The Annals of Probability
1997, Vol. 25, No. 4, 1895–1952

ON THE ERROR ESTIMATE OF THE INTEGRAL KERNEL FOR THE TROTTER PRODUCT FORMULA FOR SCHRÖDINGER OPERATORS

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The error estimate of the integral kernel for the Trotter product formula for Schrödinger operators is shown. A basic tool for doing so is the Feynman–Kac formula based on the pinned Brownian motion. This formula enables us to express the integral kernel in handleable form and hence estimate it. As a consequence the Trotter product formula in the L_p operator norm is obtained.

1. Introduction and results. It is well known (e.g., [8]) that if A and B are self-adjoint operators in a Hilbert space bounded below and $A + B$ is essentially self-adjoint, the Trotter product formula

$$\lim_{n\to\infty}\left(\exp\left(-\frac{t}{n}A\right)\exp\left(-\frac{t}{n}B\right)\right)^n = \exp(-tC) \tag{1.1}$$

and its variant

$$\lim_{n\to\infty}\left(\exp\left(-\frac{t}{2n}B\right)\exp\left(-\frac{t}{n}A\right)\exp\left(-\frac{t}{2n}B\right)\right)^n = \exp(-tC) \tag{1.2}$$

hold in strong operator topology, where C is the unique self-adjoint extension of $A + B$.

When $A + B$ is the Schrödinger operator $-(\Delta/2) + V$ in the space $L_2(\mathbb{R}^d)$ and V is a C^∞-function bounded below such that $|\partial^\alpha V(x)| \le C_\alpha(1 + |x|^2)^{((2-|\alpha|)_+)/2}$ for every multiindex α, Helffer ([3], [4]) proved the asymptotic estimate

$$\left\|\exp\left(-\frac{t}{2}V\right)\exp\left(\frac{t}{2}\Delta\right)\exp\left(-\frac{t}{2}V\right) - \exp\left(-t\left(-\frac{\Delta}{2} + V\right)\right)\right\|_2 = O(t^2) \tag{1.3}$$

as $t \to 0$, where $a_+ := a \vee 0$ and $\|\cdot\|_2$ stands for the L_2-operator norm, and, as its by-product, a variant of the Trotter product formula in L_2-operator

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Received June 1996.

AMS 1991 *subject classifications.* Primary 60H30; secondary 60H10, 60J35, 60J65.

Key words and phrases. Trotter product formula in the kernel level, Trotter product formula in L_p-operator norm, Feynman–Kac formula.

norm

$$\left\| \left(\exp\left(-\frac{t}{2n} V \right) \exp\left(\frac{t}{2n} \Delta \right) \exp\left(-\frac{t}{2n} V \right) \right)^n - \exp\left(-t \left(-\frac{\Delta}{2} + V \right) \right) \right\|_2 = \frac{1}{n} O(t^2) \tag{1.4}$$

as $n \to \infty$. Here, as $-(\Delta/2) + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, so its unique self-adjoint extension is also denoted by the same $-(\Delta/2) + V$.

The result in [5] was extended, by using probabilistic methods, to the case of more general scalar potentials V and an estimate of $e^{-(t/2)V} e^{(t/2)\Delta} e^{-(t/2)V} - e^{-t(-(\Delta/2)+V)}$ in L_p-operator norm was given; as a by-product, a variant of the Trotter product formula in L_p-operator norm was obtained. There, the integral kernels of $e^{-(t/2)V} e^{(t/2)\Delta} e^{-(t/2)V}$ and $e^{-t(-(\Delta/2)+V)}$ are expressed by the Feynman–Kac formula based on the pinned Brownian motion, and from these expressions the difference of these kernels is indeed estimated and thereby the mentioned estimate is derived. For the related L_2-results with operator-theoretic methods, we refer to [1], [2] and [6].

The aim of this paper is to estimate in the kernel level not only the difference $(e^{-(t/2n)V} e^{(t/2n)\Delta} e^{-(t/2n)V})^n - e^{-t(-(\Delta/2)+V)}$, but also $(e^{-(t/n)V} \cdot e^{(t/2n)\Delta})^n - e^{-t(-(\Delta/2)+V)}$. In view of the estimate in [5] it is natural to consider the estimate of the Trotter product formula in the kernel level. Also we improve the condition on scalar potentials V to unify those in [5] and [2].

In the following let us state our results.

Let $V: \mathbb{R}^d \to [0, \infty)$ be a continuous function. Let $0 < \delta \le 1$, $0 \le C_1, C_2 < \infty$, $0 < \gamma \le 1$, $0 \le \kappa \le 1$ and $0 \le \mu, \nu < \infty$. For V, we consider the following conditions:

$(\mathrm{A})_0$ $|V(x) - V(y)| \le C_1 |x - y|^\gamma$;

$(\mathrm{A})_1$ V is a C^1-function such that:

(i) $|\nabla V(z)| \le C_1 V(z)^{1-\delta}$,

(ii) $|\nabla V(x) - \nabla V(y)| \le C_2 |x - y|^\kappa$;

$(\mathrm{A})_2$ V is a C^1-function such that

(i) $|\nabla V(z)| \le C_1 V(z)^{1-\delta}$,

(ii) $|\nabla V(x) - \nabla V(y)| \le C_2 \{ V(x)^{(1-2\delta)+} (1 + |x - y|^\mu) + |x - y|^\nu \} |x - y|$.

Let $H = -(\Delta/2) + V$ be the Schrödinger operator and e^{-tH} be the Schrödinger semigroup, and set

$$K(t) := \exp\left(-\frac{t}{2} V \right) \exp\left(\frac{t}{2} \Delta \right) \exp\left(-\frac{t}{2} V \right),$$

$$G(t) := \exp(-tV) \exp\left(\frac{t}{2} \Delta \right).$$

We call $K(t)$ a *Kac operator*. The quantities e^{-tH}, $K(t/n)^n$ and $G(t/n)^n$ have integral kernels, which are denoted by $e^{-tH}(x, y)$, $K(t/n)^n(x, y)$ and $G(t/n)^n(x, y)$, respectively.

Let $p(t, x)$ be the heat kernel:

$$p(t, x) = \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left(-\frac{|x|^2}{2t}\right).$$

THEOREM 1.1. (i) *Under* $(\mathrm{A})_0$,

$$\begin{aligned} &\left| K\left(\frac{t}{n}\right)^n (x, y) - e^{-tH}(x, y) \right| \\ &\qquad \le p(t, x-y) \operatorname{const} C_1 \left(\frac{1}{n}\right)^{\gamma/2} t\left(|x-y|^\gamma + t^{\gamma/2}\right), \end{aligned}$$

where const *depends only on* γ *and* d.

(ii) *Under* $(\mathrm{A})_1$,

$$\begin{aligned} &\left| K\left(\frac{t}{n}\right)^n (x, y) - e^{-tH}(x, y) \right| \\ &\quad \le p(t, x-y) \\ &\qquad \times \operatorname{const}\left\{ C_1^2 t^{1+2\delta} \left(\frac{1}{n}\right)^{2\delta} + \max\left\{ C_2 \left(\frac{1}{n}\right)^{(1+\kappa)/2}, \left(C_2 \left(\frac{1}{n}\right)^{(1+\kappa)/2}\right)^2 \right\} \right. \\ &\qquad\qquad \left. \times \sum_{j=1}^{2} t^j \left(|x-y|^{j(1+\kappa)} + t^{j(1+\kappa)/2}\right) \right\}, \end{aligned}$$

where const *depends only on* δ, κ *and* d.

(iii) *Under* $(\mathrm{A})_2$,

$$\begin{aligned} &\left| K\left(\frac{t}{n}\right)^n (x, y) - e^{-tH}(x, y) \right| \\ &\quad \le p(t, x-y) \\ &\qquad \times \operatorname{const}\left\{ C_1^2 t^{1+2\delta} \left(\frac{1}{n}\right)^{2\delta} + \max\left\{ C_2 \left(\frac{1}{n}\right)^{1\wedge 2\delta}, \left(C_2 \left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^2 \right\} \right. \\ &\qquad\qquad \times \sum_{j=1}^{2} \Big[t^{j(1\wedge 2\delta)}\left(|x-y|^{2j} + t^j + |x-y|^{j(2+\mu)} + t^{j(1+\mu/2)}\right) \\ &\qquad\qquad\qquad \left. + t^j\left(|x-y|^{j(2+\nu)} + t^{j(1+\nu/2)}\right)\Big] \right\}, \end{aligned}$$

where const *depends only on* δ, μ, ν *and* d.

In Theorem 1.1 and other statements below, const's are all distinct though they are written in the same term.

THEOREM 1.2. *Under* $(A)_2$,

$$\begin{aligned}&\left|G\left(\frac{t}{n}\right)^n(x,y)-e^{-tH}(x,y)\right|\\&\quad\le p(t,x-y)\\&\qquad\times\mathrm{const}\Biggl\{\left(\frac{1}{n}\right)^{1\wedge 2\delta}\sum_{j=1}^{2}C_1^j t^{j\delta}\bigl(|x-y|^j+t^{j/2}\bigr)\\&\qquad\qquad+\max\left\{C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta},\left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^2\right\}\\&\qquad\qquad\times\biggl[t^2+\sum_{j=1}^{2}\Bigl(t^{j(1\wedge 2\delta)}\bigl(|x-y|^{2j}+t^j+|x-y|^{j(2+\mu)}+t^{j(1+\mu/2)}\bigr)\\&\qquad\qquad\qquad+t^j\bigl(|x-y|^{j(2+\nu)}+t^{j1+(\nu/2)}\bigr)\Bigr)\biggr]\Biggr\}.\end{aligned}$$

Here const *depends only on* δ, μ, ν *and* d.

REMARK 1. It is not difficult to see that the integral kernel of $e^{(t/2)\Delta}e^{-tV}$ equals $G(t)(y,x)$, and so $(e^{(t/2n)\Delta}e^{-(t/n)V})^n(x,y)=G(t/n)^n(y,x)$. Hence in Theorems 1.2 and 1.3 the statements with $G(t/n)^n(x,y)$ replaced by $(e^{(t/2n)\Delta}e^{-(t/n)V})^n(x,y)$ hold.

In [5], the condition on V was as follows: for $0<\delta\le 1$ and $m\in\{0,1,2,\dots\}$ such that $m\delta<1$ and $(m+1)\delta\ge 1$, $V\colon\mathbb{R}^d\to[0,\infty)$ is a C^m-function such that we have the following:

1. $V(z)\ge 1$;
2. $|\partial^\alpha V(z)|\le CV(z)^{1-|\alpha|\delta}$, $0\le|\alpha|\le m$;
3. $|\partial^\alpha V(x)-\partial^\alpha V(y)|\le C|x-y|^\kappa$, $|\alpha|=m$

with constants $1\le C<\infty$ and $0\le\kappa\le 1$. When $m=0$ and 1, this condition is just $(A)_0$ and $(A)_1$, respectively. When $m\ge 2$, this implies $(A)_2$ with the same δ, $C_1=\sqrt{d}\,C$, $C_2=2^{(m-3)_+}d^{m/2}C$, $\mu=m-2$ and $\nu=m-2+\kappa$, so that $(A)_2$ is an improvement.

Following Doumeki, Ichinose and Tamura [2], we can consider another group of conditions on V. Let $0\le\rho<\infty$, $0<c<\infty$ and $0\le c_1,c_2<\infty$; then we have the following conditions:

$(V)_0$ V is only a nonnegative bounded measurable function;
$(V)_1$ V is a C^1-function such that:

(i) $V(z)\ge c\langle z\rangle^\rho$,
(ii) $|\nabla V(z)|\le c_1\langle z\rangle^{(\rho-1)_+}$;

$(\mathrm{V})_2$ V is a C^2-function such that:

(i) $V(z) \geq c\langle z\rangle^{\rho}$,
(ii) $|\nabla V(z)| \leq c_1\langle z\rangle^{(\rho-1)_+}$,
(iii) $|\nabla^2 V(z)| \leq c_2\langle z\rangle^{(\rho-2)_+}$.

Here $\langle z\rangle := \sqrt{1+|z|^2}$.

Condition $(\mathrm{V})_0$ requires very little on smoothness though very much on boundedness. Condition $(\mathrm{V})_1$ implies condition $(\mathrm{A})_1$(i) with $C_1 = c_1 c^{-(1-1\wedge 1/\rho)}$ and $\delta = 1 \wedge 1/\rho$ (where, when $\rho = 0$, we interpret $1 \wedge 1/0 = 1$). However it does not give us any information on condition $(\mathrm{A})_1$(ii). In this sense, that is, on smoothness, $(\mathrm{V})_1$ is looser than $(\mathrm{A})_1$. However, condition $(\mathrm{V})_2$ is included in $(\mathrm{A})_2$. Namely, $(\mathrm{V})_2$ implies condition $(\mathrm{A})_2$ with $\delta = 1 \wedge 1/\rho$, $C_1 = c_1 c^{-(1-1\wedge 1/\rho)}$, $C_2 = c_2 2^{(\rho-3)_+}(\frac{1}{2}c^{-(1-2(1\wedge 1/\rho))_+} \vee 1)$, $\mu = 0$ and $\nu = (\rho-2)_+$.

As a corollary to Theorems 1.1 and 1.2, we can present the following theorem.

THEOREM 1.3. (i) *Under* $(\mathrm{V})_1$,

$$\begin{aligned}
&\left| K\left(\frac{t}{n}\right)^n (x,y) - e^{-tH}(x,y)\right| \\
&\qquad \leq p(t, x-y)\operatorname{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} \\
&\qquad\quad \times \Biggl\{ t^{2/((\rho\wedge 2)\vee 1)-1} + t^{1+2(1\wedge((\rho\wedge 2)\vee 1)/2\rho)} \\
&\qquad\qquad + \sum_{j=1}^{2}\Bigl[t^{j2/(2\vee\rho)}\bigl(|x-y|^{2j} + t^j\bigr) + t^j\bigl(|x-y|^{j(2\vee\rho)} + t^{j(2\vee\rho)/2}\bigr)\Bigr]\Biggr\},
\end{aligned}$$

$$\begin{aligned}
&\left| G\left(\frac{t}{n}\right)^n (x,y) - e^{-tH}(x,y)\right| \\
&\qquad \leq p(t, x-y)\operatorname{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} \\
&\qquad\quad \times \Biggl\{ t^{2/((\rho\wedge 2)\vee 1)-1} + t^2 + \sum_{j=1}^{2}\Bigl[t^{j(1\wedge((\rho\wedge 2)\vee 1)/2\rho)}\bigl(|x-y|^{j} + t^{j/2}\bigr) \\
&\qquad\qquad + t^{j2/(2\vee\rho)}\bigl(|x-y|^{2j} + t^j\bigr) \\
&\qquad\qquad + t^j\bigl(|x-y|^{j(2\vee\rho)} + t^{j(2\vee\rho)/2}\bigr)\Bigr]\Biggr\},
\end{aligned}$$

where const *depends only on* c, c_1, ρ *and* d *and* $n \gg 1$ (*in fact it may be that* $n \geq 2^{2(2\vee\rho)}$).

