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THE EXISTENCE OF AN INTERMEDIATE PHASE FOR THE CONTACT PROCESS ON TREES

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Let $\mathbb{T}_d$ be a homogeneous tree in which every vertex has d neighbors. A new proof is given that the contact process on $\mathbb{T}_d$ exhibits two phase transitions when $d \geq 3$, a behavior which distinguishes it from the contact process on $\mathbb{Z}^n$. This is the first proof which does not involve calculation of bounds on critical values, and it is much shorter than the previous proof for the binary tree, $\mathbb{T}_3$. The method is extended to prove the existence of an intermediate phase for a more general class of trees with exponential growth and certain symmetry properties, for which no such result was previously known.

0. Introduction. The contact process was first introduced by Harris (1974) and has been greatly studied since then. The extended introduction of Liggett (1996) contains an up-to-date summary of some important results, as well as numerous references to books and survey papers where further information can be found.

The contact process on a graph, $G = (V, E)$, is a continuous-time Markov process ξ_t whose state space is the collection of subsets of V, with the following transition rates for each $x \in V$:

$$\xi_t \to \begin{cases} \xi_t \setminus \{x\}, & \text{at rate } 1, \\ \xi_t \cup \{x\}, & \text{at rate } \lambda(\#\{y \in \xi_t \colon xy \in E\}). \end{cases}$$

We usually think of ξ_t as the set of site which are *occupied* by particles which are *alive* (or *active* or *infected*) at time t. So we see that particles die at rate 1 and are born at a rate equal to the number of neighbors alive multiplied by some fixed parameter λ, with the restriction that no more than one particle may occupy a given site.

The process has been most widely studied on $\mathbb{Z}^d$, although the graphical representation of the process [see Harris (1978), Liggett (1985) or Durrett (1988)] shows that the process is well defined for more general graphs. We shall use ξ_t^A to denote the contact process with starting set A and use ξ_t^x as an abbreviation for $\xi_t^{\{x\}}$, where $x \in V$. Often x will be some distinguished vertex O (in the case of $\mathbb{Z}^d$, this is the origin). We then define two critical values:

$$\lambda_1 = \inf\{\lambda \colon \mathbb{P}_\lambda(\forall\, t,\, \xi_t^O \neq \varnothing) > 0\},$$
$$\lambda_2 = \inf\{\lambda \colon \mathbb{P}_\lambda\,(\forall\, T,\, \exists\, t \geq T \text{ s.t. } O \in \xi_t^O) > 0\}.$$

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Clearly $\lambda_1 \leq \lambda_2$. It is well known that on $\mathbb{Z}^d$ there is only one phase transition, so $\lambda_1 = \lambda_2$. This is a consequence of the complete convergence theorem, proved for $\mathbb{Z}^1$ by Durrett (1980) and more generally for $\mathbb{Z}^d$ by Bezuidenhout and Grimmett (1990). However, Pemantle (1992) examined the behavior of the contact process on a tree and he showed that the behavior can be quite different.

For $d \geq 2$, let $\mathbb{T}_d$ be the homogeneous tree of degree d. That is, $\mathbb{T}_d$ is the infinite tree in which every vertex has d neighbors. [Warning: Various different conventions are used; e.g., our $\mathbb{T}_3$ is denoted T_2 by Liggett (1996) and is called a tree of degree 2 by Pemantle (1992).] Pemantle (1992) obtained upper bounds on $\lambda_1(\mathbb{T}_d)$ and lower bounds on $\lambda_2(\mathbb{T}_d)$ which are good enough to show that $\lambda_1 < \lambda_2$ in the case $d \geq 4$. This shows that the contact process on these trees can exhibit an interesting behavior that does not occur (for the symmetric process) on $\mathbb{Z}^d$: if $\lambda_1 < \lambda < \lambda_2$, then there is a positive probability that the process starting from a single particle survives globally, but it drifts off to infinity and cannot survive locally. We call this *weak survival*. In this interesting intermediate phase Durrett and Schinazi (1995) showed that there are infinitely many extremal invariant measures. The existence of such an intermediate phase on a homogeneous tree is also known to imply [Madras and Schinazi (1992)] that the second phase transition is discontinuous. The existence of two phase transitions is generally regarded [see comments to this effect in Pemantle (1992) and Liggett (1996)] as the main interest in studying the contact process on a tree.

Pemantle (1992) conjectured that $\lambda_1 < \lambda_2$ for a much broader class of trees, and such a result seems to be widely believed to be true. Very recently, Liggett (1996) settled this question for homogeneous trees by proving that the contact process on $\mathbb{T}_3$ exhibits two phase transitions. (Note that this was the only *homogeneous* tree case left open, since $\mathbb{T}_2$ is isomorphic to $\mathbb{Z}^1$ for which $\lambda_1 = \lambda_2$.)

In solving the problem for $\mathbb{T}_3$, Liggett (1996) developed ingenious methods to obtain bounds on λ_1 and λ_2 which are precise enough to separate them. The ratio of the two critical values for $\mathbb{T}_d$ seems to get closer to 1 as d gets smaller, and the values become harder to separate numerically; in the case of $\mathbb{T}_3$, rather long calculations are required to show that $\lambda_1 \leq 0.605$ and that $\lambda_2 \geq 0.609$. In Section 1 of this paper we present a quite different proof that $\lambda_1 < \lambda_2$ for $\mathbb{T}_d$, $d \geq 3$. This new method does not give numerical bounds on the critical values (which might be considered to be a disadvantage), but it is much shorter and also more clearly based on our intuition that there is so much room on the tree that if the process barely survives, then it must drift off.

