text{br}}, \quad 1 - g_1 \leq t \leq 1.$$

Set, for any $t > 1 - g_1$,

$$\gamma_t = \sup\{s < 1 : X_s^{\text{br}} \leq I_{t+g_1-1}^{\text{br}}\} = 1 - T_{-I_{t+g_1-1}^{\text{br}}}(\widehat{X}^{\text{br}}).$$

We also define, for any $0 \leq s' \leq t' \leq 1$,

$$N_\varepsilon^{\text{br}}(s', t') = \text{Card}\left\{u \in [0, s'] : X_{u-}^{\text{br}} < \inf_{[u,t']} X^{\text{br}}; \Delta X_u^{\text{br}} > \varepsilon\right\}.$$

If $1 - g_1 < t$, observe that

$$\text{Card}\left\{s \in [0, t] : X_{s-}^{\text{exc}} < \inf_{[s,t]} X^{\text{exc}}; \Delta X_s^{\text{exc}} > \varepsilon\right\} = C_1 + C_2 + C_3,$$

where

$$C_1 = \text{Card}\left\{s \in [0, \gamma_t - g_1] : X_{s-}^{\text{exc}} < \inf_{[s,t]} X^{\text{exc}}; \Delta X_s^{\text{exc}} > \varepsilon\right\} = N_\varepsilon^{\text{br}}(\gamma_t, 1),$$

$$C_2 = \text{Card}\left\{s \in (\gamma_t - g_1, 1 - g_1) : X_{s-}^{\text{exc}} < \inf_{[s,t]} X^{\text{exc}}; \Delta X_s^{\text{exc}} > \varepsilon\right\} = 0,$$

$$C_3 = \text{Card}\left\{s \in [1 - g_1, t] : X_{s-}^{\text{exc}} < \inf_{[s,t]} X^{\text{exc}}; \Delta X_s^{\text{exc}} > \varepsilon\right\}$$

$$= N_\varepsilon^{\text{br}}(t + g_1 - 1, t + g_1 - 1).$$

Thus,

$$(29) \quad \begin{aligned} &\text{Card}\left\{s \in [0, t] : X_{s-}^{\text{exc}} < \inf_{[s,t]} X^{\text{exc}}; \Delta X_s^{\text{exc}} > \varepsilon\right\} \\ &= N_\varepsilon^{\text{br}}(\gamma_t, 1) + N_\varepsilon^{\text{br}}(t + g_1 - 1, t + g_1 - 1). \end{aligned}$$

By approximation formula (19) applied to $L(\widehat{X}^{\text{br}})$, it follows that \mathbf{P} -a.s. for any $0 \leq t \leq 1$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} N_\varepsilon^{\text{br}}(\gamma_t, 1) = L_1(\widehat{X}^{\text{br}}) - L_{1-\gamma_t}(\widehat{X}^{\text{br}}) = M_{t+g_1-1}^{\text{br}}.$$

But approximation (7) of H^{exc} and approximation (20) of H^{br} imply that the limits

$$H_t^{\text{exc}} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} \text{Card} \left\{ s \in [0, t] : X_{s-}^{\text{exc}} < \inf_{[s,t]} X^{\text{exc}}; \Delta X_s^{\text{exc}} > \varepsilon \right\}$$

$$H_{t+g_1-1}^{\text{br}} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\beta_\varepsilon} N_\varepsilon^{\text{br}}(t + g_1 - 1, t + g_1 - 1)$$

hold \mathbf{P} -a.s. on a set of values of t of full Lebesgue measure in $[1 - g_1, 1]$. Then, (28) follows from (29) and the continuity of M^{br} , H^{exc} and H^{br} .