(ii) *Under* $(\mathrm{V})_2$,

$$\left|K\left(\frac{t}{n}\right)^n(x,y)-e^{-tH}(x,y)\right|$$
$$\le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{2/(2\vee\rho)}$$
$$\times\Bigg\{t^{1+2/(1\vee\rho)}+\sum_{j=1}^{2}\Big[t^{j2/(2\vee\rho)}\big(|x-y|^{2j}+t^j\big)$$
$$+t^j\big(|x-y|^{j(2\vee\rho)}+t^{j(2\vee\rho)/2}\big)\Big]\Bigg\},$$

$$\left|G\left(\frac{t}{n}\right)^n(x,y)-e^{-tH}(x,y)\right|$$
$$\le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{2/(2\vee\rho)}$$
$$\times\Bigg\{t^2+\sum_{j=1}^{2}\Big[t^{j/(1\vee\rho)}\big(|x-y|^j+t^{j/2}\big)+t^{j2/(2\vee\rho)}\big(|x-y|^{2j}+t^j\big)$$
$$+t^j\big(|x-y|^{j(2\vee\rho)}+t^{j(2\vee\rho)/2}\big)\Big]\Bigg\},$$

where const *depends only on* ρ, c, c_1, c_2 *and* d.

In Theorem 1.3 we state nothing under the condition $(\mathrm{V})_0$. In [2], this [as well as conditions $(\mathrm{V})_1$ and $(\mathrm{V})_2$] is studied and it is asserted that when $(\mathrm{V})_0$ holds, the L_2-operator norm of $K(t/n)^n-e^{-tH}$ is of order $(1/n)^{2/3}$ as $n\to\infty$ and that this order relation is locally uniform in $t\ge 0$. Their method is more operator theoretical, and for $(\mathrm{V})_0$ our method does not match it, up to the present, in order to get this (or any more profound) result. This is the reason why $(\mathrm{V})_0$ is excluded.

However, under other conditions besides $(\mathrm{V})_0$, we can give the asymptotic order of the L_P-operator norm as well as L_2-operator norm of $K(t/n)^n-e^{-tH}$ and $G(t/n)^n-e^{-tH}$ as $n\to\infty$. For this we note that for $a\ge 0$ and $1\le p\le\infty$,

$$\left\|\int_{\mathbb{R}^d}p(t,\bullet-y)|\bullet-y|^a|f(y)|\,dy\right\|_p\le C(a,d)t^{a/2}\|f\|_p,\qquad f\in L_p(\mathbb{R}^d),$$

where $C(a,d):=\int_{\mathbb{R}^d}|x|^a p(1,x)\,dx$. By combining the estimates in Theorems 1.1, 1.2 and 1.3 with this note, the following is easily confirmed.

COROLLARY 1.1. *Let $1 \le p \le \infty$. Then, for $0 \le t \le 1$ and $n \in \mathbb{N}$, the following hold.*

(i) *Under* $(\mathrm{A})_0$,

$$\left\| K\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{\gamma/2} t^{1+\gamma/2},$$

where const *depends only on* C_1, γ *and* d.

(ii) *Under* $(\mathrm{A})_1$,

$$\left\| K\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{2\delta\wedge(1+\kappa)/2} t^{1+2\delta\wedge(1+\kappa)/2},$$

where const *depends only on* C_1, C_2, δ, κ *and* d.

(iii) *Under* $(\mathrm{A})_2$,

$$\left\| K\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1+1\wedge 2\delta},$$

$$\left\| G\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1/2+\delta},$$

where const *depends only on* C_1, C_2, δ, μ, ν *and* d.

COROLLARY 1.2. *Let $1 \le p \le \infty$. Then, for $0 \le t \le 1$, the following hold.*

(i) *Under* $(\mathrm{V})_1$,

$$\left\| K\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} t^{2/((\rho\wedge 2)\vee 1)-1},$$

$$\left\| G\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} t^{2/((\rho\wedge 2)\vee 1)-1},$$

where const *depends only on* c, c_1, ρ *and* d *and* $n \gg 1$.

(ii) *Under* $(\mathrm{V})_2$,

$$\left\| K\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{2/(2\vee\rho)} t^{1+2/(2\vee\rho)},$$

$$\left\| G\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_p \le \mathrm{const}\left(\frac{1}{n}\right)^{2/(2\vee\rho)} t^{1/2+1/(1\vee\rho)},$$

where const *depends only on* ρ, c, c_1, c_2 *and* d *and* $n \ge 1$.

REMARK 2. In Corollaries 1.1 and 1.2, the statements with $G(t/n)^n$ replaced by $(e^{(t/2n)\Delta}e^{-(t/n)V})^n$ are valid. Consequently as $n \to \infty$, $(e^{-(t/2n)V}e^{(t/2n)\Delta}e^{-(t/2n)V})^n$, $(e^{-(t/n)V}e^{(t/2n)\Delta})^n$ and $(e^{(t/2n)\Delta}e^{-(t/n)V})^n$ tend to $e^{-t(-(\Delta/2)+V)}$ in L_p-operator norm at the same speed. In [6] this convergence in trace norm is stated and proved.

In Section 2 the integral kernels of $(e^{-(t/2n)V}e^{(t/2n)\Delta}e^{-(t/2n)V})^n$ and $(e^{-(t/n)V}e^{(t/2n)\Delta})^n$ are given by the Feynman–Kac formula based on the pinned Brownian motion. Our results, Theorems 1.1, 1.2 and 1.3, are proved in Sections 3, 4 and 5, respectively.

2. Feynman–Kac formula. Let (W, P_0) be a d-dimensional Weiner space: W is the totality of all continuous functions w: $[0,1] \to \mathbb{R}^d$ such that $w(0) = 0$ with the topology of uniform convergence and P_0 is the Wiener measure on W. Clearly,

$$P_0(w(t) \in dx) = p(t, x)\, dx = \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left(-\frac{|x|^2}{2t}\right) dx.$$

For simplicity, set

$$\begin{aligned} X(t, w) &:= w(t), \\ X_0(t, w) &:= X(t, w) - tX(1, w) = w(t) - tw(1). \end{aligned}$$

We start with the Feynman–Kac formula based on the pinned Brownian motion (cf. [9], [10]).

PROPOSITION 2.1.

$$\begin{aligned} \exp(-tH)(x, y) &= p(t, x - y) \\ &\quad \times E_0\left[\exp\left(-t\int_0^1 V\left(x + s(y - x) + \sqrt{t}\, X_0(s)\right) ds\right)\right], \end{aligned}$$

$$\begin{aligned} K\left(\frac{t}{n}\right)^n (x, y) &= p(t, x - y) \\ &\quad \times E_0\left[\exp\left(-\frac{t}{2}\left(\int_0^1 V\left(x + s_n^-(y - x) + \sqrt{t}\, X_0(s_n^-)\right) ds\right.\right.\right. \\ &\qquad \left.\left.\left. + \int_0^1 V\left(x + s_n^+(y - x) + \sqrt{t}\, X_0(s_n^+)\right) ds\right)\right)\right], \end{aligned}$$

$$\begin{aligned} G\left(\frac{t}{n}\right)^n (x, y) &= p(t, x - y) \\ &\quad \times E_0\left[\exp\left(-t\int_0^1 V\left(x + s_n^-(y - x) + \sqrt{t}\, X_0(s_n^-)\right) ds\right)\right]. \end{aligned}$$

Here $s_n^+ := ([ns] + 1)/n$ *and* $s_n^- := [ns]/n$.

By Proposition 2.1,

$$
\begin{aligned}
&K\left(\frac{t}{n}\right)^n(x, y) - \exp(-tH)(x, y) \\
&\quad = p(t, x-y)\Bigg(E_0\bigg[\exp\bigg(-\frac{t}{2}\bigg(\int_0^1 V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\,ds \\
&\qquad\qquad + \int_0^1 V\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\,ds\bigg)\bigg)\bigg] \qquad (2.1)\\
&\qquad\qquad - E_0\bigg[\exp\bigg(-t\int_0^1 V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds\bigg)\bigg]\Bigg) \\
&\quad =: p(t, x-y)\,k_n(t, x, y),
\end{aligned}
$$

$$
\begin{aligned}
&G\left(\frac{t}{n}\right)^n(x, y) - \exp(-tH)(x, y) \\
&\quad = p(t, x-y)\Bigg(E_0\bigg[\exp\bigg(-t\int_0^1 V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\,ds\bigg)\bigg] \qquad (2.2)\\
&\qquad\qquad - E_0\bigg[\exp\bigg(-t\int_0^1 V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds\bigg)\bigg]\Bigg) \\
&\quad =: p(t, x-y)\,g_n(t, x, y),
\end{aligned}
$$

and hence, our problem is reduced to a study of $k_n(t, x, y)$ and $g_n(t, x, y)$.

3. Proof of Theorem 1.1. For simplicity, set

$$
\begin{aligned}
v_n(t, x, y) &:= -\int_0^1 V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds \\
&\quad + \tfrac{1}{2}\bigg(\int_0^1 V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\,ds \qquad (3.1)\\
&\quad + \int_0^1 V\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\,ds\bigg).
\end{aligned}
$$

By a simple formula,

$$
e^a - e^b = (a-b)e^b + (a-b)^2\int_0^1 (1-\theta)e^{\theta a}e^{(1-\theta)b}\,d\theta,
$$

we have

$$
\begin{aligned}
&k_n(t, x, y) \\
&\quad = -tE_0\bigg[v_n(t, x, y)\exp\bigg(-\frac{t}{2}\bigg(\int_0^1 V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\,ds \\
&\qquad\qquad + \int_0^1 V\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\,ds\bigg)\bigg)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^1 (1-\theta)\, d\theta\, t^2 \\
(3.2)\qquad &\times E_0\Big[v_n(t,x,y)^2 \exp\Big(-\theta t \int_0^1 V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds \Big) \\
&\qquad \times \exp\Big(-(1-\theta)\frac{t}{2}\Big(\int_0^1 V\big(x + s_n^-(y-x) + \sqrt{t}\, X_0(s_n^-)\big)\, ds \\
&\qquad\qquad + \int_0^1 V\big(x + s_n^+(y-x) + \sqrt{t}\, X_0(s_n^+)\big)\, ds \Big)\Big)\Big] \\
&=: k_n^{(1)}(t,x,y) + k_n^{(2)}(t,x,y).
\end{aligned}
$$

In the following (up to Section 3.2), we suppose that $V\colon \mathbb{R}^d \to [0,\infty)$ is a C^1-function. By another simple formula,

$$
\begin{aligned}
V(z) - V(w) &= \langle \nabla V(w), z - w\rangle \\
&\quad + \int_0^1 \langle \nabla V(w + \theta(z-w)) - \nabla V(w), z - w\rangle\, d\theta,
\end{aligned}
$$

$v_n(t,x,y)$ is written as

$$
\begin{aligned}
v_n(t,x,y) &= -\tfrac{1}{2}\int_0^1 \Big\langle \nabla V\big(x + s_n^-(y-x) + \sqrt{t}\, X_0(s_n^-)\big), \\
&\qquad\quad (2s - s_n^- - s_n^+)(y-x) \\
&\qquad\quad + \sqrt{t}\,(2X_0(s) - X_0(s_n^-) - X_0(s_n^+))\Big\rangle\, ds \\
&\quad -\tfrac{1}{2}\int_0^1 \Big\langle \nabla V\big(x + s_n^+(y-x) + \sqrt{t}\, X_0(s_n^+)\big) \\
&\qquad\quad -\nabla V\big(x + s_n^-(y-x) + \sqrt{t}\, X_0(s_n^-)\big), \\
&\qquad\quad (s - s_n^+)(y-x) + \sqrt{t}\,\big(X_0(s) - X_0(s_n^+)\big)\Big\rangle\, ds \\
(3.3)\qquad & \\
&\quad -\tfrac{1}{2}\int_0^1 ds \int_0^1 d\theta \Big\langle \nabla V\big(x + s_n^-(y-x) + \sqrt{t}\, X_0(s_n^-) \\
&\qquad\qquad + \theta\big((s - s_n^-)(y-x) + \sqrt{t}\,(X_0(s) - X_0(s_n^-))\big)\big) \\
&\qquad\qquad -\nabla V\big(x + s_n^-(y-x) + \sqrt{t}\, X_0(s_n^-)\big), \\
&\qquad\qquad (s - s_n^-)(y-x) + \sqrt{t}\,\big(X_0(s) - X_0(s_n^-)\big)\Big\rangle \\
&\quad -\tfrac{1}{2}\int_0^1 ds \int_0^1 d\theta \Big\langle \nabla V\big(x + s_n^+(y-x) + \sqrt{t}\, X_0(s_n^+) \\
&\qquad\qquad + \theta\big((s - s_n^+)(y-x) + \sqrt{t}\,(X_0(s) - X_0(s_n^+))\big)\big) \\
&\qquad\qquad -\nabla V\big(x + s_n^+(y-x) + \sqrt{t}\, X_0(s_n^+)\big), \\
&\qquad\qquad (s - s_n^+)(y-x) + \sqrt{t}\,\big(X_0(s) - X_0(s_n^+)\big)\Big\rangle \\
&=: \sum_{j=1}^4 v_n^{(j)}(t,x,y).
\end{aligned}
$$

In view of (3.2) and (3.3), set

$$
\begin{aligned}
k_n^{(1j)}(t,x,y) &=: -tE_0\biggl[v_n^{(j)}(t,x,y) \\
&\qquad \times \exp\biggl(-\frac{t}{2}\biggl(\int_0^1 V\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)\,ds \\
&\qquad\qquad + \int_0^1 V\bigl(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\bigr)\,ds\biggr)\biggr)\biggr],
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
k_n^{(2j)}(t,x,y) &\\
&:= \int_0^1 \theta\,d\theta\, 4t^2 E_0\biggl[v_n^{(j)}(t,x,y)^2 \\
&\qquad \times \exp\biggl(-\theta\frac{t}{2}\biggl(\int_0^1 V\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)\,ds \\
&\qquad\qquad + \int_0^1 V\bigl(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\bigr)\,ds\biggr)\biggr)\biggr].
\end{aligned}
\tag{3.5}
$$

Then clearly

$$
k_n^{(1)}(t,x,y) = \sum_{j=1}^4 k_n^{(1j)}(t,x,y),
$$

$$
\bigl|k_n^{(2)}(t,x,y)\bigr| \le \sum_{j=1}^4 k_n^{(2j)}(t,x,y),
$$

and hence

$$
\begin{aligned}
|k_n(t,x,y)| \le{}& \bigl|k_n^{(11)}(t,x,y)\bigr| + k_n^{(21)}(t,x,y) \\
&+ \sum_{j=2}^4 \bigl|k_n^{(1j)}(t,x,y)\bigr| + \sum_{j=2}^4 k_n^{(2j)}(t,x,y).
\end{aligned}
\tag{3.6}
$$

3.1. *Case* $(\mathrm{A})_2$. In this subsection, we assume condition $(\mathrm{A})_2$.