One further advantage of the new method is that it can be adapted to prove the existence of an intermediate phase for a more general class of trees, and this is done in Section 2. It seems very unlikely that the same result could be obtained by giving precise numerical bounds on λ_1 and λ_2 for each of the trees in the class: accurate upper bounds on critical values are usually very hard to obtain and their precision is significantly limited by the complexity of the calculations that can be carried out. Even with a great deal of work, the upper bound of 0.605 that Liggett (1996) gives for $\lambda_1(\mathbb{T}_3)$ is some way from the non-rigorous estimate of $\lambda_1 \approx 0.542$ by Tretyakov and Konno (1996). Moreover, the computations used to obtain this upper bound are specific to $\mathbb{T}_3$, and they would need to be reworked with an arbitrarily high degree of precision to cover other trees.

1. Homogeneous trees. In this section we shall consider the tree $\mathbb{T}_d$, in which every vertex has degree d, $d \geq 3$, and in which there is a distinguished vertex O, called the *root*. Although it is customary to consider all the other vertices as lying below the root, it will greatly simplify some of our calculations if we arrange the tree so that *every* vertex has one neighbor above it and $d-1$ neighbors below it. We can then assign a *level* to each vertex in such a way that the root has level 0 and any vertex in level l has one neighbor in level $l-1$ and $d-1$ neighbors in level $l+1$. For $n \in \mathbb{Z}$, we shall use $\mathscr{L}_n$ to denote the set of all vertices in level n. Of course, each set $\mathscr{L}_n$ is infinite. We use $l(x)$ to denote the level of a vertex x.

Having arranged the vertices in levels, we now define the *weight* of a vertex x by

$$w_\alpha(x) = \alpha^{l(x)}, \tag{1.0}$$

where $\alpha > 0$ is to be specified later; we shall often use $w(x)$ as an abbreviation for $w_\alpha(x)$. The weight of a set of vertices is defined to be the sum of the weights of all the vertices in the set. This arrangement of the tree and assignment of weights appears in Liggett (1996).

Now the tree seen from a vertex, say x_n, at level n, looks exactly like the tree seen from the root with all the weights multiplied by a factor of α^n. Hence if ξ_t^x is the contact process on $\mathbb{T}_d$ starting from a vertex x, then a very simple coupling argument shows that $\mathbb{E}(w(\xi_t^{x_n})) = \alpha^n \mathbb{E}(w(\xi_t^O))$, or equivalently, for any vertex x,

$$\mathbb{E}\big(w(\xi_t^x)\big) = w(x)\mathbb{E}\big(w(\xi_t^O)\big). \tag{1.1}$$

Of course, it is exactly this property which will enable us to set up a useful supermartingale. The additivity of the contact process [see, for instance, Liggett (1985)] easily implies that for any set of vertices, A,

$$\mathbb{E}\big(w(\xi_t^A)\big) \leq \sum_{x \in A} \mathbb{E}\big(w(\xi_t^x)\big). \tag{1.2}$$

For clarity in what follows let $f_s(A) = \mathbb{E}(w(\xi_s^A))$, so if A is a random set, then $f_s(A)$ is a random variable. Now let $\mathscr{F}_t = \sigma(\xi_u^o\colon u \leq t)$. Then we see fairly easily that for any $s, t \geq 0$,

$$\begin{aligned} \mathbb{E}\big(w(\xi_{t+s}^o) \mid \mathscr{F}_t\big) &= f_s(\xi_t^o) \\ &\leq \sum_{x \in \xi_t^o} \mathbb{E}\big(w(\xi_s^x)\big) \\ &= \sum_{x \in \xi_t^o} w(x)\mathbb{E}\big(w(\xi_s^o)\big) \\ &= w(\xi_t^o)\mathbb{E}\big(w(\xi_s^o)\big), \end{aligned} \tag{1.3}$$

since the first equality follows from the Markov property, the inequality is an application of (1.2) and the next equality an application of (1.1).

These straightforward observations about the behavior of the weighting function enable us to establish the following result which will be central to the proof of the main theorem of this section.

PROPOSITION 1.0. *Let* ξ_t^o *be the contact process with parameter* λ *on a homogeneous tree and suppose that for some* $t_0 > 0$ *and some weighting function* w_α *of the form given by* (1.0),

$$\mathbb{E}\big(w_\alpha(\xi_{t_0}^o)\big) = \beta < 1. \tag{1.4}$$

Then

$$\mathbb{P}(\exists\, T\, s.t.\, \forall\, t \geq T, O \notin \xi_t^o) = 1, \tag{1.5}$$

so, a fortiori, $\lambda_2 \geq \lambda$.