It remains to show that $M^{\text{br}} + H^{\text{br}}$ reaches its infimum at the unique time g_1 . If $s > g_1$, then $M_s^{\text{br}} = 0$ by (i) and $H_s^{\text{br}} = H_{s-g_1}^{\text{exc}} > 0$. If $g_1 > s$, then

$$\inf_{[0,s]} X^{\text{br}} > X_{g_1}^{\text{br}} = \inf_{[0,1]} X^{\text{br}}$$

and (i) implies that $M_s^{\text{br}} > 0$. Finally, (ii) follows from the obvious fact $M_{g_1}^{\text{br}} = H_{g_1}^{\text{br}} = 0$. \square

We now explain how the auxiliary processes X^{br} , H^{br} and M^{br} are used in the proof of Theorem 3.1: Let $(W_t^{\text{br},p}; t \in [0, 1])$ be a process whose distribution is the law of $(1/a_p W_{[pt]}; 0 \leq t \leq 1)$ under $\mathbf{P}(\cdot | W_p = -1)$. Simultaneously with $W^{\text{br},p}$, we can introduce the processes $H^{\text{br},p}$, $M^{\text{br},p}$, $\widehat{W}^{\text{br},p}$, $L^{\text{br},p}$ and $\widehat{L}^{\text{br},p}$ (which can be written as functionals of $W^{\text{br},p}$) that are such that

$$(W^{\text{br},p}, H^{\text{br},p}, M^{\text{br},p}, \widehat{W}^{\text{br},p}, L^{\text{br},p}, \widehat{L}^{\text{br},p})$$

has the same law as

$$\left(\frac{1}{a_p} W_{[pt]}, \frac{a_p}{p} H_{[pt]}(W), \frac{a_p}{p} M_{[pt]}, \frac{1}{a_p} \widehat{W}_{[pt]}, \frac{a_p}{p} L_{[pt]}(W), \frac{a_p}{p} L_{[pt]}(\widehat{W}^p) \right)_{0 \leq t \leq 1}$$

under $\mathbf{P}(\cdot | W_p = -1)$. We have the following proposition.

PROPOSITION 4.3. *Under the assumptions of Theorem 3.1, we have*

$$(W^{\text{br},p}, H^{\text{br},p}, M^{\text{br},p}) \xrightarrow[p \rightarrow \infty]{d} (X^{\text{br}}, H^{\text{br}}, M^{\text{br}}).$$

Let us complete the proof of Theorem 3.1 thanks to Proposition 4.3 whose proof is postponed to the next section.

PROOF THEOREM 3.1. First, it easy to deduce from Proposition 2.1, from the definition of the discrete Vervaat transform and from Proposition 4.1 that

$$(30) \quad (V(W^{\text{br},p}), V(M^{\text{br},p} + H^{\text{br},p})) \stackrel{\text{law}}{=} \left(\left(\frac{1}{a_p} W_{[pt]}^{\text{exc},p} \right)_{0 \leq t \leq 1}, \left(\frac{a_p}{p} H_{[pt]}^{\text{exc},p} \right)_{0 \leq t \leq 1} \right).$$

Then, we need to prove some continuity property of V : Let $(\omega_n)_{n \geq 0}$ be a sequence of paths in $\mathbb{D}([0, 1], \mathbb{R})$ that converges to ω for the Skorokhod topology. If ω is continuous, then the convergence holds uniformly on $[0, 1]$:

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |\omega_n(t) - \omega(t)| = 0$$

(see Jacod and Shiryaev [14], Chapter VI). Then, if we assume furthermore that ω attains its minimum at a unique instant, it is easily seen that $\lim g_1(\omega_n) = g_1(\omega)$. Thus,

$$(31) \quad \lim_{n \rightarrow +\infty} \sup_{t \in [0,1]} |V(\omega_n)(t) - V(\omega)(t)| = 0.$$

This shows that V is continuous at any continuous path ω in $\mathbb{D}([0, 1], \mathbb{R})$ that attains its minimum at a unique time. This observation combined with Proposition 4.2(ii) and Proposition 4.3, shows that

$$(32) \quad V(M^{\text{br},p} + H^{\text{br},p}) \xrightarrow{d} V(M^{\text{br}} + H^{\text{br}}).$$

Then Theorem 3.1 follows from (30) and from Proposition 4.2 (iii). \square

4.3. Proof of Proposition 4.3. We first prove the following lemma for unconditioned processes.

LEMMA 4.4. Under the assumptions of Theorem 3.1, the following joint convergence holds:

$$\left(\frac{1}{a_p} W_{[pt]}, \frac{a_p}{p} L_{[pt]}(W), \frac{a_p}{p} H_{[pt]}(W); t \geq 0 \right) \xrightarrow[p \rightarrow \infty]{d} (X, L, H).$$

PROOF. A classical result on random walks shows that assumption (H) implies the convergence of $(1/a_p W_{[pt]}; t \geq 0)$ to X in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ (see Jacod and Shiryaev [14], Chapter VII). Theorem 2.3.2 in [11] [recalled in (10)] shows that the rescaled height process $(\frac{a_p}{p} H_{[pt]}(W); t \geq 0)$ converges to H in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ under assumption (H).