We first state the following lemma, which is easily seen from condition $(\mathrm{A})_2$(ii).

LEMMA 3.1.

$$
\begin{aligned}
&\bigl|v_n^{(2)}(t,x,y)\bigr| \\
&\quad \le \frac{C_2}{2}\int_0^1 \Bigl\{V\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)^{(1-2\delta)_+} \\
&\qquad\qquad \times\Bigl(1+\bigl|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,\bigl(X_0(s_n^+)-X_0(s_n^-)\bigr)\bigr|^{\mu}\Bigr) \\
&\qquad\qquad \times\bigl|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,\bigl(X_0(s_n^+)-X_0(s_n^-)\bigr)\bigr|
\end{aligned}
$$

$$
\begin{aligned}
&\times\left|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\right| \\
&+\left|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\right|^{1+\nu} \\
&\qquad\times\left|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\right|\Big\}\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\left|v_n^{(3)}(t,x,y)\right| \\
&\quad\le \frac{C_2}{2}\int_0^1\Big\{V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)^{(1-2\delta)_+} \\
&\qquad\times\Big(1+\left|(s-s_n^-)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^-))\right|^{\mu}\Big) \\
&\qquad\times\left|(s-s_n^-)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^-))\right|^{2} \\
&\qquad+\left|(s-s_n^-)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^-))\right|^{2+\nu}\Big\}\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\left|v_n^{(4)}(t,x,y)\right| \\
&\quad\le \frac{C_2}{2}\int_0^1\Big\{V\big(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\big)^{(1-2\delta)_+} \\
&\qquad\times\Big(1+\left|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\right|^{\mu}\Big) \\
&\qquad\times\left|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\right|^{2} \\
&\qquad+\left|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\right|^{2+\nu}\Big\}\,ds.
\end{aligned}
$$

By this and the moment estimate (3.10) below, the following is shown.

LEMMA 3.2. *Let* $m\in\mathbb{N}$, $0<\theta\le 1$ *and* $j=2,3,4$. *Then*

$$
\begin{aligned}
&\theta t^m E_0\bigg[\left|v_n^{(j)}(t,x,y)\right|^m \exp\bigg(-\theta\frac{t}{2}\bigg(\int_0^1 V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)\,ds \\
&\qquad\qquad+\int_0^1 V\big(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\big)\,ds\bigg)\bigg)\bigg] \\
&\quad\le \mathrm{const}\left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^m \\
&\qquad\times\Big\{\theta^{1-m(1-2\delta)_+}t^{m(1\wedge 2\delta)}\big(|x-y|^{2m}+t^m+|x-y|^{m(2+\mu)}+t^{m(1+\mu/2)}\big) \\
&\qquad\qquad+\theta t^m\big(|x-y|^{m(2+\nu)}+t^{m(1+\nu/2)}\big)\Big\}.
\end{aligned}
$$

Here const *depends only on* δ, μ, ν, d *and* m.

PROOF. We do the proof only for $j = 2$. By the Jensen and then the Hölder inequalities, Lemma 3.1 implies that

$$
\begin{aligned}
&\theta t^m E_0\bigg[\big|v_n^{(2)}(t,x,y)\big|^m \\
&\qquad \times\exp\bigg(-\theta\frac{t}{2}\bigg(\int_0^1 V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)\,ds \\
&\qquad\qquad + \int_0^1 V\big(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\big)\,ds\bigg)\bigg)\bigg] \\
&\quad\le \theta t^m \frac{1}{3}\left(\frac{3C_2}{2}\right)^m \\
&\qquad \times E_0\bigg[\int_0^1 \Big\{V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)^{m(1-2\delta)_+} \\
&\qquad\qquad \times\exp\bigg(-\theta\frac{t}{2}\int_0^1 V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)\,ds\bigg) \\
&\qquad\qquad \times\Big(1+\big|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\big|^{m\mu}\Big) \\
&\qquad\qquad \times\big|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\big|^m \\
&\qquad\qquad \times\big|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\big|^m \\
&\qquad\qquad +\big|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\big|^{m(1+\nu)} \\
&\qquad\qquad\quad \times\big|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\big|^m\Big\}\,ds\bigg].
\end{aligned}
\tag{3.7}
$$

Here we note that

$$
t^b e^{-t} \le \left(\frac{b}{e}\right)^b, \qquad t \ge 0,\ b \ge 0, \tag{3.8}
$$

where, when $b = 0$ we understand $(0/e)^0 := 1$. Since, by this

$$
\begin{aligned}
&V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)^{m(1-2\delta)_+} \\
&\quad \times\exp\bigg(-\theta\frac{t}{2}\int_0^1 V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)\,ds\bigg) \\
&\qquad \le V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)^{m(1-2\delta)_+} \\
&\qquad\qquad \times\exp\bigg(-\theta\frac{t}{2}\frac{1}{n}V\big(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\big)\bigg) \\
&\qquad \le \theta^{-m(1-2\delta)_+}\left(\frac{t}{n}\right)^{-m(1-2\delta)_+}\left(\frac{2m(1-2\delta)_+}{e}\right)^{m(1-2\delta)_+},
\end{aligned}
$$

the right-hand side of (3.7) is dominated by

$$
\begin{aligned}
&\frac{1}{3}\left(\frac{3C_2}{2}\right)^m\Biggl\{\theta^{1-m(1-2\delta)_+}t^{m(1\wedge 2\delta)}\left(\frac{1}{n}\right)^{-m(1-2\delta)_+}\left(\frac{2m(1-2\delta)_+}{e}\right)^{m(1-2\delta)_+}\\
&\qquad\times\int_0^1\Biggl(E_0\Bigl[\bigl|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\bigr|^m\\
&\qquad\qquad\times\bigl|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\bigr|^m\Bigr]\\
&\qquad\qquad+E_0\Bigl[\bigl|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\bigr|^{m(1+\mu)}\\
&\qquad\qquad\times\bigl|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\bigr|^m\Bigr]\Biggr)\,ds\\
&\qquad+\theta t^m\int_0^1 E_0\Bigl[\bigl|(s_n^+-s_n^-)(y-x)+\sqrt{t}\,(X_0(s_n^+)-X_0(s_n^-))\bigr|^{m(1+\nu)}\\
&\qquad\qquad\times\bigl|(s-s_n^+)(y-x)+\sqrt{t}\,(X_0(s)-X_0(s_n^+))\bigr|^m\Bigr]\,ds\Biggr\}.
\end{aligned}
\tag{3.9}
$$

To estimate this, we use the following moment estimate: for $0\le s<u\le 1$ and $a>0$,

$$
\begin{aligned}
&E_0\Bigl[\bigl|(u-s)z+\sqrt{t}\,(X_0(u)-X_0(s))\bigr|^a\Bigr]\\
&\qquad\le 3^{(a-1)_+}\Bigl\{|u-s|^a|z|^a+2t^{a/2}|u-s|^{a/2}E_0\bigl[|X(1)|^a\bigr]\Bigr\}.
\end{aligned}
\tag{3.10}
$$

By the Schwarz inequality and this, (3.9) is further dominated as follows:

$$
\begin{aligned}
&\le\frac{1}{3}\left(\frac{3C_2}{2}\right)^m\Biggl\{\theta^{1-m(1-2\delta)_+}t^{m(1\wedge 2\delta)}\left(\frac{2m(1-2\delta)_+}{e}\right)^{m(1-2\delta)_+}\left(\frac{1}{n}\right)^{-m(1-2\delta)_+}\\
&\qquad\times\Biggl[3^{2m-1}\left(\left(\frac{1}{n}\right)^{2m}|x-y|^{2m}+2t^m\left(\frac{1}{n}\right)^m E_0\bigl[|X(1)|^{2m}\bigr]\right)\\
&\qquad\quad+\Biggl(3^{2m(1+\mu)-1}\Biggl(\left(\frac{1}{n}\right)^{2m(1+\mu)}|x-y|^{2m(1+\mu)}\\
&\qquad\qquad\qquad+2t^{m(1+\mu)}\left(\frac{1}{n}\right)^{m(1+\mu)}E_0\bigl[|X(1)|^{2m(1+\mu)}\bigr]\Biggr)\\
&\qquad\quad\times 3^{2m-1}\left(\left(\frac{1}{n}\right)^{2m}|x-y|^{2m}+2t^m\left(\frac{1}{n}\right)^m E_0\bigl[|X(1)|^{2m}\bigr]\right)\Biggr)^{1/2}\Biggr]\\
&\qquad+\theta t^m\Biggl(3^{2m(1+\nu)-1}\Biggl(\left(\frac{1}{n}\right)^{2m(1+\nu)}|x-y|^{2m(1+\nu)}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + 2t^{m(1+\nu)}\left(\frac{1}{n}\right)^{m(1+\nu)} E_0\left[|X(1)|^{2m(1+\nu)}\right]\Bigg)\\
&\qquad \times 3^{2m-1}\left(\left(\frac{1}{n}\right)^{2m}|x-y|^{2m} + 2t^m\left(\frac{1}{n}\right)^m E_0\left[|X(1)|^{2m}\right]\right)\Bigg)^{1/2}\Bigg\}\\
&\le \frac{1}{3}\left(\frac{3C_2}{2}\right)^m\Bigg\{\theta^{1-m(1-2\delta)_+}\left(\frac{2m(1-2\delta)_+}{e}\right)^{m(1-2\delta)_+} t^{m(1\wedge 2\delta)}\left(\frac{1}{n}\right)^{m(1\wedge 2\delta)}\\
&\qquad \times\Bigg[3^{2m-1}\Big(\Big(|x-y|^{2m} + 2t^m E_0\left[|X(1)|^{2m}\right]\Big)\\
&\qquad\quad + 3^{m(2+\mu)-1}\Big(\Big(|x-y|^{2m(1+\mu)} + 2t^{m(1+\mu)}E_0\left[|X(1)|^{2m(1+\mu)}\right]\Big)\\
&\qquad\qquad\qquad \times\Big(|x-y|^{2m} + 2t^m E_0\left[|X(1)|^{2m}\right]\Big)\Big)^{1/2}\Bigg]\\
&\qquad + \theta t^m\left(\frac{1}{n}\right)^m 3^{m(2+\nu)-1}\\
&\qquad \times\Big(\Big(|x-y|^{2m(1+\nu)} + 2t^{m(1+\nu)}E_0\left[|X(1)|^{2m(1+\nu)}\right]\Big)\\
&\qquad\qquad \times\Big(|x-y|^{2m} + 2t^m E_0\left[|X(1)|^{2m}\right]\Big)\Big)^{1/2}\Bigg\}\\
&\le \left(\frac{1}{n}\right)^{m(1\wedge 2\delta)}\frac{1}{9}\left(\frac{27C_2}{2}\right)^m\\
&\quad \times\Bigg\{\theta^{1-m(1-2\delta)_+}\left(\frac{2m(1-2\delta)_+}{e}\right)^{m(1-2\delta)_+} t^{m(1\wedge 2\delta)}\\
&\qquad \times\Bigg[|x-y|^{2m} + 2t^m E_0\left[|X(1)|^{2m}\right]\\
&\qquad\quad + 3^{m\mu}\Big(|x-y|^{m(2+\mu)} + \sqrt{2}\,t^{m/2}|x-y|^{m(1+\mu)}E_0\left[|X(1)|^{2m}\right]^{1/2}\\
&\qquad\qquad + \sqrt{2}\,t^{(m(1+\mu))/2}|x-y|^m E_0\left[|X(1)|^{2m(1+\mu)}\right]^{1/2}\\
&\qquad\qquad + 2t^{m(1+\mu/2)}E_0\left[|X(1)|^{2m}\right]^{1/2} E_0\left[|X(1)|^{2m(1+\mu)}\right]^{1/2}\Big)\Bigg]\\
&\qquad + \theta 3^{m\nu}t^m\Big(|x-y|^{m(2+\nu)} + \sqrt{2}\,t^{m/2}|x-y|^{m(1+\nu)}E_0\left[|X(1)|^{2m}\right]^{1/2}\\
&\qquad\qquad + \sqrt{2}\,t^{(m(1+\nu))/2}|x-y|^m E_0\left[|X(1)|^{2m(1+\nu)}\right]^{1/2}\\
&\qquad\qquad + 2t^{m(1+\nu/2)}E_0\left[|X(1)|^{2m}\right]^{1/2} E_0\left[|X(1)|^{2m(1+\nu)}\right]^{1/2}\Big)\Bigg\}.
\end{aligned}
$$

Consequently, noting that

$$t^{1/2}|x-y|^{1+\mu} \le \frac{1}{2+\mu}t^{1+\mu/2} + \frac{1+\mu}{2+\mu}|x-y|^{2+\mu},$$
$$t^{(1+\mu)/2}|x-y| \le \frac{1+\mu}{2+\mu}t^{1+\mu/2} + \frac{1}{2+\mu}|x-y|^{2+\mu},$$

we obtain the desired estimate. □

CLAIM 3.1.

$$\begin{aligned}\sum_{j=2}^{4}\bigl|k_n^{(1j)}(t,x,y)\bigr| + \sum_{j=2}^{4}k_n^{(2j)}(t,x,y) &\le \mathrm{const}\max\left\{C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}, \left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^2\right\}\\ &\quad\times\sum_{j=1}^{2}\Bigl\{t^{j(1\wedge 2\delta)}\bigl(|x-y|^{2j} + t^j + |x-y|^{j(2+\mu)} + t^{j(1+\mu/2)}\bigr)\\ &\qquad + t^j\bigl(|x-y|^{j(2+\nu)} + t^{j(1+\nu/2)}\bigr)\Bigr\}.\end{aligned}$$

Here const *depends only on* δ, μ, ν *and* d.