PROOF. Let $X_n = w(\xi_{nt_0}^o)$ and let $\tilde{\mathscr{F}}_n = \mathscr{F}_{nt_0}$. Then (1.3) demonstrates that

$$\mathbb{E}\big(X_{n+1} | \tilde{\mathscr{F}}_n\big) \leq X_n \mathbb{E}\big(w(\xi_{t_0}^o)\big),$$

so condition (1.4) shows that $(\beta^{-n} X_n)$ is a supermartingale [relative to $(\tilde{\mathscr{F}}_n)$]. Since it is non-negative it converges almost surely, so it is certainly bounded a.s. Since $\beta < 1$, this implies that

$$X_n \to 0 \quad \text{a.s.}$$

However, if $X_n \to 0$ then, because $w(O) = 1$, for sufficiently large n we must have $O \notin \xi_{nt_0}^O$. This virtually establishes (1.5). Suppose now that (1.5) fails to hold. Then there is a positive probability, p say, that for infinitely many values of n, there exists some $t \in ((n-1)t_0, nt_0)$ with $O \in \xi_t^O$. However, for each such t, $\mathbb{P}(O \in \xi_{nt_0}^O | \mathscr{F}_t) \geq e^{-t_0}$; since this bound does not depend on n, it is straightforward to define an appropriate sequence of stopping times and apply a generalized Borel–Cantelli lemma [see, e.g., Williams (1991), page 124] to show that, with probability p, $O \in \xi_{nt_0}^O$ infinitely often. This contradicts our previous conclusion and completes the proof. □

We shall also need one technical result about the behavior of the weighting function.

LEMMA 1.1. *Let* $\xi_t^{O,\lambda}$ *be the contact process with parameter* λ *on* $\mathbb{T}_d$ ($d \geq 3$), *let* $w = w_\alpha$ *be a weighting function as above and let* T *be some fixed time. Then the function*

$$\lambda \to \mathbb{E}\big(w(\xi_T^{O,\lambda})\big)$$

is continuous.

Note that although results about the continuity of certain expectations and probabilities, as functions of various parameters, are rather standard, some care is required. If the weighting function were very fast growing, then the conclusion might fail to hold. The proof which we shall give for the homogeneous tree can be easily adapted for general graphs of bounded degree with weighting functions which grow no faster than exponentially.

PROOF OF LEMMA 1.1. Fix some large Λ; we shall prove continuity for $\lambda \le \Lambda$. We recall the graphical representation of the contact process in which, for any two neighboring vertices x and y, arrows (representing births) are drawn from x to y at the arrival times of a Poisson process of rate λ.

Let x be a vertex of $\mathbb{T}_d$ at distance n from the root. Let the unique (non-self-intersecting) path from O to x be $x_0 x_1 \cdots x_n$ (so $x_0 = O$ and $x_n = x$). Let a *birth path from O to x* be a sequence of times $0 \le t_1 \le \cdots \le t_n$ such that at time t_i there is an arrow from x_{i-1} to x_i in the graphical representation. If site x is occupied at or before time T, then there must exist a birth path from O to x with $t_n \le T$. Now the expected number of such paths is easily computed as

$$\int \cdots \int_{0 \le t_1 \le \cdots \le t_n \le T} \lambda^n \, dt_1 \cdots dt_n = \frac{\lambda^n T^n}{n!}.$$

So for any $\lambda \le \Lambda$, the probability that x is occupied at or before time T is at most $\Lambda^n T^n / n!$. We shall use this fact to show that we can restrict attention to a finite graph.

Let $\varepsilon > 0$ be given. Let $\mathscr{D}_n$ be the set of vertices at distance n from the root. Unlike $\mathscr{L}_n$, which is infinite, $|\mathscr{D}_n| = d(d-1)^{n-1}$. Note that $D_n \subset \bigcup_{k=-n}^{n} \mathscr{L}_k$. Let $\gamma = \max(\alpha, 1/\alpha)$, so that any vertex in $\mathscr{D}_n$ has weight at most γ^n. Let $\mathscr{B}_n$ be the set of vertices at distance less than n from the root. The probability that any vertex outside $\mathscr{B}_n$ is occupied before time T is at most $|\mathscr{D}_n| \Lambda^n T^n / n!$. Since $w(\mathscr{B}_n) \le \gamma^n d(d-1)^{n-1}$, if N is sufficiently large then

$$w(\mathscr{B}_N) \frac{|\mathscr{D}_N| \Lambda^N T^N}{N!} \le \varepsilon. \tag{1.6}$$

Likewise, using the fact that $w(\mathscr{D}_n) \le \gamma^n d(d-1)^{n-1}$, if N is sufficiently large then

$$\sum_{n \ge N} w(\mathscr{D}_n) \frac{\Lambda^n T^n}{n!} \le \varepsilon. \tag{1.7}$$

Choose N so that both (1.6) and (1.7) hold. Let $\bar{\xi}_t^{O,\lambda}$ be the contact process, with parameter $\lambda < \Lambda$, restricted to the induced subgraph with vertex set $\mathscr{B}_N$. We can couple $\bar{\xi}_t^{O,\lambda}$ and $\xi_t^{O,\lambda}$ together in a natural way so that they are equal at time T if $\xi_t^{O,\lambda}$ has not had a birth outside $\mathscr{B}_N$ before then. We therefore see that (1.6) implies that

$$\left| \mathbb{E}\big(w\big(\mathscr{B}_N \cap \bar{\xi}_T^{O,\lambda}\big)\big) - \mathbb{E}\big(w\big(\mathscr{B}_N \cap \xi_T^{O,\lambda}\big)\big) \right| \le \varepsilon.$$

Also, (1.7) implies that

$$\mathbb{E}\big(w\big(\xi_T^{O,\lambda} \setminus \mathscr{B}_N\big)\big) \le \varepsilon,$$

so for any $\lambda \le \Lambda$,

$$\left|\mathbb{E}\left(w\left(\xi_T^{O,\lambda}\right)\right) - \mathbb{E}\left(w\left(\bar{\xi}_T^{O,\lambda}\right)\right)\right| \le 2\varepsilon. \tag{1.8}$$

Using a standard technique, we can couple together two processes on a finite graph, $\bar{\xi}_t^{O,\lambda}$ and $\bar{\xi}_t^{O,\mu}$, in such a way that, at time T, the processes differ with a probability which tends to zero as $\mu \to \lambda$. In this way we can show that $\mathbb{E}(w(\bar{\xi}_T^{O,\lambda}))$ is a continuous function of λ; combining this fact with (1.8) completes the proof. □

The other main ingredient which we shall need in our proof is a result of Morrow, Schinazi and Zhang (1994), which builds on previous work of Madras and Schinazi (1992).