As a first step toward the proof of the convergence of rescaled process, it is also proved in [11] (see Theorem 2.2.1) that

$$(33) \quad \left(\frac{a_p}{p} L_{[pt]}(W) \right)_{t \geq 0} \xrightarrow[p \rightarrow \infty]{fd} (L_t)_{t \geq 0}.$$

As L is a continuous nondecreasing process, a standard argument show that the convergence (33) actually holds in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. Thus, the laws of the processes

$$\left(\frac{1}{a_p} W_{[pt]}, \frac{a_p}{p} L_{[pt]}(W), \frac{a_p}{p} H_{[pt]}(W); t \geq 0 \right)$$

are tight in the space of probability measures on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3)$.

If we look carefully at Theorem 2.2.1 in [11], we see that the proof actually gives a stronger result than the weak convergence of the finite-dimensional marginals of the rescaled height process: By the Skorokhod representation theorem, we can find a sequence of random walks W'^p , $p \geq 1$, each with the same law as W , and a Lévy process X' with

$$\left(\frac{1}{a_p} W'^p_{[pt]}; t \geq 0 \right) \rightarrow (X'_t; t \geq 0),$$

P-a.s. for the Skorokhod topology. Then the proof of Theorem 2.2.1 in [11] shows that

$$(34) \quad \frac{a_p}{p} L_{[pt]}(W'^p) \rightarrow L_t(X') \quad \text{and} \quad \frac{a_p}{p} H_{[pt]}(W'^p) \rightarrow H_t(X')$$

in probability for every $t \geq 0$ (with an evident notation for $L(X')$ and $H(X')$). It follows that the only possible weak limit for the laws of

$$\left(\frac{1}{a_p} W_{[pt]}, \frac{a_p}{p} L_{[pt]}(W), \frac{a_p}{p} H_{[pt]}(W); t \geq 0 \right)$$

is that of (X, L, H) and the lemma is proved. \square

LEMMA 4.5. *Under the assumptions of Theorem 3.1, for any $t < 1$, we have*

$$(W_s^{\text{br},p}, L_s^{\text{br},p}, H_s^{\text{br},p})_{0 \leq s \leq t} \xrightarrow[p \rightarrow \infty]{d} (X_s^{\text{br}}, L_s(X^{\text{br}}), H_s^{\text{br}})_{0 \leq s \leq t}.$$

PROOF. Set $f(n, k) = \mathbf{P}(W_n = k)$, $n \in \mathbb{N}, k \in \mathbb{Z}$. Let F be any bounded continuous functional on $\mathbb{D}([0, t], \mathbb{R}^3)$. The Markov property at time $[pt]$ under $\mathbf{P}(\cdot | W_p = -1)$ implies that

$$(35) \quad \begin{aligned} & \mathbf{E}[F(W_s^{\text{br},p}, L_s^{\text{br},p}, H_s^{\text{br},p}; 0 \leq s \leq t)] \\ &= \mathbf{E} \left[\frac{f(p - [pt], -1 - W_{[pt]})}{f(p, -1)} \right. \\ & \quad \left. \times F \left(\frac{1}{a_p} W_{[ps]}, \frac{a_p}{p} L_{[ps]}(W), \frac{a_p}{p} H_{[ps]}(W); 0 \leq s \leq t \right) \right]. \end{aligned}$$

Since we assume (H) and since μ (and thus ν) is aperiodic, we can apply the Gnedenko local limit theorem to ν in order to get

$$\lim_{p \rightarrow \infty} \sup_{k \in \mathbb{Z}} |a_p f(p - [pt], k) - p_{1-t}(k/a_p)| = 0$$