PROOF. Recall (3.4) and (3.5). By Lemma 3.2,

$$\begin{aligned}\bigl|k_n^{(1j)}(t,x,y)\bigr| &\le \mathrm{const}\, C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\\ &\quad\times\Bigl\{t^{1\wedge 2\delta}\bigl(|x-y|^2 + t + |x-y|^{2+\mu} + t^{1+\mu/2}\bigr)\\ &\qquad + t\bigl(|x-y|^{2+\nu} + t^{1+\nu/2}\bigr)\Bigr\},\\ k_n^{(2j)}(t,x,y) &\le 2\,\mathrm{const}\left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^2\\ &\quad\times\left\{\frac{1}{1\wedge 2\delta}t^{2(1\wedge 2\delta)}\bigl(|x-y|^4 + t^2 + |x-y|^{4+2\mu} + t^{2+\mu}\bigr)\right.\\ &\qquad \left. + t^2\bigl(|x-y|^{4+2\nu} + t^{2+\nu}\bigr)\right\}.\end{aligned}$$

From this the claim follows immediately. □

Next we estimate $k_n^{(11)}(t,x,y)$ and $k_n^{(21)}(t,x,y)$.

Recalling (3.3), we rewrite $v_n^{(1)}(t,x,y)$ as

$$
\begin{aligned}
v_n^{(1)}&(t,x,y)\\
&= -\frac{1}{2}\sum_{l=1}^{n}\int_{(l-1)/n}^{l/n}\Big\langle \nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\Big(\frac{l-1}{n}\Big)\Big),\\
&\qquad\qquad \Big(2s-\frac{2l-1}{n}\Big)(y-x)\\
&\qquad\qquad +\sqrt{t}\Big(2X_0(s)-X_0\Big(\frac{l-1}{n}\Big)-X_0\Big(\frac{l}{n}\Big)\Big)\Big\rangle ds\\
&= -\frac{1}{2}\sum_{l=1}^{n}\int_{0}^{1/n}\Big\langle \nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\Big(\frac{l-1}{n}\Big)\Big),\\
&\qquad\qquad \Big(2s-\frac{1}{n}\Big)(y-x)\\
&\qquad\qquad +\sqrt{t}\Big(2X_0\Big(s+\frac{l-1}{n}\Big)-X_0\Big(\frac{l-1}{n}\Big)-X_0\Big(\frac{l}{n}\Big)\Big)\Big\rangle ds
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{l=1}^{n}\Big(\Big\langle \nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\Big(\frac{l-1}{n}\Big)\Big),\\
&\qquad\qquad \int_0^{1/n}\Big(2s-\frac{1}{n}\Big)\,ds(y-x)\Big\rangle\\
&\qquad +\sqrt{t}\int_0^{1/n}\Big\langle \nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\Big(\frac{l-1}{n}\Big)\Big),\\
&\qquad\qquad 2X_0\Big(s+\frac{l-1}{n}\Big)-X_0\Big(\frac{l-1}{n}\Big)-X_0\Big(\frac{l}{n}\Big)\Big\rangle ds\Big)\\
&= -\frac{\sqrt{t}}{2}\sum_{l=1}^{n}\int_0^{1/n}\Big\langle \nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\Big(\frac{l-1}{n}\Big)\Big),\\
&\qquad\qquad 2X_0\Big(s+\frac{l-1}{n}\Big)-X_0\Big(\frac{l-1}{n}\Big)-X_0\Big(\frac{l}{n}\Big)\Big\rangle ds,
\end{aligned}
$$

and so

$$
\begin{aligned}
v_n^{(1)}&(t,x,y)^2\\
&= \frac{t}{4}\sum_{l=1}^{n}\Big(\int_0^{1/n}\Big\langle \nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\Big(\frac{l-1}{n}\Big)\Big),\\
&\qquad\qquad 2X_0\Big(s+\frac{l-1}{n}\Big)-X_0\Big(\frac{l-1}{n}\Big)-X_0\Big(\frac{l}{n}\Big)\Big\rangle ds\Big)^2
\end{aligned}
\tag{3.12}
$$

$$+\frac{t}{2}\sum_{1\le l<m\le n}\int_0^{1/n}\left\langle \nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right),\right.$$
$$\left. 2X_0\left(s+\frac{l-1}{n}\right)-X_0\left(\frac{l-1}{n}\right)-X_0\left(\frac{l}{n}\right)\right\rangle ds$$
$$\times\int_0^{1/n}\left\langle \nabla V\left(x+\frac{m-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{m-1}{n}\right)\right),\right.$$
$$\left. 2X_0\left(s+\frac{m-1}{n}\right)-X_0\left(\frac{m-1}{n}\right)-X_0\left(\frac{m}{n}\right)\right\rangle ds.$$

Substituting (3.11) and (3.12) into (3.4) and (3.5), respectively, we have

$$\begin{aligned} k_n^{(11)}(t,x,y) &= \frac{t^{3/2}}{2}\exp\left(-\frac{t}{2n}(V(x)+V(y))\right) \\ &\quad\times\sum_{l=1}^{n}E_0\left[\int_0^{1/n}\left\langle \nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right),\right.\right. \\ &\qquad\left. 2X_0\left(s+\frac{l-1}{n}\right)-X_0\left(\frac{l-1}{n}\right)-X_0\left(\frac{l}{n}\right)\right\rangle ds \\ &\qquad\left.\times\exp\left(-\frac{t}{n}\sum_{j=1}^{n-1}V\left(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{j}{n}\right)\right)\right)\right], \end{aligned} \tag{3.13}$$

$$\begin{aligned} &k_n^{(21)}(t,x,y) \\ &\quad=\int_0^1\theta\,d\theta\,t^3\exp\left(-\frac{\theta t}{2n}(V(x)+V(y))\right) \\ &\quad\times\left\{\sum_{l=1}^{n}E_0\left[\left(\int_0^{1/n}\left\langle \nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right),\right.\right.\right.\right. \\ &\qquad\left.\left. 2X_0\left(s+\frac{l-1}{n}\right)-X_0\left(\frac{l-1}{n}\right)-X_0\left(\frac{l}{n}\right)\right\rangle ds\right)^2 \\ &\qquad\left.\times\exp\left(-\frac{\theta t}{n}\sum_{j=1}^{n-1}V\left(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{j}{n}\right)\right)\right)\right] \\ &\quad+2\sum_{1\le l<m\le n}E_0\left[\int_0^{1/n}\left\langle \nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right),\right.\right. \\ &\qquad\left. 2X_0\left(s+\frac{l-1}{n}\right)-X_0\left(\frac{l-1}{n}\right)-X_0\left(\frac{l}{n}\right)\right\rangle ds \end{aligned} \tag{3.14}$$

$$\times \int_0^{\frac{1}{n}} \left\langle \nabla V\left(x + \frac{m-1}{n}(y-x) + \sqrt{t}\, X_0\left(\frac{m-1}{n}\right)\right), \right.$$
$$\left. 2X_0\left(s + \frac{m-1}{n}\right) - X_0\left(\frac{m-1}{n}\right) - X_0\left(\frac{m}{n}\right)\right\rangle ds$$
$$\left.\left. \times \exp\left(- \frac{\theta t}{n} \sum_{j=1}^{n-1} V\left(x + \frac{j}{n}(y-x) + \sqrt{t}\, X_0\left(\frac{j}{n}\right)\right)\right)\right]\right\}.$$

Here we have used that

$$\frac{1}{2}\left(\int_0^1 V\left(x + s_n^-(y-x) + \sqrt{t}\, X_0(s_n^-)\right) ds \right.$$
$$\left. + \int_0^1 V\left(x + s_n^+(y-x) + \sqrt{t}\, X_0(s_n^+)\right) ds \right)$$
$$= \frac{1}{2n}(V(x) + V(y)) + \frac{1}{n}\sum_{j=1}^{n-1} V\left(x + \frac{j}{n}(y-x) + \sqrt{t}\, X_0\left(\frac{j}{n}\right)\right).$$

PROPOSITION 3.1. *For* $\xi, \eta \in \mathbb{R}^d$ *and* $0 < t_0 \le 1$, *let* $(X_\xi^{t_0,\eta}(t))_{0 \le t \le t_0}$ *be the solution of the following SDE*:

$$dX_t = dw_t + \frac{\eta - X_t}{t_0 - t}\, dt, \qquad 0 \le t \le t_0,$$
$$X_0 = \xi$$

(*cf*. [7], *pages* 243–244). *Then we have the following*:

(i) $(X_\xi^{t_0,\eta}(t))_{0 \le t \le t_0} \sim_L (\xi + (t/t_0)(\eta - \xi) + w(t) - (t/t_0)w(t_0))_{0 \le t \le t_0}$. *In particular* $(X_0^{1,0}(t))_{0 \le t \le 1} \sim_L (X_0(t))_{0 \le t \le 1}$.

(ii) $P_0(X_\xi^{t_0,\eta}(\bullet + t_1) \in * | F_{t_1}) = P_0(X_{\xi_1}^{t_0 - t_1,\eta}(\bullet) \in *)|_{\xi_1 = X_\xi^{t_0,\eta}(t_1)}$ $(0 \le t_1 < t_0 \le 1)$, *where* F_τ *is the sub-σ-field generated by* $w(t)$, $0 \le t \le \tau$.

(iii) $(X_\xi^{t_0,\eta}(t))_{0 \le t \le t_0} \sim_L (X_\eta^{t_0,\xi}(t_0 - t))_{0 \le t \le t_0}$.

By virtue of Proposition 3.1, $k_n^{(11)}(t,x,y)$ and $k_n^{(21)}(t,x,y)$ are rewritten as

$$k_n^{(11)}(t,x,y)$$
$$= \frac{t^{3/2}}{2} \exp\left(- \frac{t}{2n}(V(x) + V(y))\right)$$
$$\times \sum_{l=1}^{n} E_0\left[\exp\left(- \frac{t}{n}\sum_{j=1}^{l-1} V\left(x + \frac{j}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{j}{n}\right)\right)\right)\right. \tag{3.15}$$
$$\times E_0\left[\int_0^{1/n} \left\langle \nabla V\left(x + \frac{l-1}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{l-1}{n}\right)\right), \right.\right.$$

$$2X_0^{1,0}\Big(s+\frac{l-1}{n}\Big)-X_0^{1,0}\Big(\frac{l-1}{n}\Big)-X_0^{1,0}\Big(\frac{l}{n}\Big)\Big\rangle ds$$

$$\times\exp\Big(-\frac{t}{n}\sum_{j=l}^{n-1}V\Big(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{j}{n}\Big)\Big)\Big)\Big|F_{(l-1)/n}\Big]\Big],$$

$k_n^{(21)}(t,x,y)$

$$=\int_0^1\theta\,d\theta\,t^3\exp\Big(-\frac{\theta t}{2n}(V(x)+V(y))\Big)$$

$$\times\Big\{\sum_{l=1}^{n}E_0\Big[\exp\Big(-\frac{\theta t}{n}\sum_{j=1}^{l-1}V\Big(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{j}{n}\Big)\Big)\Big)$$

$$\times E_0\Big[\Big(\int_0^{1/n}\Big\langle\nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{l-1}{n}\Big)\Big),$$

$$2X_0^{1,0}\Big(s+\frac{l-1}{n}\Big)-X_0^{1,0}\Big(\frac{l-1}{n}\Big)-X_0^{1,0}\Big(\frac{l}{n}\Big)\Big\rangle ds\Big)^2$$

$$\times\exp\Big(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Big(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{j}{n}\Big)\Big)\Big)\Big|F_{(l-1)/n}\Big]\Big]$$

(3.16)

$$+2\sum_{1\le l<m\le n}E_0\Big[\exp\Big(-\frac{\theta t}{n}\sum_{j=1}^{m-1}V\Big(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{j}{n}\Big)\Big)\Big)$$

$$\times\int_0^{1/n}\Big\langle\nabla V\Big(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{l-1}{n}\Big)\Big),$$

$$2X_0^{1,0}\Big(s+\frac{l-1}{n}\Big)-X_0^{1,0}\Big(\frac{l-1}{n}\Big)-X_0^{1,0}\Big(\frac{l}{n}\Big)\Big\rangle ds$$

$$\times E_0\Big[\int_0^{1/n}\Big\langle\nabla V\Big(x+\frac{m-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{m-1}{n}\Big)\Big),$$

$$2X_0^{1,0}\Big(s+\frac{m-1}{n}\Big)-X_0^{1,0}\Big(\frac{m-1}{n}\Big)-X_0^{1,0}\Big(\frac{m}{n}\Big)\Big\rangle ds$$

$$\times\exp\Big(-\frac{\theta t}{n}\sum_{j=m}^{n-1}V\Big(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Big(\frac{j}{n}\Big)\Big)\Big)\Big|F_{(m-1)/n}\Big]\Big]\Big\}.$$

LEMMA 3.3. *Let* $1 \le l \le n$, $m \in \mathbb{N}$ *and* $0 < \theta \le 1$. *Then*

$$
\begin{aligned}
&E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Bigl(\frac{l-1}{n}\Bigr)\Bigr),\\
&\qquad 2X_0^{1,0}\Bigl(s+\frac{l-1}{n}\Bigr)-X_0^{1,0}\Bigl(\frac{l-1}{n}\Bigr)-X_0^{1,0}\Bigl(\frac{l}{n}\Bigr)\Bigr\rangle ds\Biggr)^m\\
&\quad\times\exp\Biggl(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Bigl(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Bigl(\frac{j}{n}\Bigr)\Bigr)\Biggr)\Bigg|\mathcal{F}_{(l-1)/n}\Biggr]\\
&\quad= E_0\Biggl[E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigr),\\
&\qquad\qquad 2X_\eta^{1/n,\xi}(s)-\xi-\eta\Bigr\rangle ds\Biggr)^m\Biggr]\\
&\qquad\times E_0\Biggl[\exp\Biggl(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Bigl(x+\frac{j}{n}(y-x)\\
&\qquad\qquad+\sqrt{t}\,X_\eta^{1-l/n,0}\Bigl(\frac{j-l}{n}\Bigr)\Bigr)\Biggr)\Biggr]\Bigg|_{\eta=X_0^{1-(l-1)/n,\xi}(1-l/n)}\Biggr]\Bigg|_{\xi=X_0^{1,0}((l-1)/n)}.
\end{aligned}
$$