THEOREM 1.2 [Morrow, Schinazi and Zhang (1994)]. *Let ξ_t^O be the contact process on $\mathbb{T}_d$ for $d \ge 3$. Then, at the critical value λ_1 we have*

$$1 \le \mathbb{E}\left(|\xi_t^O|\right) \le C(d),$$

where $C(d)$ is a constant depending only on d.

Let us now state and prove the main result of this section.

THEOREM 1.3. *For the contact process on a homogeneous tree $\mathbb{T}_d$, in which every vertex has degree $d \ge 3$, we have*

$$\lambda_1 < \lambda_2.$$

PROOF. Let ξ_t^O be the contact process at the first critical value, λ_1. Let $w(\cdot)$ be the weighting function as defined by (1.0) with a choice of $\alpha = 1/\sqrt{d-1}$. We consider how $w(\xi_t^O)$ behaves as t increases; roughly speaking the idea is that the expected number of particles remains bounded, but because they must spread out, the expected weight decreases.

As in Lemma 1.1, let $\mathscr{D}_n$ be the set of $d(d-1)^{n-1}$ vertices at distance n from the root. Rather trivially we have

$$\mathbb{E}\left(w\left(\xi_t^o\right)\right) = \sum_{n \ge 0} \sum_{x \in \mathscr{D}_n} w(x)\mathbb{P}\left(x \in \xi_t^O\right). \tag{1.9}$$

Let a_n be the average weight of a site in $\mathscr{D}_n$, that is, $a_n = w(\mathscr{D}_n)/|\mathscr{D}_n|$. By symmetry, all the sites in $\mathscr{D}_n$ are equally likely to be occupied at time t, and we make use of this fact to rewrite (1.9) as

$$\mathbb{E}\left(w\left(\xi_t^o\right)\right) = \sum_{n \ge 0} a_n \mathbb{E}\left(|\xi_t^O \cap \mathscr{D}_n|\right). \tag{1.10}$$

It is straightforward to evaluate a_n precisely: $\mathscr{D}_n$ contains sites in $\mathscr{L}_n$, $\mathscr{L}_{n-2}, \dots, \mathscr{L}_{-n}$ with

$$
\begin{aligned}
|\mathscr{D}_n \cap \mathscr{L}_n| &= (d-1)^n,\\
|\mathscr{D}_n \cap \mathscr{L}_{n-2}| &= (d-2)(d-1)^{n-2},\\
|\mathscr{D}_n \cap \mathscr{L}_{n-4}| &= (d-2)(d-1)^{n-3},\\
&\vdots\\
|\mathscr{D}_n \cap \mathscr{L}_{n+2-2i}| &= (d-2)(d-1)^{n-i}, \qquad (1.11)\\
&\vdots\\
|\mathscr{D}_n \cap \mathscr{L}_{-n+4}| &= (d-2)(d-1)^{1},\\
|\mathscr{D}_n \cap \mathscr{L}_{-n+2}| &= (d-2)(d-1)^{0},\\
|\mathscr{D}_n \cap \mathscr{L}_{-n}| &= (d-1)^{0}.
\end{aligned}
$$

Note that the pattern varies slightly at the start and finish.

So we see from our choice of $\alpha = 1/\sqrt{d-1}$ that all the levels make precisely the same contribution of $(d-2)(d-1)^{(n-2)/2}$ to $w(\mathscr{D}_n)$ except for the first and last levels, which each contribute $(d-1)^{n/2}$. Adding all these contributions shows that (for $n \geq 1$),

$$w(\mathscr{D}_n) = (2(d-1) + (n-1)(d-2))(d-1)^{(n-2)/2}.$$

Dividing by the number of vertices in $\mathscr{D}_n$ we obtain

$$a_n = \frac{2(d-1) + (n-1)(d-2)}{d(d-1)^{n/2}}.$$

So (using the fact that $d \geq 3$) $a_n \to 0$ as $n \to \infty$.

We shall now return to estimating the quantity on the r.h.s. of (1.10). Let $\varepsilon > 0$. Choose N such that if $n \geq N$, then $a_n \leq \varepsilon / C(d)$, where $C(d)$ is the constant appearing in Theorem 1.2. Then we split up the r.h.s. of (1.10) as

$$\sum_{0 \leq n < N} \sum_{x \in \mathscr{D}_n} w(x) \mathbb{P}\big(x \in \xi_t^O\big) + \sum_{n \geq N} a_n \mathbb{E}\big(|\xi_t^O \cap \mathscr{D}_n|\big). \qquad (1.12)$$