(see [6]). This result combined with (35), the continuity of $x \rightarrow p_{1-t}(x)$ and Lemma 4.4 gives

$$\begin{aligned} & \lim_{p \rightarrow \infty} \mathbf{E}[F(W_s^{\text{br},p}, L_s^{\text{br},p}, H_s^{\text{br},p}; 0 \leq s \leq t)] \\ &= \mathbf{E}\left[\frac{p_{1-t}(-X_t)}{p_1(0)} F(X_s, L_s, H_s; 0 \leq s \leq t)\right] \end{aligned}$$

and the lemma follows from (16). \square

Next, we need to prove the following lemma:

LEMMA 4.6. *Under the assumptions of Theorem 3.1, we have*

$$(\widehat{L}^{\text{br},p}, \widehat{W}^{\text{br},p}, W^{\text{br},p}) \xrightarrow[p \rightarrow \infty]{d} (L(\widehat{X}^{\text{br}}), \widehat{X}^{\text{br}}, X^{\text{br}}).$$

PROOF. First, let us show that $W^{\text{br},p}$ converges to X^{br} in distribution in $\mathbb{D}([0, 1], \mathbb{R})$. From Lemma 4.5 and the usual tightness criterion, we only need to prove

$$(36) \quad \lim_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P}\left(\sup_{s \in [1-\delta, 1]} |W_s^{\text{br},p} - W_1^{\text{br},p}| > \eta\right) = 0$$

for any $\eta > 0$. Notice that the two variables

$$\sup_{[p(1-\delta)] \leq k \leq p} |W_k - W_p| \quad \text{and} \quad \sup_{0 \leq k \leq p - [p(1-\delta)]} |W_k|$$

have the same law under $\mathbf{P}(\cdot | W_p = -1)$. Thus

$$(37) \quad \mathbf{P}\left(\sup_{s \in [1-\delta, 1]} |W_s^{\text{br},p} - W_1^{\text{br},p}| > \eta\right) \leq \mathbf{P}\left(\sup_{s \in [0, \delta+1/p]} |W_s^{\text{br},p}| > \eta\right).$$

But Lemma 4.5 implies, for any $\eta > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P}\left(\sup_{s \in [0, \delta+1/p]} |W_s^{\text{br},p}| > \eta\right) = 0.$$

Then, (36) follows from (37).

We now prove

$$(38) \quad L^{\text{br},p} \xrightarrow[p \rightarrow \infty]{d} L(X^{\text{br}}).$$

First, from Lemma 4.5 we have, for any $t < 1$,

$$(L_s^{\text{br},p})_{0 \leq s \leq t} \xrightarrow[p \rightarrow \infty]{d} (L_s(X^{\text{br}}))_{0 \leq s \leq t}$$

in distribution in $\mathbb{D}([0, t], \mathbb{R})$. Next, recall that \mathbf{P} -a.s. there exists a small interval $(1 - \delta, 1]$ on which $L(X^{\text{br}})$ is constant and equal to $L_1(X^{\text{br}})$. So, we only need to prove

$$\lim_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P} \left(\sup_{s \in [1-\delta, 1]} |L_s^{\text{br},p} - L_1^{\text{br},p}| > \eta \right) = 0$$

for any $\eta > 0$. But this is immediate from the observation that

$$\lim_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P} \left(\sup_{s \in [1-\delta, 1]} W_s^{\text{br},p} = \sup_{s \in [0, 1]} W_s^{\text{br},p} \right) = 0,$$

which itself follows from the convergence of $W^{\text{br},p}$ to X^{br} .

Since $W^{\text{br},p}$ (resp. X^{br}) has the same law as $\widehat{W}^{\text{br},p}$ (resp. \widehat{X}^{br}), the lemma is equivalent to

$$(L^{\text{br},p}, W^{\text{br},p}, \widehat{W}^{\text{br},p}) \xrightarrow[p \rightarrow \infty]{d} (L(X^{\text{br}}), X^{\text{br}}, \widehat{X}^{\text{br}}).$$

First notice that the laws of $(L^{\text{br},p}, W^{\text{br},p}, \widehat{W}^{\text{br},p})$ are tight in the space of all probability measures on $\mathbb{D}([0, 1], \mathbb{R}^3)$. We only need to prove the convergence of the finite dimensional marginals. By Lemma 4.5, we see that the only possible weak limit of the laws of $(L^{\text{br},p}, W^{\text{br},p})$ is the law of $(L(X^{\text{br}}), X^{\text{br}})$. Since X^{br} has no fixed discontinuities, we have for any t_1, \dots, t_n in $[0, 1]$

$$(L_{t_i}^{\text{br},p}, W_{t_i}^{\text{br},p})_{1 \leq i \leq n} \xrightarrow[p \rightarrow \infty]{d} (L_{t_i}(X^{\text{br}}), X_{t_i}^{\text{br}})_{1 \leq i \leq n}.$$