PROOF. By Proposition 3.1, this equality is derived in the following way. The left-hand side is

$$
\begin{aligned}
&E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\Bigl(\frac{l-1}{n}\Bigr)\Bigr),\\
&\qquad 2X_0^{1,0}\Bigl(s+\frac{l-1}{n}\Bigr)-X_0^{1,0}\Bigl(\frac{l-1}{n}\Bigr)-X_0^{1,0}\Bigl(\frac{1}{n}+\frac{l-1}{n}\Bigr)\Bigr\rangle ds\Biggr)^m\\
&\quad\times\exp\Biggl(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Bigl(x+\frac{j}{n}(y-x)\\
&\qquad\qquad+\sqrt{t}\,X_0^{1,0}\Bigl(\frac{j-l+1}{n}+\frac{l-1}{n}\Bigr)\Bigr)\Biggr)\Bigg|\mathcal{F}_{(l-1)/n}\Biggr]\\
&\quad= E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigr),\\
&\qquad 2X_\xi^{1-(l-1)/n,0}(s)-\xi-X_\xi^{1-(l-1)/n,0}\Bigl(\frac{1}{n}\Bigr)\Bigr\rangle ds\Biggr)^m
\end{aligned}
$$

$$
\begin{aligned}
&\times \exp\Biggl(-\frac{\theta t}{n}\sum_{j=l}^{n-1} V\biggl(x+\frac{j}{n}(y-x)\\
&\qquad\qquad +\sqrt{t}\, X_\xi^{1-(l-1)/n,0}\Bigl(\frac{j-l+1}{n}\Bigr)\biggr)\Biggr)\Biggr]\Biggr|_{\xi=X_0^{1,0}((l-1)/n)}\\
&\qquad\qquad\qquad\qquad [\text{by Proposition 3.1(ii)}]\\
&= E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigr),\\
&\qquad 2X_\xi^{1-(l-1)/n,0}(s)-\xi-X_\xi^{1-(l-1)/n,0}\Bigl(\frac{1}{n}\Bigr)\Bigr\rangle ds\Biggr)^m\\
&\qquad \times E_0\Biggl[\exp\Biggl(-\frac{\theta t}{n}\sum_{j=l}^{n-1} V\biggl(x+\frac{j}{n}(y-x)\\
&\qquad\qquad +\sqrt{t}\, X_\xi^{1-(l-1)/n,0}\Bigl(\frac{j-l+1}{n}\Bigr)\biggr)\Biggr)\,\Bigg|\,F_{1/n}\Biggr]\Biggr]\Biggr|_{\xi=X_0^{1,0}((l-1)/n)}\\
&\qquad\qquad\qquad \Bigl[\text{since } \sigma\Bigl(X_\xi^{1-(l-1)/n,0}(s);\, 0\le s\le \frac{1}{n}\Bigr)\subset F_{1/n}\Bigr]\\
&= E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigr),\\
&\qquad 2X_\xi^{1-(l-1)/n,0}(s)-\xi-X_\xi^{1-(l-1)/n,0}\Bigl(\frac{1}{n}\Bigr)\Bigr\rangle ds\Biggr)^m\\
&\qquad \times E_0\Biggl[\exp\Biggl(-\frac{\theta t}{n}\sum_{j=l}^{n-1} V\biggl(x+\frac{j}{n}(y-x)\\
&\qquad\qquad +\sqrt{t}\, X_\eta^{1-(l/n),0}\Bigl(\frac{j-l}{n}\Bigr)\biggr)\Biggr)\Biggr]\Biggr|_{\eta=X_\xi^{1-(l-1)/n,0}(1/n)}\Biggr]\Biggr|_{\xi=X_0^{1,0}((l-1)/n)}\\
&\qquad\qquad\qquad\qquad [\text{by Proposition 3.1(ii)}]\\
&= E_0\Biggl[\Biggl(\int_0^{1/n}\Bigl\langle \nabla V\Bigl(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigr),\\
&\qquad 2X_0^{1-(l-1)/n,\xi}\Bigl(1-\frac{l-1}{n}-s\Bigr)-\xi\\
&\qquad\qquad -X_0^{1-(l-1)/n,\xi}\Bigl(1-\frac{l}{n}\Bigr)\Bigr\rangle\, ds\Biggr)^m
\end{aligned}
$$

$$E_0\Bigg[\exp\Bigg(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Bigg(x+\frac{j}{n}(y-x)$$

$$+\sqrt{t}\,X_\eta^{1-(l/n),0}\Big(\frac{j-l}{n}\Big)\Bigg)\Bigg)\Bigg]\Bigg|_{\eta=X_0^{1-(l-1)/n,\xi}(1-l/n)}\Bigg]\Bigg|_{\xi=X_0^{1,0}((l-1)/n)}$$

[by Proposition 3.1(iii)]

$$=E_0\Bigg[E_0\Bigg[\exp\Bigg(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Bigg(x+\frac{j}{n}(y-x)$$

$$+\sqrt{t}\,X_\eta^{1-l/n,0}\Big(\frac{j-l}{n}\Big)\Bigg)\Bigg)\Bigg]\Bigg|_{\eta=X_0^{1-(l-1)/n,\xi}(1-l/n)}$$

$$\times E_0\Bigg[\Bigg(\int_0^{1/n}\Bigg\langle\nabla V\Bigg(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigg),$$

$$2X_0^{1-(l-1)/n,\xi}\Big(\frac{1}{n}-s+1-\frac{l}{n}\Big)$$

$$-\xi-X_0^{1-(l-1)/n,\xi}\Big(1-\frac{l}{n}\Big)\Bigg\rangle ds\Bigg)^m\Bigg|F_{1-l/n}\Bigg]\Bigg]\Bigg|_{\xi=X_0^{1,0}((l-1)/n)}$$

$$\Bigg[\text{since } X_0^{1-(l-1)/n,\xi}\Big(1-\frac{l}{n}\Big)\text{ is }F_{1-l/n}\text{-measurable}\Bigg]$$

$$=E_0\Bigg[E_0\Bigg[\exp\Bigg(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\Bigg(x+\frac{j}{n}(y-x)$$

$$+\sqrt{t}\,X_\eta^{1-l/n,0}\Big(\frac{j-l}{n}\Big)\Bigg)\Bigg)\Bigg]\Bigg|_{\eta=X_0^{1-(l-1)/n,\xi}(1-l/n)}$$

$$\times E_0\Bigg[\Bigg(\int_0^{1/n}\Bigg\langle\nabla V\Bigg(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigg),$$

$$2X_\eta^{1/n,\xi}\Big(\frac{1}{n}-s\Big)-\xi-\eta\Bigg\rangle ds\Bigg)^m\Bigg]\Bigg|_{\eta=X_0^{1-(l-1)/n,\xi}(1-l/n)}\Bigg]\Bigg|_{\xi=X_0^{1,0}((l-1)/n)}$$

[by Proposition 3.1(iii)]

and that equals the right-hand side. □

LEMMA 3.4. *Let $1\le l\le n$ and $\xi,\eta\in\mathbb{R}^d$. Then*

$$E_0\Bigg[\int_0^{1/n}\Bigg\langle\nabla V\Bigg(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\Bigg),2X_\eta^{1/n,\xi}(s)-\xi-\eta\Bigg\rangle ds\Bigg]=0,$$

$$E_0\left[\left(\int_0^{1/n}\left\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right), 2X_\eta^{1/n,\xi}(s)-\xi-\eta\right\rangle ds\right)^2\right]$$
$$=\frac{1}{3n^3}\left|\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right)\right|^2.$$

PROOF. By Proposition 3.1(i), we indeed compute these as follows. The first expression is

$$\begin{aligned}
&E_0\Bigg[\int_0^{1/n}\Bigg\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right),\\
&\qquad 2\left(\eta+ns(\xi-\eta)+w(s)-nsw\left(\frac{1}{n}\right)\right)-\xi-\eta\Bigg\rangle ds\Bigg]\\
&=E_0\Bigg[\int_0^{1/n}\Bigg\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right),\\
&\qquad (2ns-1)(\xi-\eta)+2\left(w(s)-nsw\left(\frac{1}{n}\right)\right)\Bigg\rangle ds\Bigg]\\
&=\left\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right),\int_0^{1/n}(2ns-1)\,ds(\xi-\eta)\right\rangle\\
&\quad+\int_0^{1/n}\left\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right), 2E_0\left[w(s)-nsw\left(\frac{1}{n}\right)\right]\right\rangle ds\\
&=0.
\end{aligned}$$

The second expression is

$$\begin{aligned}
&E_0\Bigg[\Bigg(\int_0^{1/n}\Bigg\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right),\\
&\qquad (2ns-1)(\xi-\eta)+2\left(w(s)-nsw\left(\frac{1}{n}\right)\right)\Bigg\rangle ds\Bigg)^2\Bigg]\\
&=4E_0\left[\left(\int_0^{1/n}\left\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right), w(s)-nsw\left(\frac{1}{n}\right)\right\rangle ds\right)^2\right]\\
&=\frac{4}{n^2}E_0\left[\left(\int_0^{1}\left\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right), w\left(\frac{\sigma}{n}\right)-\sigma w\left(\frac{1}{n}\right)\right\rangle d\sigma\right)^2\right]\\
&\qquad\qquad\left(\text{by the change of variables: } s=\frac{\sigma}{n}\right)\\
&=\frac{4}{n^3}E_0\left[\left(\int_0^{1}\left\langle\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,\xi\right), w(\sigma)-\sigma w(1)\right\rangle d\sigma\right)^2\right]
\end{aligned}$$

$$\left[\text{by the scaling property: } \left(w\left(\frac{\sigma}{n}\right)\right)_{0\le\sigma\le 1} \sim_L \left(\frac{1}{\sqrt{n}}\, w(\sigma)\right)_{0\le\sigma\le 1}\right]$$

$$= \frac{4}{n^3}\left|\nabla V\left(x + \frac{l-1}{n}(y-x) + \sqrt{t}\,\xi\right)\right|^2$$

$$\times E_0\left[\left(\int_0^1 (w_1(\sigma) - \sigma w_1(1))\, d\sigma\right)^2\right]$$

$$= \frac{1}{3n^3}\left|\nabla V\left(x + \frac{l-1}{n}(y-x) + \sqrt{t}\,\xi\right)\right|^2. \qquad \square$$

By Lemmas 3.3 and 3.4, we claim the following.

CLAIM 3.2.

$$k_n^{(11)}(t, x, y) = 0,$$

$$k_n^{(21)}(t, x, y) \le \frac{C_1^2}{6}\frac{1}{\delta}\left(\frac{4(1-\delta)}{e}\right)^{2(1-\delta)} t^{1+2\delta}\left(\frac{1}{n}\right)^{2\delta}.$$

PROOF. Combining Lemma 3.3 with Lemma 3.4, we have

$$E_0\Bigg[\int_0^{1/n}\Bigg\langle \nabla V\left(x + \frac{l-1}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{l-1}{n}\right)\right),$$
$$2X_0^{1,0}\left(s + \frac{l-1}{n}\right) - X_0^{1,0}\left(\frac{l-1}{n}\right) - X_0^{1,0}\left(\frac{l}{n}\right)\Bigg\rangle ds$$
$$\times \exp\left(-\frac{t}{n}\sum_{j=l}^{n-1} V\left(x + \frac{j}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{j}{n}\right)\right)\right)\Bigg|\mathcal{F}_{(l-1)/n}\Bigg]$$
$$= 0,$$

$$E_0\Bigg[\Bigg(\int_0^{1/n}\Bigg\langle \nabla V\left(x + \frac{l-1}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{l-1}{n}\right)\right),$$
$$2X_0^{1,0}\left(s + \frac{l-1}{n}\right) - X_0^{1,0}\left(\frac{l-1}{n}\right) - X_0^{1,0}\left(\frac{l}{n}\right)\Bigg\rangle ds\Bigg)^2$$
$$\times \exp\left(-\frac{\theta t}{n}\sum_{j=l}^{n-1} V\left(x + \frac{j}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{j}{n}\right)\right)\right)\Bigg|\mathcal{F}_{(l-1)/n}\Bigg]$$
$$= \frac{1}{3n^3}\left|\nabla V\left(x + \frac{l-1}{n}(y-x) + \sqrt{t}\, X_0^{1,0}\left(\frac{l-1}{n}\right)\right)\right|^2$$

$$\times E_0\left[E_0\left[\exp\left(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\left(x+\frac{j}{n}(y-x)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.+\sqrt{t}\,X_\eta^{1-l/n,0}\left(\frac{j-l}{n}\right)\right)\right)\right]\right|_{\eta=X_0^{1-(l-1)/n,\xi}(1-l/n)}\right]\right|_{\xi=X_0^{1,0}((l-1)/n)}$$

$$=\frac{1}{3n^3}\left|\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\left(\frac{l-1}{n}\right)\right)\right|^2$$

$$\times E_0\left[\exp\left(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\left(x+\frac{j}{n}(y-x)\right.\right.\right.$$

$$\left.\left.\left.\left.+\sqrt{t}\,X_\xi^{1-(l-1)/n,0}\left(\frac{j-l+1}{n}\right)\right)\right)\right]\right|_{\xi=X_0^{1,0}((l-1)/n)}$$

[by Proposition 3.1(ii)]

$$=E_0\left[\frac{1}{3n^3}\left|\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\left(\frac{l-1}{n}\right)\right)\right|^2\right.$$

$$\left.\times\exp\left(-\frac{\theta t}{n}\sum_{j=l}^{n-1}V\left(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\left(\frac{j}{n}\right)\right)\right)\right|F_{(l-1)/n}\right]$$

[by Proposition 3.1(ii) again].