By Theorem 1.2 and the choice of N, the second term is at most ε for any value of t. The fact that the process dies out at λ_1 (an easy corollary to Theorem 1.2) implies that for any x, $\mathbb{P}(x \in \xi_t^o) \to 0$ as $t \to \infty$ (in fact, this happens uniformly in x). Consequently, for t sufficiently large, the first term in (1.12) is at most ε. So, for some value t_0 we see that

$$\mathbb{E}\Big(w\big(\xi_{t_0}^O\big)\Big) \leq 2\varepsilon.$$

This reasoning applied to the contact process at λ_1. However, we now apply Lemma 1.1 to obtain that for some $\lambda^* > \lambda_1$ we have

$$\mathbb{E}\Big(w\big(\xi_{t_0}^{O,\lambda^*}\big)\Big) \leq 3\varepsilon.$$

Since ε was arbitrary we can choose it in such a way that $3\varepsilon < 1$. We can then apply Proposition 1.0 to show that $\lambda_1 < \lambda^* \le \lambda_2$, which completes the proof. □

2. Inhomogeneous trees. The only trees for which it is already known that two phase transitions exist are the homogeneous trees in which every vertex has degree $d \ge 3$. As indicated in the Introduction, this was first proved by Pemantle (1992) for $d \ge 4$ and by Liggett (1996) for $d = 3$. It is reasonable to suppose, however, that for a much larger class of graphs there is enough room for the process to move faster than it grows and for an intermediate phase to exist. Pemantle (1992) defines a class of trees with *strongly exponential growth* and conjectures that, for the contact process on such trees, either $\lambda_1 = \lambda_2 = 0$ (which will happen if the tree grows too fast) or $\lambda_1 < \lambda_2$. Madras and Schinazi (1992) also make a conjecture of this kind, which relates the behavior of the contact process on a graph to the behavior of the branching random walk.

We shall constructively describe a class of graphs for which we can extend Theorem 1.3 and show that $\lambda_1 < \lambda_2$. Let $H = (V, E)$ be a finite graph with distinguished vertices $u_1, \dots, u_h$, where $h \ge 2$. Say that a map $\pi \colon V \to V$ is a strong automorphism of H if it is a graph automorphism in the usual sense and also $\pi(\{u_1 \cdots u_h\}) = \{u_1 \cdots u_h\}$. We call H an *isotropic block* if the following two conditions hold.

CONDITION 1. Given distinct $i, j, k \le h$, there exists a strong automorphism of H, π, for which $\pi(u_i) = u_i$ and $\pi(u_j) = u_k$.

CONDITION 2. Given $i, j \le h$, there exists a strong automorphism of H, π, with $\pi(u_i) = u_j$.

Note that unless $h = 2$, Condition 1 implies Condition 2. Note further that we do not require $u_1, \dots, u_j$ to be distinct, although if two of them are the same, then Condition 1 implies that they all are. These two conditions look rather strange; we explain their significance once we have explained how these isotropic blocks are used to construct infinite graphs. In practice, most of the examples one thinks of, and all of the small examples which we shall give, satisfy the following stronger and simpler Condition S which is easily seen to imply Conditions 1 and 2.

CONDITION S. Given any $\rho \in S_h$, the symmetric group on $\{1, \dots, h\}$, there exists an automorphism of H, π, with $\pi(v_i) = v_{\rho(i)}$ for all $i \le h$.

The graphs which we shall consider will be built up of isotropic blocks in the following way. Let H and K be two isotropic blocks with distinguished vertices $u_1, \dots, u_h$ and $v_1, \dots, v_k$, respectively, and—to avoid an absurd example—suppose that at least one of these lists consists of distinct vertices. Let $G_0(H, K)$ be a copy of H. Obtain $G_1(H, K)$ by taking h copies of K and identifying v_1 in the ith copy of K with u_i in G_0. To obtain $G_2(H, K)$, we take each "unused" distinguished vertex (all the v_i's for $i \geq 2$) and identify it with a vertex u_1 in a new copy of H; so we are adding $h(k-1)$ new copies of H to the graph. Repeat this process, so that at stage $2n$ we are adding (by vertex identification) a separate copy of H to each of the $h(h-1)^{n-1}(k-1)^n$ unused distinguished vertices, and at stage $2n+1$ we are adding $h(h-1)^n(k-1)^n$ copies of K. We obtain an infinite graph, $G = G(H, K)$, as a limit. Examples are given after the statement of the main result of this section.

In the graph $G(H, K)$, each copy of H is attached to h copies of K (one at each distinguished vertex, $u_1, \dots, u_h$) and each copy of K is attached to k copies of H. If H and K are both trees, then G will also be a tree. In the important special case where H is a singleton (so $u_1 = \cdots = u_h$) we are really just attaching each copy of K to $h-1$ other copies of K at each distinguished vertex.

Another way to view $G(H, K)$ is by arranging it as a rooted tree in such a way that each block of type H has one block of type K as its parent and $h-1$ such blocks as children. Each block of type K will have one block of type H as a parent and $k-1$ such blocks as children. This view will be of use later in our proof.

A graph constructed in the above fashion, $G = G(H, K)$, we shall call an *isotropic block tree*. Although this construction is slightly cumbersome, it does have the advantage of sufficient generality to include a number of significant examples, while still ensuring that enough of the useful properties of the homogeneous trees are retained. Condition 2 of isotropic blocks ensures that isotropic block trees are periodic in the strong sense that if $G = G(H, K)$ and if H_1 and H_2 (or, likewise, K_1 and K_2) are two copies of H used in the construction of G, then there is an automorphism of G taking H_1 to H_2; in other words, the graph looks the same seen from any copy of H. Condition 1 of isotropic blocks says that if we hold one distinguished vertex fixed, any other two distinguished vertices are interchangeable. We can think of the tree as having branches hanging off the distinguished vertices, and with a little thought one can see that Condition 1 ensures that isotropic block trees are indeed isotropic in the sense that any path of blocks leading away from, say, the first copy of H, to infinity, looks the same.