For the same reason $\widehat{X}_t^{\text{br}} = -X_{1-t}^{\text{br}}$, \mathbf{P} -a.s. for any t in $[0, 1]$. So, we get

$$(L_{t_i}^{\text{br},p}, W_{t_i}^{\text{br},p}, W_1^{\text{br},p} - W_{1-t_i}^{\text{br},p})_{1 \leq i \leq n} \xrightarrow[p \rightarrow \infty]{d} (L_{t_i}(X^{\text{br}}), X_{t_i}^{\text{br}}, \widehat{X}_{t_i}^{\text{br}})_{1 \leq i \leq n}.$$

But we have, for any t in $[0, 1]$, the convergence in probability

$$W_1^{\text{br},p} - W_{1-t}^{\text{br},p} - \widehat{W}_t^{\text{br},p} \xrightarrow[p \rightarrow \infty]{} 0$$

because $W_1^{\text{br},p} - W_{1-t}^{\text{br},p} - \widehat{W}_t^{\text{br},p}$ has the same law as $(W_{p-[pt]} - W_{[p(1-t)]})/a_p$ under $\mathbf{P}(\cdot | W_p = -1)$. Thus, we have

$$(L_{t_i}^{\text{br},p}, W_{t_i}^{\text{br},p}, \widehat{W}_{t_i}^{\text{br},p})_{1 \leq i \leq n} \xrightarrow[p \rightarrow \infty]{d} (L_{t_i}(X^{\text{br}}), X_{t_i}^{\text{br}}, \widehat{X}_{t_i}^{\text{br}})_{1 \leq i \leq n},$$

that implies the desired result. \square

Next, we claim that the two following lemmas imply Proposition 4.3.

LEMMA 4.7. *Under the assumptions of Theorem 3.1, we have*

$$(W^{\text{br},p}, M^{\text{br},p}) \xrightarrow[p \rightarrow \infty]{d} (X^{\text{br}}, M^{\text{br}}).$$

LEMMA 4.8. *Under the assumptions of Theorem 3.1, the laws of the processes $(H^{\text{br},p})$ are tight in the space of all probability measures on $\mathbb{D}([0, 1], \mathbb{R})$.*

END OF PROOF OF PROPOSITION 4.3. The previous two lemmas imply that the laws of $(W^{\text{br},p}, H^{\text{br},p}, M^{\text{br},p})$ are tight in the space of all probability measures on $\mathbb{D}([0, 1], \mathbb{R}^2)$. Let us assume that a subsequence of the sequence $(W^{\text{br},p}, H^{\text{br},p})$ converges in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$ to a certain process (A, B) . By Lemma 4.5, it follows that

$$(A_s, B_s)_{0 \leq s \leq t} \stackrel{\text{law}}{=} (X_s^{\text{br}}, H_s^{\text{br}})_{0 \leq s \leq t},$$

for any $t < 1$. Also Lemma 4.7 implies $A \stackrel{\text{law}}{=} X^{\text{br}}$. Then, observe that $H_1^{\text{br},p} = \widehat{L}_1^{\text{br},p}$, $p \geq 1$, and that $H_1^{\text{br}} = L_1(\widehat{X}^{\text{br}})$. From Lemma 4.6, we get

$$(A, B_1) \stackrel{\text{law}}{=} (X^{\text{br}}, L_1(\widehat{X}^{\text{br}})) = (X^{\text{br}}, H_1^{\text{br}}).$$

This is more than enough to conclude that

$$(A, B) \stackrel{\text{law}}{=} (X^{\text{br}}, H^{\text{br}}).$$

So we have

$$(W^{\text{br},p}, H^{\text{br},p}) \xrightarrow[p \rightarrow \infty]{d} (X^{\text{br}}, H^{\text{br}}).$$