Substituting these into (3.15) and (3.16), respectively, we see

$$k_n^{(11)}(t,x,y)=0,$$

$$k_n^{(21)}(t,x,y)=\int_0^1\theta\,d\theta\,t^3\exp\left(-\frac{\theta t}{2n}(V(x)+V(y))\right)$$

$$\times\sum_{l=1}^{n}E_0\left[\frac{1}{3n^3}\left|\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\left(\frac{l-1}{n}\right)\right)\right|^2\right.$$

$$\left.\times\exp\left(-\frac{\theta t}{n}\sum_{j=1}^{n-1}V\left(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0^{1,0}\left(\frac{j}{n}\right)\right)\right)\right]$$

$$=\frac{1}{3n^3}\sum_{l=1}^{n}\int_0^1\theta\,d\theta\,t^3$$

$$\times E_0\left[\left|\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right)\right|^2\right.$$

$$\times \exp\left(-\frac{\theta t}{2n}\sum_{j=1}^{n} V\left(x+\frac{j-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{j-1}{n}\right)\right)\right)$$
$$\times \exp\left(-\frac{\theta t}{2n}\sum_{j=1}^{n} V\left(x+\frac{j}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{j}{n}\right)\right)\right)\Bigg].$$

Here, recall condition $(\mathrm{A})_2$(i) and inequality (3.8). By these,

$$\begin{aligned}
&\left|\nabla V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right)\right|^2\\
&\quad\times \exp\left(-\frac{\theta t}{2n}\sum_{j=1}^{n} V\left(x+\frac{j-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{j-1}{n}\right)\right)\right)\\
&\quad\le C_1^2 V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right)^{2(1-\delta)}\\
&\qquad\times \exp\left(-\frac{\theta t}{2n} V\left(x+\frac{l-1}{n}(y-x)+\sqrt{t}\,X_0\left(\frac{l-1}{n}\right)\right)\right)\\
&\quad\le C_1^2\left(\frac{\theta t}{n}\right)^{-2(1-\delta)}\left(\frac{4(1-\delta)}{e}\right)^{2(1-\delta)},
\end{aligned}$$

and hence

$$\begin{aligned}
&k_n^{(21)}(t,x,y)\\
&\quad\le \frac{1}{3n^3}\sum_{l=1}^{n}\int_0^1 \theta\, d\theta\, t^3 C_1^2 \theta^{-2(1-\delta)} t^{-2(1-\delta)}\left(\frac{1}{n}\right)^{-2(1-\delta)}\left(\frac{4(1-\delta)}{e}\right)^{2(1-\delta)}\\
&\quad= \frac{1}{3}\int_0^1 \theta^{2\delta-1}\, d\theta\, C_1^2 t^{1+2\delta}\left(\frac{1}{n}\right)^{2\delta}\left(\frac{4(1-\delta)}{e}\right)^{2(1-\delta)}\\
&\quad= \frac{C_1^2}{6}\frac{1}{\delta}\left(\frac{4(1-\delta)}{e}\right)^{2(1-\delta)} t^{1+2\delta}\left(\frac{1}{n}\right)^{2\delta}.
\end{aligned}$$

We complete the proof of Claim 3.2. □

For the proof of Theorem 1.1 in Case $(\mathrm{A})_2$, by (3.6), Claims 3.1 and 3.2, we have the statement of Theorem 1.1 at once.

3.2. *Case* $(\mathrm{A})_1$. In this subsection, we assume condition $(\mathrm{A})_1$.

By using condition $(\mathrm{A})_1$(ii) instead of $(\mathrm{A})_2$(ii), Lemma 3.1 is replaced by the following.

LEMMA 3.5.

$$\left|v_n^{(2)}(t,x,y)\right| \le \frac{C_2}{2}\int_0^1 \left|(s_n^+ - s_n^-)(y-x) + \sqrt{t}\left(X_0(s_n^+) - X_0(s_n^-)\right)\right|^{\kappa} \times \left|(s - s_n^+)(y-x) + \sqrt{t}\left(X_0(s) - X_0(s_n^+)\right)\right| ds,$$

$$\left|v_n^{(3)}(t,x,y)\right| \le \frac{C_2}{2}\int_0^1 \left|(s - s_n^-)(y-x) + \sqrt{t}\left(X_0(s) - X_0(s_n^-)\right)\right|^{1+\kappa} ds,$$

$$\left|v_n^{(4)}(t,x,y)\right| \le \frac{C_2}{2}\int_0^1 \left|(s - s_n^+)(y-x) + \sqrt{t}\left(X_0(s) - X_0(s_n^+)\right)\right|^{1+\kappa} ds.$$

By this and (3.10), the following is easily seen.

LEMMA 3.6. *Let* $m \in \mathbb{N}$ *and* $j = 2, 3, 4$. *Then*

$$t^m E_0\left[\left|v_n^{(j)}(t,x,y)\right|^m\right] \le \mathrm{const}\left(C_2\left(\frac{1}{n}\right)^{(1+\kappa)/2}\right)^m t^m\left(|x-y|^{m(1+\kappa)} + t^{m(1+\kappa)/2}\right).$$

Here const *depends only on* κ, d *and* m.

CLAIM 3.3.

$$\sum_{j=2}^{4}\left|k_n^{(1j)}(t,x,y)\right| + \sum_{j=2}^{4} k_n^{(2j)}(t,x,y) \le \mathrm{const}\max\left\{C_2\left(\frac{1}{n}\right)^{(1+\kappa)/2}, \left(C_2\left(\frac{1}{n}\right)^{(1+\kappa)/2}\right)^2\right\} \times \sum_{j=1}^{2} t^j\left(|x-y|^{j(1+\kappa)} + t^{j(1+\kappa)/2}\right).$$

Here const *depends only on* κ *and* d.

PROOF. By (3.4) and (3.5),

$$\left|k_n^{(1j)}(t,x,y)\right| \le tE_0\left[\left|v_n^{(j)}(t,x,y)\right|\right],$$

$$k_n^{(2j)}(t,x,y) \le \int_0^1 \theta\, d\theta\, 4t^2 E_0\left[\left|v_n^{(j)}(t,x,y)\right|^2\right] = 2t^2 E_0\left[\left|v_n^{(j)}(t,x,y)\right|^2\right],$$

and hence, by Lemma 3.6, the claim follows immediately. □

Since condition $(\mathrm{A})_1$(i) is the same as $(\mathrm{A})_2$(i), Claim 3.2 is valid.

For the proof of Theorem 1.1 in Case $(\mathrm{A})_1$, by (3.6), Claims 3.3 and 3.2, we have the statement of Theorem 1.1 at once.

3.3. *Case* $(\mathrm{A})_0$. In this subsection, we assume condition $(\mathrm{A})_0$. Since, by (3.1)

$$\begin{aligned} v_n(t, x, y) = -\tfrac{1}{2}\int_0^1 \Big(& V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big) \\ & - V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\Big)\, ds \\ -\tfrac{1}{2}\int_0^1 \Big(& V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big) \\ & - V\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\Big)\, ds, \end{aligned}$$

we have by condition $(\mathrm{A})_0$,

$$\begin{aligned} |v_n(t, x, y)| \le{} & \frac{C_1}{2}\int_0^1 \big|(s - s_n^-)(y-x) + \sqrt{t}\,(X_0(s) - X_0(s_n^-))\big|^\gamma\, ds \\ & + \frac{C_1}{2}\int_0^1 \big|(s - s_n^+)(y-x) + \sqrt{t}\,(X_0(s) - X_0(s_n^+))\big|^\gamma\, ds. \end{aligned}$$

Note that

$$|k_n(t, x, y)| \le E_0\big[t|v_n(t, x, y)|\big].$$

Hence, by combining these with (3.10), we see

$$\begin{aligned} |k_n(t, x, y)| &\le C_1 t\left(\left(\frac{1}{n}\right)^\gamma |x-y|^\gamma + 2t^{\gamma/2}\left(\frac{1}{n}\right)^{\gamma/2} E_0\big[|X(1)|^\gamma\big]\right) \\ &\le C_1\left(\frac{1}{n}\right)^{\gamma/2} \max\Big\{1, 2E_0\big[|X(1)|^\gamma\big]\Big\}\, t\big(|x-y|^\gamma + t^{\gamma/2}\big), \end{aligned}$$

which completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2. Throughout this section we assume condition $(\mathrm{A})_2$.

4.1. *Proof for* $n = 1$. Though our aim is to estimate $(G(t/n)^n - e^{-tH})(x, y)$ or $g_n(t, x, y)$ [recall (2.2)], we first view it for $n = 1$.

CLAIM 4.1.

$$\begin{aligned} &\big|(G(t) - e^{-tH})(x, y)\big| \\ &\qquad = p(t, x-y)\,|g_1(t, x, y)| \\ &\qquad \le p(t, x-y) \end{aligned}$$

$$\times \operatorname{const}\Bigg\{ \frac{C_1}{2} t^{\delta} |x-y| + \left(\frac{C_1}{2}\right)^2 t^{2\delta}\big(|x-y|^2 + t\big) + \sum_{j=1}^{2} C_2^j \Big(t^{j(1\wedge 2\delta)}\big(|x-y|^{2j} + t^j + |x-y|^{j(2+\mu)} + t^{j(1+\mu/2)}\big) + t^j\big(|x-y|^{j(2+\nu)} + t^{j(1+\nu/2)}\big)\Big)\Bigg\}$$

where const *depends only on* δ, μ, ν *and* d.

PROOF. As $v_n(t,x,y)$ and $v_n^{(j)}(t,x,y)$ $(1 \le j \le 4)$, we set

$$u(t,x,y) := -\int_0^1 V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds + V(x)$$

$$u^{(1)}(t,x,y) := \tfrac{1}{2}\langle \nabla V(x), x-y\rangle$$

$$u^{(2)}(t,x,y) := -\sqrt{t}\left\langle \nabla V(x), \int_0^1 X_0(s)\,ds \right\rangle$$

$$u^{(3)}(t,x,y) := -\int_0^1 ds \int_0^1 d\theta \Big\langle \nabla V\Big(x + \theta\big(s(y-x) + \sqrt{t}\,X_0(s)\big)\Big) - \nabla V(x), s(y-x) + \sqrt{t}\,X_0(s)\Big\rangle .$$

Note that $u(t,x,y) = \sum_{j=1}^{3} u^{(j)}(t,x,y)$. Then

$$\begin{aligned} g_1(t,x,y) &= -tE_0\big[u(t,x,y)\exp(-tV(x))\big] - \int_0^1 (1-\theta)\,d\theta\, t^2 \\ &\quad \times E_0\Big[u(t,x,y)^2 \exp\Big(-\theta t\int_0^1 V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds\Big) \times \exp(-(1-\theta)tV(x))\Big] \\ &=: g_1^{(1)}(t,x,y) + g_1^{(2)}(t,x,y) \end{aligned}$$

and

$$g_1^{(1)}(t,x,y) = \sum_{j=1}^{3} -tE_0\big[u^{(j)}(t,x,y)e^{-tV(x)}\big] =: \sum_{j=1}^{3} g_1^{(1j)}(t,x,y)$$

$$\begin{aligned} \big|g_1^{(2)}(t,x,y)\big| &\le \sum_{j=1}^{3}\int_0^1 \theta\,d\theta\,3t^2 E_0\big[u^{(j)}(t,x,y)^2 e^{-\theta tV(x)}\big] \\ &=: \sum_{j=1}^{3} g_1^{(2j)}(t,x,y). \end{aligned}$$

Since

$$E_0\left[u^{(2)}(t,x,y)\right] = -\sqrt{t}\left\langle \nabla V(x), \int_0^1 E_0[X_0(s)]\,ds\right\rangle = 0,$$

$g^{(12)}(t,x,y) = 0$. By condition $(\mathrm{A})_2$(i),

$$\left|u^{(1)}(t,x,y)\right| \le \frac{C_1}{2} V(x)^{1-\delta}|x-y|,$$

$$\left|u^{(2)}(t,x,y)\right| \le C_1 V(x)^{1-\delta} t^{1/2}\left|\int_0^1 X_0(s)\,ds\right|,$$

and by this and (3.8),

$$\begin{aligned}
&\left|u^{(1)}(t,x,y)\right|^m e^{-\theta t V(x)} \\
&\quad\le \left(\frac{C_1}{2}\right)^m \left(\frac{m(1-\delta)}{e}\right)^{m(1-\delta)} \theta^{-m(1-\delta)} t^{-m(1-\delta)}|x-y|^m, \\
&\left|u^{(2)}(t,x,y)\right|^m e^{-\theta t V(x)} \\
&\quad\le C_1^m \left(\frac{m(1-\delta)}{e}\right)^{m(1-\delta)} \theta^{-m(1-\delta)} t^{-m(1-\delta)+m/2}\left|\int_0^1 X_0(s)\,ds\right|^m.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|g_1^{(11)}(t,x,y)\right| &\le \frac{C_1}{2}\left(\frac{1-\delta}{e}\right)^{1-\delta} t^{\delta}|x-y|, \\
g_1^{(21)}(t,x,y) &\le \left(\frac{C_1}{2}\right)^2 \frac{3}{2\delta}\left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} t^{2\delta}|x-y|^2, \\
g_1^{(22)}(t,x,y) &\le C_1^2 \frac{3}{2\delta}\left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} t^{1+2\delta} E_0\left[\left|\int_0^1 X_0(s)\,ds\right|^2\right] \\
&= C_1^2 \frac{3}{2\delta}\left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} \frac{d}{12} t^{1+2\delta} \\
&= \left(\frac{C_1}{2}\right)^2 \frac{d}{2\delta}\left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} t^{1+2\delta}.
\end{aligned}$$

By condition $(\mathrm{A})_2$(i),

$$\begin{aligned}
\left|u^{(3)}(t,x,y)\right| \le C_2 \int_0^1 \Big\{ V(x)^{(1-2\delta)_+} \Big(&\left|s(y-x)+\sqrt{t}X_0(s)\right|^2 \\
&+ \left|s(y-x)+\sqrt{t}X_0(s)\right|^{2+\mu}\Big) \\
&+ \left|s(y-x)+\sqrt{t}X_0(s)\right|^{2+\nu}\Big\}\,ds,
\end{aligned}$$

and by this and (3.8),

$$\begin{aligned}
&\left|u^{(3)}(t,x,y)\right|^m e^{-\theta t V(x)} \\
&\quad\le \frac{1}{3}(3C_2)^m\left\{\left(\frac{m(1-2\delta)_+}{e}\right)^{m(1-2\delta)_+}\theta^{-m(1-2\delta)_+}t^{-m(1-2\delta)_+}\right. \\
&\qquad\qquad\times\left(\int_0^1\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{2m}ds\right. \\
&\qquad\qquad\quad\left.+\int_0^1\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{m(2+\mu)}ds\right) \\
&\qquad\qquad\quad\left.+\int_0^1\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{m(2+\nu)}ds\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|g_1^{(13)}(t,x,y)\right| &\le C_2\left\{\left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+}t^{1\wedge 2\delta}\right. \\
&\qquad\times\left(\int_0^1 E_0\left[\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^2\right]ds\right. \\
&\qquad\quad\left.+\int_0^1 E_0\left[\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{2+\mu}\right]ds\right) \\
&\qquad\quad\left.+t\int_0^1 E_0\left[\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{2+\nu}\right]ds\right\} \\
&\le \mathrm{const}\; C_2\left\{t^{1\wedge 2\delta}\left(|x-y|^2+t+|x-y|^{2+\mu}+t^{1+\mu/2}\right)\right. \\
&\qquad\left.+t\left(|x-y|^{2+\nu}+t^{1+\nu/2}\right)\right\},
\end{aligned}$$

$$\begin{aligned}
&g_1^{(23)}(t,x,y) \\
&\quad\le \frac{1}{2}(3C_2)^2\left\{\left(\frac{2(1-2\delta)_+}{e}\right)^{2(1-2\delta)_+}\frac{1}{1\wedge 2\delta}t^{2(1\wedge 2\delta)}\right. \\
&\qquad\qquad\times\left(\int_0^1 E_0\left[\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^4\right]ds\right. \\
&\qquad\qquad\quad\left.+\int_0^1 E_0\left[\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{4+2\mu}\right]ds\right) \\
&\qquad\qquad\quad\left.+t^2\int_0^1 E_0\left[\left|s(y-x)+\sqrt{t}\,X_0(s)\right|^{4+2\nu}\right]ds\right\} \\
&\quad\le \mathrm{const}\; C_2^2\left\{t^{2(1\wedge 2\delta)}\left(|x-y|^4+t^2+|x-y|^{4+2\mu}+t^{2+\mu}\right)\right. \\
&\qquad\qquad\left.+t^2\left(|x-y|^{4+2\nu}+t^{2+\nu}\right)\right\},
\end{aligned}$$

where in the last lines in each inequality we have used (3.10), and const depends only on δ, μ, ν, and d.