In order to show that $\lambda_1 < \lambda_2$ we shall need the number of blocks added at stage n (in the construction of G) to grow exponentially with n. With this in mind, we say that $G(H, K)$ has *exponential growth* if $(h-1)(k-1) \geq 2$. We are now in a position to state our general result.

THEOREM 2.0. *Let G be an isotropic block tree with exponential growth. Then the contact process on G has an intermediate phase in the sense that*

$$\lambda_1 < \lambda_2.$$

This theorem, and the results leading up to it can be proved in a similar way to the corresponding results for homogeneous trees, although extra work is needed. Before discussing this matter in more detail, let us give some examples of isotropic block trees.

EXAMPLE 1. Let H be a single vertex and let K be a path of length m (i.e., K has m edges) with its two endvertices distinguished. Then $G(H, K)$ is the same graph as that obtained from the homogeneous tree of degree h by replacing each edge by a path of length m. Of course, it has exponential growth if $h \geq 3$.

EXAMPLE 2. Let H be as in Example 1 and let K be a star in which a central vertex is joined by edges to k outer vertices. The outer vertices of K are all distinguished. Then $G(H, K)$ is a graph which can be arranged so that every vertex has one parent and in which all the vertices in the same generation have the same number of children, this number alternating between h and k.

EXAMPLE 3. Let H be a triangle with all three vertices distinguished and let K be a star with the k outer vertices distinguished. This gives a simple example of a graph which cannot be described by a single isotropic block, yet which has sufficiently good properties to show $\lambda_1 < \lambda_2$.

The first of these families of examples is of interest because it is very unlikely that one could use explicit bounds on λ_1 and λ_2 to show that all the trees in the family exhibit an intermediate phase. For fixed h, as the length of the path, m, tends to infinity, it seems reasonable to suppose that the critical probability for survival (λ_1) converges to that for $\mathbb{Z}_1$, $\lambda_c(\mathbb{Z}_1)$. Since $\lambda_1 \leq \lambda_2 \leq \lambda_c(\mathbb{Z}_1)$, λ_1/λ_2 will be converging to 1 as $m \to \infty$ and hence the critical values will become increasingly hard to separate. This is consistent with the fact that a homogeneous tree of degree d looks less and less like $\mathbb{Z}_1$ as $d \to \infty$ and the critical values are hardest to separate numerically in the case of $\mathbb{T}_3$. They become easier to separate as $d \to \infty$ and, in fact, it is known [Pemantle (1992)] that, as $d \to \infty$, $\lambda_1(\mathbb{T}_d)/\lambda_2(\mathbb{T}_d) \to 0$.

Let us make one more observation about the graphs we have constructed. As seen from the starting block $G_0(H, K)$, the graph $G(H, K)$ looks the same as it does seen from any block which is a copy of H. However, sometimes one considers rooted trees where the situation is slightly different. For example, consider the simple rooted tree mentioned in Pemantle (1992) in which one starts with a root, the root has n children each of which has only one child. Each of the n grandchildren of the root has n children and so on, with the number of children per parent alternating between 1 and n from generation to generation. Call this tree R_n. If this tree were continued backward appropriately, then we would obtain a tree, R'_n, say, which is a special case of Example 2 (and also a special case of Example 1), but R_n itself is not of the form $G(H, K)$: in particular, the root is the only vertex of degree n. However, the fact that R'_n has an intermediate phase implies that the same must be true for R_n. Consider the contact proces on R'_n in a phase of weak survival: it can survive but drifts off to infinity, so it must be able to survive when restricted to any single branch of the tree (where a branch, loosely speaking, is obtained by severing the tree at one edge or vertex), so in particular when restricted to R_n. Yet, if it cannot survive strongly on the whole tree, it certainly cannot survive strongly on a branch, so it must exhibit weak survival when restricted to a branch. [This type of argument is easy to make precise; see Pemantle (1992), Lemma 6.5.] Hence the rooted tree R_n has an intermediate phase too.

Let us remark that the simplicity of the argument sketched in the previous paragraph—to show that survival on the tree implies survival on a branch *under the assumption that there is an intermediate phase of weak survival*—is, in a sense, slightly misleading. Without the assumption, it is rather harder to show this result; it is proved in Morrow, Schinazi and Zhang (1994) for homogeneous trees, and their proof can be extended to give a proof for isotropic block trees. This more difficult result (i.e., without the assumption of a phase of weak survival) is still necessary since the result is needed in their proof of Theorem 1.2 and hence is an ingredient in our proof of the existence of the intermediate phase.

We now turn to consider the proof of Theorem 2.0. To begin, analogues of Proposition 1.0, Lemma 1.1 and Theorem 1.2 need to be proven in the more general setting of isotropic block trees. We shall not give all the details since the modifications required are of a similar flavor in each case. However, in order to show the kind of work one needs to do, we will give the essential details for one result, the analogue of a proposition of Madras and Schinazi (1992) which preceded the proof of Theorem 1.2 [Morrow, Schinazi and Zhang (1994)].