Together with Lemma 4.7, this implies that the only possible weak limit of the laws of $(W^{\text{br},p}, H^{\text{br},p}, M^{\text{br},p})$ is the law of $(X^{\text{br}}, H^{\text{br}}, M^{\text{br}})$. That completes the proof of Proposition 4.3. \square

PROOF OF LEMMA 4.7. We can apply Skorokhod’s representation theorem to replace the weak convergence of Lemma 4.6 by an a.s. convergence. For convenience, we keep the same notation for the processes and the underlying probability space, so we can suppose

$$(39) \quad (\widehat{L}^{\text{br},p}, \widehat{W}^{\text{br},p}, W^{\text{br},p}) \xrightarrow[p \rightarrow \infty]{} (L(\widehat{X}^{\text{br}}), \widehat{X}^{\text{br}}, X^{\text{br}})$$

P-a.s. for the Skorokhod topology in $\mathbb{D}([0, 1], \mathbb{R}^3)$.

For any $p \geq 1$, we define the process $(B_x^p; x \geq 0)$ by

$$B_x^p = \widehat{L}_1^{\text{br},p} - \widehat{L}_{1 \wedge T_x}^{\text{br},p}(\widehat{W}^{\text{br},p}).$$

We get from the definition of M the following inequality:

$$(40) \quad \sup_{t \in [0, 1]} |M_t^{\text{br}, p} - B_{-\inf[0, t]}^{p, p} W^{\text{br}, p}| \leq \frac{ap}{p},$$

because

$$|p \wedge T_{-\inf[0, t]} W^{\text{br}, p}(\widehat{W}^{\text{br}, p}) - \gamma_p([pt])| \leq 1.$$

We claim next that

$$(41) \quad (B_x^p)_{x \geq 0} \xrightarrow{p \rightarrow \infty} (B_x)_{x \geq 0},$$

P-a.s. for the Skorokhod topology in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$: Lemma 2.10, page 304, Chapter VI in [14] shows for any $x \geq 0$ that the functional $1 \wedge T_x(\cdot)$, defined on $\mathbb{D}([0, 1], \mathbb{R})$ is continuous with respect to the Skorokhod topology at any path ω satisfying $x \notin J(\omega)$, where

$$J(\omega) = \{y > 0 : T_{y+}(\omega) > T_y(\omega)\}.$$

An elementary argument shows that for any $x \geq 0$, $\mathbf{P}(x \in J(X)) = 0$. Then, we can use the absolute continuity relation (16) to deduce that **P**-a.s. x is not in $J(X^{\text{br}})$. Hence

$$(42) \quad \mathbf{P}\text{-a.s.}, \quad 1 \wedge T_q(\widehat{W}^{\text{br}, p}) \rightarrow 1 \wedge T_q(\widehat{X}^{\text{br}}), \quad q \in \mathbb{Q}_+.$$

Since $L(\widehat{X}^{\text{br}})$ is continuous, a standard argument implies that **P**-a.s. $\widehat{L}^{\text{br}, p}$ converges to $L(\widehat{X}^{\text{br}})$ uniformly on $[0, 1]$. Then, by (42), it follows that

$$(43) \quad (B_{q_1}^p, \dots, B_{q_n}^p) \rightarrow (B_{q_1}, \dots, B_{q_n}), \quad \mathbf{P}\text{-a.s.}$$

for any positive rational numbers q_1, q_2, \dots, q_n . Next, observe that B^p and B are nondecreasing processes and that B is continuous [cf. Proposition 4.2(i)] so (43) implies the desired claim by a standard argument.

It remains to prove that (41) implies the lemma: Since I^{br} is continuous, (39) implies that

$$\lim_{p \rightarrow \infty} \sup_{t \in [0, 1]} \left| \inf_{t \in [0, t]} W^{\text{br}, p} - I_t^{\text{br}} \right| = 0 \quad \text{a.s.}$$

This, combined with (41), shows that **P**-a.s. $M^{\text{br}, p}$ converges to M^{br} , uniformly on $[0, 1]$. As M^{br} is continuous, a standard argument (see [14], Proposition 1.23, page 293) implies that $(W^{\text{br}, p}, M^{\text{br}, p})$ converges almost surely to $(X^{\text{br}}, M^{\text{br}})$ for the Skorokhod topology in $\mathbb{D}([0, 1], \mathbb{R}^2)$. That completes the proof of the lemma. \square