Consequently, combining all the above, we have the desired estimate and the proof is complete. □

4.2. *Decomposition of* $(G(t/n)^n - e^{-tH})(x, y)$. In the following (up to the end of Section 4) let $n \geq 2$. In this subsection we decompose $(G(t/n)^n - e^{-tH})(x, y)$ into the sum of three terms.

As an operator, it is observed that

$$\begin{aligned}
&G\left(\frac{t}{n}\right)^n - \exp(-tH) \\
&\quad = \exp\left(-\frac{t}{2n}V\right)K\left(\frac{t}{n}\right)^{n-1}\exp\left(-\frac{t}{2n}V\right)\exp\left(\frac{t}{2n}\Delta\right) - \exp(-tH) \\
&\quad = \exp\left(-\frac{t}{2n}V\right)\left(K\left(\frac{n-1}{n}t\frac{1}{n-1}\right)^{n-1} - \exp\left(-\frac{n-1}{n}tH\right)\right) \\
&\qquad \times \exp\left(-\frac{t}{2n}V\right)\exp\left(\frac{t}{2n}\Delta\right) \\
&\qquad + \left[\exp\left(-\frac{t}{2n}V\right), \exp\left(-\frac{n-1}{n}tH\right)\right]\exp\left(-\frac{t}{2n}V\right)\exp\left(\frac{t}{2n}\Delta\right) \\
&\qquad + \exp\left(-\frac{n-1}{n}tH\right)\left(G\left(\frac{t}{n}\right) - \exp\left(-\frac{t}{n}H\right)\right),
\end{aligned}$$

where in general $[A, B] := AB - BA$ (the commutator of A and B). Viewing integral kernels in both sides, we have

$$\begin{aligned}
&\left(G\left(\frac{t}{n}\right)^n - \exp(-tH)\right)(x, y) \\
&\quad = \int_{\mathbb{R}^d} \exp\left(-\frac{t}{2n}V(x)\right)\left(K\left(\frac{n-1}{n}t\frac{1}{n-1}\right)^{n-1} - \exp\left(-\frac{n-1}{n}tH\right)\right)(x, z) \\
&\qquad \times \exp\left(-\frac{t}{2n}V(z)\right)p\left(\frac{t}{n}, z-y\right)\,dz \\
&\qquad + \int_{\mathbb{R}^d}\left[\exp\left(-\frac{t}{2n}V\right), \exp\left(-\frac{n-1}{n}tH\right)\right](x, z) \\
&\qquad \times \exp\left(-\frac{t}{2n}V(z)\right)p\left(\frac{t}{n}, z-y\right)\,dz \\
&\qquad + \int_{\mathbb{R}^d}\exp\left(-\frac{n-1}{n}tH\right)(x, z)\left(G\left(\frac{t}{n}\right) - \exp\left(-\frac{t}{n}H\right)\right)(z, y)\,dz,
\end{aligned}$$

and hence

$$
\begin{aligned}
&\left|\left(G\left(\frac{t}{n}\right)^{n} - \exp(-tH)\right)(x, y)\right| \\
&\quad \le \int_{\mathbb{R}^d} \left|\left(K\left(\frac{n-1}{n} t \frac{1}{n-1}\right)^{n-1} - \exp\left(-\frac{n-1}{n} tH\right)\right)(x, z)\right| \\
&\qquad \times p\left(\frac{t}{n}, z - y\right) dz \\
&\qquad + \int_{\mathbb{R}^d} \left|\left[\exp\left(-\frac{t}{2n} V\right), \exp\left(-\frac{n-1}{n} tH\right)\right](x, z)\right| p\left(\frac{t}{n}, z - y\right) dz \\
&\qquad + \int_{\mathbb{R}^d} p\left(\frac{n-1}{n} t, x - z\right) \left|\left(G\left(\frac{t}{n}\right) - \exp\left(-\frac{t}{n} H\right)\right)(z, y)\right| dz \\
&\quad =: I_1 + I_2 + I_3,
\end{aligned} \tag{4.1}
$$

where the last line in the inequality comes from the fact

$$
0 \le \exp\left(-\frac{n-1}{n} tH\right)(x, z) \le p\left(\frac{n-1}{n} t, x - z\right).
$$

In the next three subsections we estimate I_1, I_2 and I_3, respectively. Before closing this subsection, we present the following (whose proof is routine, and so we omit it).

PROPOSITION 4.1. *Let $a \ge 0$ and $u, s > 0$. Then*

$$
\begin{aligned}
&\int_{\mathbb{R}^d} p(u, x - y)\, p(s, y) |y|^a \, dy \\
&\qquad = p(u + s, x) \int_{\mathbb{R}^d} p(1, y) \left|\sqrt{\frac{us}{u+s}}\, y + \frac{s}{u+s} x\right|^a dy.
\end{aligned}
$$

4.3. *Estimate of I_1.*

CLAIM 4.2.

$$
\begin{aligned}
I_1 \le{}& p(t, x - y) \\
&\times \mathrm{const}\Bigg\{ C_1^2 t^{1+2\delta} \left(\frac{1}{n}\right)^{2\delta} + \max\left\{ C_2 \left(\frac{1}{n}\right)^{1\wedge 2\delta}, \left(C_2 \left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^2 \right\} \\
&\qquad \times \sum_{j=1}^{2} \Big[t^{j(1\wedge 2\delta)} \left(|x - y|^{2j} + t^j + |x - y|^{j(2+\mu)} + t^{j(1+\mu/2)}\right) \\
&\qquad\qquad + t^j \left(|x - y|^{j(2+\nu)} + t^{j(1+\nu/2)}\right)\Big] \Bigg\}.
\end{aligned}
$$

Here const *depends only on δ, μ, ν and d.*

PROOF. By Theorem 1.1 and Proposition 4.1, this estimate is obtained in the following way:

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} \left| \left(K\left(\frac{n-1}{n} t \frac{1}{n-1}\right)^{n-1} - \exp\left(-\frac{n-1}{n} tH\right)\right)(x, z) \right| p\left(\frac{t}{n}, z-y\right) dz \\
&\le \int_{\mathbb{R}^d} p\left(\frac{n-1}{n} t, x-z\right) p\left(\frac{t}{n}, z-y\right) dz \\
&\qquad \times \mathrm{const} \Bigg\{ C_1^2 \left(\frac{n-1}{n} t\right)^{1+2\delta} \left(\frac{1}{n-1}\right)^{2\delta} \\
&\qquad\qquad + \max\left\{ C_2 \left(\frac{1}{n-1}\right)^{1\wedge 2\delta}, \left(C_2 \left(\frac{1}{n-1}\right)^{1\wedge 2\delta}\right)^2 \right\} \\
&\qquad\qquad \times \sum_{j=1}^{2} \Bigg[\left(\frac{n-1}{n} t\right)^{j(1\wedge 2\delta)} \\
&\qquad\qquad \times \left(|x-z|^{2j} + \left(\frac{n-1}{n} t\right)^{j} + |x-z|^{j(2+\mu)} + \left(\frac{n-1}{n} t\right)^{j(1+\mu/2)} \right) \\
&\qquad\qquad\qquad + \left(\frac{n-1}{n} t\right)^{j} \left(|x-z|^{j(2+\nu)} + \left(\frac{n-1}{n} t\right)^{j(1+\nu/2)} \right) \Bigg] \Bigg\} \\
&= \mathrm{const}\, C_1^2 \left(\frac{n-1}{n} t\right)^{1+2\delta} \left(\frac{1}{n-1}\right)^{2\delta} p(t, x-y) \\
&\quad + \mathrm{const} \max\left\{ C_2 \left(\frac{1}{n-1}\right)^{1\wedge 2\delta}, \left(C_2 \left(\frac{1}{n-1}\right)^{1\wedge 2\delta}\right)^2 \right\} \\
&\quad \times \sum_{j=1}^{2} \Bigg[\left(\frac{n-1}{n} t\right)^{j(1\wedge 2\delta)} \Bigg(\int_{\mathbb{R}^d} p\left(\frac{n-1}{n} t, x-z\right) p\left(\frac{t}{n}, z-y\right) |x-z|^{2j}\, dz \\
&\qquad\qquad + \left(\frac{n-1}{n} t\right)^{j} p(t, x-y) \\
&\qquad\qquad + \int_{\mathbb{R}^d} p\left(\frac{n-1}{n} t, x-z\right) p\left(\frac{t}{n}, z-y\right) |x-z|^{j(2+\mu)}\, dz \\
&\qquad\qquad + \left(\frac{n-1}{n} t\right)^{j(1+\mu/2)} p(t, x-y) \Bigg) \\
&\qquad + \left(\frac{n-1}{n} t\right)^{j} \Bigg(\int_{\mathbb{R}^d} p\left(\frac{n-1}{n} t, x-z\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad \times p\left(\frac{t}{n}, z-y\right)|x-z|^{j(2+\nu)}\,dz \\
&\qquad\qquad\qquad + \left(\frac{n-1}{n}t\right)^{j(1+\nu/2)} p(t, x-y)\Bigg)\Bigg] \\
&= \mathrm{const}\, C_1^2\left(\frac{n-1}{n}t\right)^{1+2\delta}\left(\frac{1}{n-1}\right)^{2\delta} p(t, x-y) \\
&\quad + \mathrm{const}\max\left\{C_2\left(\frac{1}{n-1}\right)^{1\wedge 2\delta}, \left(C_2\left(\frac{1}{n-1}\right)^{1\wedge 2\delta}\right)^2\right\} \\
&\quad \times \sum_{j=1}^{2}\Bigg[\left(\frac{n-1}{n}t\right)^{j(1\wedge 2\delta)} p(t, x-y) \\
&\qquad \times \Bigg(\int_{\mathbb{R}^d} p(1, z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z + \frac{n-1}{n}(x-y)\right|^{2j} dz \\
&\qquad\qquad + \left(\frac{n-1}{n}t\right)^{j} \\
&\qquad\quad + \int_{\mathbb{R}^d} p(1, z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z + \frac{n-1}{n}(x-y)\right|^{j(2+\mu)} dz \\
&\qquad\qquad\qquad + \left(\frac{n-1}{n}t\right)^{j(1+\mu/2)}\Bigg) \\
&\qquad + \left(\frac{n-1}{n}t\right)^{j} p(t, x-y) \\
&\qquad \times \Bigg(\int_{\mathbb{R}^d} p(1, z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z + \frac{n-1}{n}(x-y)\right|^{j(2+\nu)} dz \\
&\qquad\qquad\qquad + \left(\frac{n-1}{n}t\right)^{j(1+\nu/2)}\Bigg)\Bigg] \\
&\le p(t, x-y)\Bigg\{\mathrm{const}\, C_1^2 t^{1+2\delta}\left(\frac{2}{n}\right)^{2\delta} \\
&\qquad\qquad + \mathrm{const}\max\left\{C_2\left(\frac{2}{n}\right)^{1\wedge 2\delta}, \left(C_2\left(\frac{2}{n}\right)^{1\wedge 2\delta}\right)^2\right\} \\
&\quad \times \sum_{j=1}^{2}\Bigg[t^{j(1\wedge 2\delta)}\Bigg(\int_{\mathbb{R}^d} p(1, z)2^{2j-1}\left(t^j|z|^{2j} + |x-y|^{2j}\right) dz + t^j
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^d} p(1,z)2^{j(2+\mu)-1}\bigl(t^{j(1+\mu/2)}|z|^{j(2+\mu)}+|x-y|^{j(2+\mu)}\bigr)\,dz+t^{j(1+\mu/2)}\Bigr)\\
&\qquad+t^j\Bigl(\int_{\mathbb{R}^d} p(1,z)2^{j(2+\nu)-1}\bigl(t^{j(1+\nu/2)}|z|^{j(2+\nu)}\\
&\qquad\qquad+|x-y|^{j(2+\nu)}\bigr)\,dz+t^{j(1+\nu/2)}\Bigr)\Bigr]\Bigr\}\\
&\le p(t,x-y)\mathrm{const}\biggl\{C_1^2t^{1+2\delta}\Bigl(\frac{1}{n}\Bigr)^{2\delta}+\max\biggl\{C_2\Bigl(\frac{1}{n}\Bigr)^{1\wedge 2\delta},\Bigl(C_2\Bigl(\frac{1}{n}\Bigr)^{1\wedge 2\delta}\Bigr)^2\biggr\}\\
&\qquad\times\sum_{j=1}^{2}\Bigl[t^{j(1\wedge 2\delta)}\bigl(|x-y|^{2j}+t^j+|x-y|^{j(2+\mu)}+t^{j(1+\mu/2)}\bigr)\\
&\qquad\qquad+t^j\bigl(|x-y|^{j(2+\nu)}+t^{j(1+\nu/2)}\bigr)\Bigr]\biggr\}.\quad\square
\end{aligned}
$$

4.4. *Estimate of I_3.*

CLAIM 4.3.