PROPOSITION 2.1. *Let $G = G(H, K)$ be an isotropic block tree with exponential growth. Let A be the vertex set of the graph $G_1(H, K)$, that is, the graph obtained from one copy of H and h copies of K. Let ξ_t^A be the contact process on G with parameter λ and starting set A. Then there exist constants α and C such that for all $t \geq 0$,*

$$\exp(\alpha t) \leq \mathbb{E}\left(|\xi_t^A|\right) \leq C \exp(\alpha t). \tag{2.0}$$

Moreover, α is a continuous function of λ and, for fixed $b \geq a > 0$, we can use the same value of C for all $\lambda \in [a, b]$.

PROOF. Let $m_t = \mathbb{E}(|\xi_t^A|)$. Using the fact that every vertex of G is equivalent (via an automorphism) to some vertex of A, and using the additivity of the process, we see that

$$m_{t+s} \leq m_t m_s.$$

A standard subadditivity argument leads us to conclude that

$$\lim_{t\to\infty} \frac{1}{t} \log m_t \quad \text{exists and equals} \quad \inf_{t>0} \frac{1}{t} \log m_t. \tag{2.1}$$

Call this limit α. It is immediate from (2.1) that $\mathbb{E}(m_t) \geq e^{\alpha t}$ for all t, establishing the first inequality of (2.0).

The upper bound on m_t is more involved. We shall consider the way the tree $G(H, K)$ is built up from isotropic blocks of two *types*, H and K. We shall say two blocks are adjacent if a vertex from each was identified in the construction of G. So each block of type H is adjacent to exactly h blocks of type K, and each block of type K is adjacent to exactly k blocks of type H.

It is easy to see that if the upper bound in (2.0) holds, it holds (with a different constant) when A is replaced by any other finite set. So, interchanging the roles of H and K if necessary, we may assume that the distinguished vertices of H, $u_1, \dots, u_h$, are distinct. This ensures that different blocks of type K are disjoint, which will be convenient.

Now suppose that there are n particles at time t, that is, $|\xi_t^A| = n$. Letting $c_1 = \max(|H|, |K|)$, at least n/c_1 different blocks contain a particle. It is not hard to see that this implies that at least $n/2c_1$ blocks of type H either contain a particle or have an adjacent block of type K which contains a particle. Among these, we can find a collection of at least $n/2kc_1$ blocks of type H such that no two of them are adjacent to the same block of type K. Looked at another way, there are at least $n/2kc_1$ *disjoint* copies of A each of which contains at least one occupied site. We can choose a constant c_2 so that if the contact process is run on a copy of A (only) starting with just one occupied site, then with probability at least c_2, the whole of A is occupied at time 1. In order to put these facts together, we introduce one more piece of notation: given a finite set of sites, S, let $a(S)$ be the maximum number of disjoint copies of A that we can find all of whose vertices are in S. Then the preceding argument, together with the additivity of the process, shows that

$$\mathbb{E}\big(a(\xi_{t+1}^A)\big) \geq \frac{c_2}{2kc_1}\mathbb{E}\big(|\xi_t^A|\big). \tag{2.2}$$

When we go from $G_1(H, K)$ to $G_2(H, K)$ in the construction of G, we add $h(k-1)$ copies of H to A. Generally these copies are disjoint, but if all the distinguished vertices of K are the same, then some of these copies will have vertices in common. In any case, let H' be any one of these $h(k-1)$ copies of H. Let the set of all those vertices which can only be reached from $G_0(H, K)$ by passing through H' be called a *branch adjacent to* A. So there are $h(k-1)$ branches adjacent to A, all of them isomorphic, and the vertex set of G is the union of A and all these branches; apart from the “outer” vertices of A, this is a disjoint union. If A' is any copy of A we can also consider the $h(k-1)$ branches adjacent to A'.

Let B be the union of A, together with b of its branches. It is not hard to see by isotropy that, for any s,

$$\mathbb{E}\big(|\xi_s^A \cap B|\big) \geq \frac{b}{h(k-1)}\mathbb{E}\big(|\xi_s^A|\big). \tag{2.3}$$

Now suppose at time $t+1$ we pick $a(\xi_{t+1}^A)$ disjoint copies of A all of whose sites are occupied and delete all the other particles on the tree. An easy edge-counting argument shows that there must be at least $(h(k-1)-2) \times a(\xi_{t+1}^A)$ branches which are adjacent to occupied copies of A but which are themselves unoccupied (with the exception of the one vertex they share with A). The latter condition on the branches ensures that any two such branches are disjoint (with the possible exception of a single vertex shared between the two branches and the same copy of A).

Consider running the process from one of the occupied copies of A which has b unoccupied branches. Then (2.3) implies that, after a time s, the expected number of particles on A and those b branches is at least $bm_s/h(k-1)$. Any two different copies of A which are both occupied at time $t+1$ have disjoint unoccupied branches, so additivity and the argument of the preceding paragraph show that

$$m_{t+s+1} \geq \frac{(h(k-1)-2)\mathbb{E}\big(a\big(\xi_{t+1}^A\big)\big)}{h(k-1)} m_s. \tag{2.4}$$

Combining this with (2.2) we see that

$$m_{t+s+1} \geq \frac{c_2}{2kc_1} \frac{h(k-1)-2}{h(k-1)} m_s m_t.$$

Since G has exponential growth, $h(k-1)-2 \geq 1$ and so we see that, for some $c_3 > 0$,

$$m_{t+s+1} \geq c_3 m_s m_t. \tag{2.5}$$

Letting $\tilde{m}_t = c_3 m_t / e^{\alpha t} e^{\alpha}$, (2.5) becomes

$$\tilde{m}_{t+s+1} \geq \tilde{m}_t \tilde{m}_s. \tag{2.6}$$

Equation (2.1) implies that $\lim_{t\to\infty}(1/t)\log \tilde{m}_t = 0$. From this and (2.6) we see that we must have $\tilde{m}_t \leq 1$ for all t and hence

$$m_t \leq \frac{e^{\alpha}}{c_3} e^{\alpha t},$$

establishing the second inequality of (2.0).