PROOF OF LEMMA 4.8. By Lemma 4.5, it is sufficient to show

$$(44) \quad \lim_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P} \left(\sup_{s \in [1-\delta, 1]} |H_s^{\text{br}, p} - H_1^{\text{br}, p}| > \eta \right) = 0$$

for any $\eta > 0$. Recall that

$$G_p = \inf \left\{ 0 \leq k \leq p : W_k = \inf_{0 \leq j \leq k} W_j \right\}$$

and that

$$W_k^{([p\varepsilon])} = W_{[p\varepsilon]+k} - W_{[p\varepsilon]}, \quad k \geq 0.$$

Let $\delta, \varepsilon > 0$ and set $A_p = \{[p\varepsilon] \leq G_p \leq p - [p\delta]\}$. Observe that on A_p ,

$$\inf_{[p\varepsilon] \leq i \leq [p\varepsilon]+k} H_i(W) = 0, \quad 1 - [p\delta] - [p\varepsilon] \leq k \leq 1 - [p\varepsilon].$$

Then, by (12), it follows that

$$(45) \quad H_{[p\varepsilon]+k}(W) = H_k(W^{([p\varepsilon])}), \quad 1 - [p\delta] - [p\varepsilon] \leq k \leq 1 - [p\varepsilon].$$

But it is easy to see that under $\mathbf{P}(\cdot | W_p = -1)$, $(W_i^{([p\varepsilon])}; 0 \leq i \leq 1 - [p\varepsilon])$ and $(W_i; 0 \leq i \leq 1 - [p\varepsilon])$ have the same law. Thus, by (45), we have, for any $a > 0$,

$$(46) \quad \begin{aligned} & \mathbf{P} \left(\sup_{1 - [p\delta] - [p\varepsilon] \leq k \leq 1 - [p\varepsilon]} |H_{[p\varepsilon]+k}(W) - H_p(W)| > a \mid W_p = -1 \right) \\ & \leq \mathbf{P}(A_p^c | W_p = -1) \\ & \quad + \mathbf{P} \left(\sup_{1 - [p\delta] - [p\varepsilon] \leq k \leq 1 - [p\varepsilon]} |H_k(W) - H_{p - [p\varepsilon]}(W)| > a \mid W_p = -1 \right). \end{aligned}$$

Recall that under $\mathbf{P}(\cdot | W_p = -1)$, the instant G_p is uniformly distributed on $\{1, \dots, p\}$. So,

$$(47) \quad \mathbf{P}(A_p^c | W_p = -1) \leq \delta + \varepsilon.$$

Set $\delta_p = [p\delta]/p$ and $\varepsilon_p = [p\varepsilon]/p$, and take $a = a_p \eta/p$ in (46) in order to get

$$(48) \quad \begin{aligned} & \mathbf{P} \left(\sup_{s \in [1 - \delta, 1]} |H_s^{\text{br}, p} - H_1^{\text{br}, p}| > \eta \right) \\ & \leq \mathbf{P} \left(\sup_{s \in [1 - \varepsilon_p - \delta_p, 1 - \varepsilon_p]} |H_s^{\text{br}, p} - H_{1 - \varepsilon_p}^{\text{br}, p}| > \eta \right) + \delta + \varepsilon. \end{aligned}$$

By Lemma 4.5, it follows that

$$\lim_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P} \left(\sup_{s \in [1 - \varepsilon_p - \delta_p, 1 - \varepsilon_p]} |H_s^{\text{br}, p} - H_{1 - \varepsilon_p}^{\text{br}, p}| > \eta \right) = 0.$$

Thus

$$\limsup_{\delta \rightarrow 0} \limsup_{p \rightarrow +\infty} \mathbf{P} \left(\sup_{s \in [1-\delta, 1]} |H_s^{\text{br}, p} - H_1^{\text{br}, p}| > \eta \right) \leq \varepsilon,$$

which yields the desired result by letting ε go to 0. \square

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ECOLE NORMALE SUPÉRIEURE DE CACHAN
C.M.L.A.
61, AVENUE DU PRÉSIDENT WILSON
94230 CACHAN
E-MAIL: thomas.duquesne@cmla.ens-cachan.fr