The dependency of C on λ is via the choice of c_2 and the term e^{α}; both of these terms can be uniformly bounded for all $\lambda \in [a, b]$ in such a way that we can make (2.0) hold with the same choice of C, as claimed. Finally, the continuity of α as a function of λ follows exactly as in Madras and Schinazi (1992): (2.0) can be used to express α as both the supremum and the infimum of families of continuous functions. □

The above proof is an adaptation of that of Madras and Schinazi (1992) with substantial technical modifications to deal with the extra complexity of the general isotropic block tree. One can make similar modifications to the proof of Morrow, Schinazi and Zhang (1994) to show that, at the critical value λ_1, the exponent appearing in Proposition 2.1, α, is equal to 0. This establishes an analogue of Theorem 1.2 for isotropic block trees.

As indicated in the remarks preceding its proof, it is not hard to check that Lemma 1.1 holds fairly generally, although the proof needs some modifications; we omit the straightforward details.

In order to state and prove a more general version of Proposition 1.0, and prove Theorem 2.0, it is necessary to define a weighting function for an isotropic block tree. As remarked earlier, given such a tree, $G = G(H, K)$, we can arrange it so that each block of type H has one block of type K as its parent and $h - 1$ such blocks as children. Each block of type K has one block of type H as a parent and $k - 1$ such blocks as children. We assign levels to the blocks in such a way that the levels only change every other generation: the initial copy of H has level 0 and each copy of H at level i has $(h-1)(k-1)$ grandchildren at level $i+1$ and one grandparent at level $i-1$. The level of a block of type K is declared to be equal to the level of the blocks of type H immediately below it. Given a block B with level $l(B)$ (B is regarded as a subgraph of G; it is either a copy of H or a copy of K), we define its weight to be

$$w_\alpha(B) = \alpha^{l(B)}.$$

The weight of a set of vertices, S, is defined to be the sum of the weights of all the blocks whose intersection with S is non-empty. Note that this is rather generous, since some vertices belong to more than one block. Having made these definitions, it is easy to establish the following result, whose proof is a very slight extension of the proof of Proposition 1.0.

PROPOSITION 2.2. *Let $G = (H, K)$ be an isotropic block tree arranged in the manner described above and let w_α be a weighting function. Let B consist of those vertices in the initial copy of H, $G_0(H, K)$, together with those vertices in the copy of K immediately above it in the arrangement of $G(H, K)$. Let ξ_t^B be the contact process on G with parameter λ and starting set B. Suppose that for some $t_0 > 0$,*

$$\mathbb{E}\Big(w_\alpha\big(\xi_{t_0}^B\big)\Big) = \beta < 1.$$

Then

$$\mathbb{P}\big(\exists\, T\, s.t.\, \forall\, t \geq T,\, B \cap \xi_t^B = \varnothing\big) = 1,$$

so, a fortiori, $\lambda_2 \geq \lambda$.

With all the ingredients in place, it is very easy to imitate the proof of Theorem 1.3 [with a choice of $\alpha = 1/\sqrt{(h-1)(k-1)}$] to prove Theorem 2.0, although the precise details are rather more tedious and we omit them. Let us also comment that one consequence of Theorem 2.0 is that all isotropic block trees with exponential growth have a type of discontinuous second phase transition; the proof of this is an easy adaptation of the corresponding proof for homogeneous trees [Madras and Schinazi (1992)].

The proof that we have given for the existence of two phase transitions seems to rely strongly on properties of periodicity and isotropy possessed by the graphs in question. As we have already commented, an intermediate phase should exist for a much broader class of graphs. As well as making a conjecture to this effect, Pemantle (1992) discusses two specific types of trees for which the result should hold. One is a class of *periodic trees*, constructed in a similar way to our construction of isotropic block trees. However, there are many examples of these periodic trees which are not sufficiently isotropic to look like an isotropic block tree (or even like a branch of an isotropic block tree). One of the simplest examples is a rooted tree in which all vertices in the same generation have the same number of offspring, that number cycling from two to three to four and back to two again. The fact that we cannot prove $\lambda_1 < \lambda_2$ even for such a simple graph indicates a significant limitation on the power of our current techniques.

The second particular class of graphs discussed by Pemantle (1992) consists of the Galton–Watson trees generated by a branching process. Although any one such tree is likely to be highly irregular, the space of all such trees (for a given branching process) is stochastically regular in the sense that the descendants of any two particles have the same distribution. Given a branching process with a positive chance of survival, it is not hard to see that, conditional on survival, the critical values are deterministic (i.e., the same for almost all trees generated by the process) so one can talk about critical values for the branching process itself [see Pemantle (1992) for one result along these lines]. We have not been able to show that $\lambda_1 < \lambda_2$ for any (non-trivial) branching process, but it is hoped that the techniques of this paper will prove to be a useful tool in doing so.

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