$$
\begin{aligned}
I_3 &\le p(t,x-y)\\
&\quad\times\mathrm{const}\sum_{j=1}^{2}\biggl\{\Bigl(\frac{C_1}{2}\Bigr)^j\Bigl(\frac{1}{n}\Bigr)^{j(\delta+1/2)}t^{j\delta}\bigl(|x-y|^j+t^{j/2}\bigr)\\
&\qquad+\Bigl(C_2\Bigl(\frac{1}{n}\Bigr)^{1\wedge 2\delta}\Bigr)^j\Bigl(\frac{1}{n}\Bigr)^j\Bigl[t^{j(1\wedge 2\delta)}\bigl(|x-y|^{2j}+t^j\\
&\qquad\qquad+|x-y|^{j(2+\mu)}+t^{j(1+\mu/2)}\bigr)\\
&\qquad\qquad+t^j\bigl(|x-y|^{j(2+\nu)}+t^{j(1+\nu/2)}\bigr)\Bigr]\biggr\}.
\end{aligned}
$$

Here const *depends only on δ, μ, ν and d.*

PROOF. By Claim 4.1 and Proposition 4.1, this estimate is obtained in the following way:

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}^d} p\Bigl(\frac{n-1}{n}t, x-z\Bigr)\Bigl|\Bigl(G\Bigl(\frac{t}{n}\Bigr)-e^{-(t/n)H}\Bigr)(z,y)\Bigr|\,dz\\
&\le \int_{\mathbb{R}^d} p\Bigl(\frac{n-1}{n}t, x-z\Bigr)p\Bigl(\frac{t}{n}, z-y\Bigr)\mathrm{const}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg\{\frac{C_1}{2}\left(\frac{t}{n}\right)^{\delta}|z-y| + \left(\frac{C_1}{2}\right)^2\left(\frac{t}{n}\right)^{2\delta}\left(|z-y|^2+\frac{t}{n}\right) \\
&\quad + \sum_{j=1}^{2} C_2^j\Bigg[\left(\frac{t}{n}\right)^{j(1\wedge 2\delta)}\left(|z-y|^{2j}+\left(\frac{t}{n}\right)^{j}+|z-y|^{j(2+\mu)}+\left(\frac{t}{n}\right)^{j(1+\mu/2)}\right) \\
&\qquad\qquad + \left(\frac{t}{n}\right)^{j}\left(|z-y|^{j(2+\nu)}+\left(\frac{t}{n}\right)^{j(1+\nu/2)}\right)\Bigg]\Bigg\}\,dz \\
&= \mathrm{const}\Bigg\{\frac{C_1}{2}\left(\frac{t}{n}\right)^{\delta}\int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right)p\left(\frac{t}{n}, z-y\right)|z-y|\,dz \\
&\quad + \left(\frac{C_1}{2}\right)^2\left(\frac{t}{n}\right)^{2\delta}\Bigg(\int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right)p\left(\frac{t}{n}, z-y\right)|z-y|^2\,dz \\
&\qquad\qquad + \frac{t}{n}p(t, x-y)\Bigg) \\
&\quad + \sum_{j=1}^{2}\Bigg[\left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^{j} t^{j(1\wedge 2\delta)} \\
&\qquad \times\Bigg(\int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right)p\left(\frac{t}{n}, z-y\right)|z-y|^{2j}\,dz \\
&\qquad\quad + \left(\frac{t}{n}\right)^{j} p(t, x-y) \\
&\qquad\quad + \int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right)p\left(\frac{t}{n}, z-y\right)|z-y|^{j(2+\mu)}\,dz \\
&\qquad\qquad + \left(\frac{t}{n}\right)^{j(1+\mu/2)} p(t, x-y)\Bigg) \\
&\quad + \left(C_2\frac{1}{n}\right)^{j} t^{j}\Bigg(\int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right)p\left(\frac{t}{n}, z-y\right)|z-y|^{j(2+\nu)}\,dz \\
&\qquad\qquad + \left(\frac{t}{n}\right)^{j(1+\nu/2)} p(t, x-y)\Bigg)\Bigg]\Bigg\} \\
&= \mathrm{const}\Bigg\{\frac{C_1}{2}\left(\frac{t}{n}\right)^{\delta} p(t, x-y)\int_{\mathbb{R}^d} p(1, z)\left|\sqrt{\frac{1}{n}\,\frac{n-1}{n}t}\,z+\frac{1}{n}(x-y)\right|dz \\
&\quad + \left(\frac{C_1}{2}\right)^2\left(\frac{t}{n}\right)^{2\delta} p(t, x-y)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_{\mathbb{R}^d} p(1,z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z+\frac{1}{n}(x-y)\right|^2 dz+\frac{t}{n}\right)\\
&+\sum_{j=1}^{2}\Bigg[\left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^j t^{j(1\wedge 2\delta)}p(t,x-y)\\
&\qquad\times\Bigg(\int_{\mathbb{R}^d} p(1,z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z+\frac{1}{n}(x-y)\right|^{2j} dz+\left(\frac{t}{n}\right)^j\\
&\qquad\quad+\int_{\mathbb{R}^d} p(1,z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z+\frac{1}{n}(x-y)\right|^{j(2+\mu)} dz\\
&\qquad\qquad+\left(\frac{t}{n}\right)^{j(1+\mu/2)}\Bigg)\\
&\qquad+\left(C_2\frac{1}{n}\right)^j t^j p(t,x-y)\\
&\qquad\times\Bigg(\int_{\mathbb{R}^d} p(1,z)\left|\sqrt{\frac{1}{n}\frac{n-1}{n}t}\,z+\frac{1}{n}(x-y)\right|^{j(2+\nu)} dz\\
&\qquad\qquad+\left(\frac{t}{n}\right)^{j(1+\nu/2)}\Bigg)\Bigg]\Bigg\}\\
&\le p(t,x-y)\\
&\quad\times\text{const}\sum_{j=1}^{2}\Bigg\{\left(\frac{C_1}{2}\right)^j\left(\frac{t}{n}\right)^{j(\delta+1/2)} t^{j\delta}\left(|x-y|^j+t^{j/2}\right)\\
&\qquad+\left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^j\left(\frac{1}{n}\right)^j\\
&\qquad\times\Big[t^{j(1\wedge 2\delta)}\left(|x-y|^{2j}+t^j+|x-y|^{j(2+\mu)}+t^{j(1+\mu/2)}\right)\\
&\qquad\qquad+t^j\left(|x-y|^{j(2+\nu)}+t^{j(1+\nu/2)}\right)\Big]\Bigg\}.\qquad\square
\end{aligned}
$$

4.5. *Estimate of* I_2. We start with the following lemma.

LEMMA 4.1. *Suppose V satisfies condition* $(\mathrm{A})_2$.

(i) *If V is moreover a C^2-function, then we have the equality*:

$$
\begin{aligned}
&(V(y)-V(x))\,E_0\left[\exp\left(-t\int_0^1 V\left(x+s(y-x)+\sqrt{t}\,X_0(s)\right)ds\right)\right]\\
&\qquad=\tfrac{1}{2}E_0\Bigg[\exp\left(-t\int_0^1 V\left(x+s(y-x)+\sqrt{t}\,X_0(s)\right)ds\right)
\end{aligned}
$$

$$\times\biggl(t\int_0^1(1-2s)\,\Delta V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds$$
$$-t^2\biggl\langle\int_0^1\nabla V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds,$$
$$\int_0^1(1-2s)\,\nabla V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\biggr\rangle$$
$$-2\int_0^1\bigl\langle\nabla V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr),\,x-y\bigr\rangle\,ds\biggr)\biggr].$$

(ii) *Even though V is not a C^2-function, we have the inequality*:

$$\biggl|(V(y)-V(x))\,E_0\biggl[\exp\biggl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\biggr)\biggr]\biggr|$$
$$\le\frac{1}{2}\biggl\{C_2 2\sqrt{d}\biggl(\frac{(1-2\delta)_+}{e}\biggr)^{(1-2\delta)_+}t^{1\wedge 2\delta}+C_2\sqrt{d}\,t$$
$$+C_1^2\biggl(\frac{2(1-\delta)}{e}\biggr)^{2(1-\delta)}t^{2\delta}+C_1 2\biggl(\frac{1-\delta}{e}\biggr)^{1-\delta}t^{-1+\delta}|x-y|\biggr\}.$$

PROOF. (i) Let $V\colon\mathbb{R}^d\to[0,\infty)$ be a C^2-function satisfying $(\mathrm{A})_2$. As before, for simplicity set $H:=-(\Delta/2)+V$. Note that the integral kernel of $[e^{-tH},V]$ is given by the following expression:

$$\begin{aligned}[\exp(-tH),V](x,y)&=p(t,x-y)(V(y)-V(x))\\ &\quad\times E_0\biggl[\exp\biggl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\biggr)\biggr].\end{aligned}\tag{4.2}$$

Since $He^{-tH}=e^{-tH}H$, $[e^{-tH},V]$ is rewritten as

$$\begin{aligned}[e^{-tH},V]=e^{-tH}V-Ve^{-tH}&=e^{-tH}\biggl(H+\frac{\Delta}{2}\biggr)-\biggl(H+\frac{\Delta}{2}\biggr)e^{-tH}\\ &=e^{-tH}\frac{\Delta}{2}-\frac{\Delta}{2}e^{-tH},\end{aligned}$$

and hence, for $f\in C_0^\infty(\mathbb{R}^d)$

$$[e^{-tH},V]f(x)=e^{-tH}\frac{\Delta}{2}f(x)-\frac{\Delta}{2}e^{-tH}f(x).\tag{4.3}$$

On the other hand, by the usual Feynman–Kac formula,

$$\exp(-tH)f(x)=E_0\biggl[\exp\biggl(-t\int_0^1 V\bigl(x+\sqrt{t}\,X(s)\bigr)\,ds\biggr)f\bigl(x+\sqrt{t}\,X(1)\bigr)\biggr],$$

it is observed that

$$
\begin{aligned}
&\Delta \exp(-tH) f(x) \\
&\quad = E_0\bigg[\exp\left(-t\int_0^1 V\big(x+\sqrt{t}\,X(s)\big)\,ds\right) \\
&\qquad\quad \times\bigg\{t^2\left|\int_0^1 \nabla V\big(x+\sqrt{t}\,X(s)\big)\,ds\right|^2 f\big(x+\sqrt{t}\,X(1)\big) \\
&\qquad\qquad -2t\left\langle\int_0^1 \nabla V\big(x+\sqrt{t}\,X(s)\big)\,ds, \nabla f\big(x+\sqrt{t}\,X(1)\big)\right\rangle \\
&\qquad\qquad -t\int_0^1 \Delta V\big(x+\sqrt{t}\,X(s)\big)\,ds\; f\big(x+\sqrt{t}\,X(1)\big) + \Delta f\big(x+\sqrt{t}\,X(1)\big)\bigg\}\bigg] \\
&\quad = \exp(-tH)\,\Delta f(x) \\
&\qquad + E_0\bigg[\exp\left(-t\int_0^1 V\big(x+\sqrt{t}\,X(s)\big)\,ds\right) \\
&\qquad\qquad \times\bigg\{\bigg(-t\int_0^1 \Delta V\big(x+\sqrt{t}\,X(s)\big)\,ds \\
&\qquad\qquad\qquad + t^2\left|\int_0^1 \nabla V\big(x+\sqrt{t}\,X(s)\big)\,ds\right|^2\bigg) f\big(x+\sqrt{t}\,X(1)\big) \\
&\qquad\qquad\quad -2t\left\langle\int_0^1 \nabla V\big(x+\sqrt{t}\,X(s)\big)\,ds, \nabla f\big(x+\sqrt{t}\,X(1)\big)\right\rangle\bigg\}\bigg].
\end{aligned}
$$

Combining this with (4.3), we see

$$
\begin{aligned}
&[\exp(-tH), V]\, f(x) \\
&\quad = E_0\bigg[\exp\left(-t\int_0^1 V\big(x+\sqrt{t}\,X(s)\big)\,ds\right) \\
&\qquad\quad \times\bigg(\frac{t}{2}\int_0^1 \Delta V\big(x+\sqrt{t}\,X(s)\big)\,ds \\
&\qquad\qquad - \frac{t^2}{2}\left|\int_0^1 \nabla V\big(x+\sqrt{t}\,X(s)\big)\,ds\right|^2\bigg) f\big(x+\sqrt{t}\,X(1)\big)\bigg] \\
&\qquad + E_0\bigg[\exp\left(-t\int_0^1 V\big(x+\sqrt{t}\,X(s)\big)\,ds\right) \\
&\qquad\qquad \times t\left\langle\int_0^1 \nabla V\big(x+\sqrt{t}\,X(s)\big)\,ds, \nabla f\big(x+\sqrt{t}\,X(1)\big)\right\rangle\bigg].
\end{aligned}
\tag{4.4}
$$

Now we compute the second term of (4.4). By integration by parts, and the formula, $(\partial/\partial y_i)p(t,x-y) = p(t,x-y)(x_i - y_i)/t$,

$$
\begin{aligned}
&E_0\Bigl[\exp\Bigl(-t\int_0^1 V\bigl(x+\sqrt{t}\,X(s)\bigr)\,ds\Bigr)\\
&\qquad\times t\Bigl\langle \int_0^1 \nabla V\bigl(x+\sqrt{t}\,X(s)\bigr)\,ds, \nabla f\bigl(x+\sqrt{t}\,X(1)\bigr)\Bigr\rangle\Bigr]\\
&\quad= \int_{\mathbb{R}^d} p(t,x-y)\Bigl\langle E_0\Bigl[\exp\Bigl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr)\\
&\qquad\qquad\times t\int_0^1 \nabla V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr], \nabla f(y)\Bigr\rangle\,dy\\
&\quad= \sum_{i=1}^d \int_{\mathbb{R}^d} p(t,x-y)\,E_0\Bigl[\exp\Bigl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr)\\
&\qquad\qquad\times t\int_0^1 \partial_i V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr]\,\partial_i f(y)\,dy\\
&\quad= -\sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial y_i}\Bigl(p(t,x-y)\\
&\qquad\qquad\times E_0\Bigl[\exp\Bigl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr)\\
&\qquad\qquad\qquad\times t\int_0^1 \partial_i V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr]\Bigr) f(y)\,dy\\
&\quad= -\sum_{i=1}^d \int_{\mathbb{R}^d} p(t,x-y) f(y)\\
&\qquad\qquad\times\Bigl\{(x_i-y_i)\,E_0\Bigl[\exp\Bigl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr)\\
&\qquad\qquad\qquad\times\int_0^1 \partial_i V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr]\\
&\qquad\qquad\quad+E_0\Bigl[\exp\Bigl(-t\int_0^1 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr)\\
&\qquad\qquad\qquad\times\Bigl(-t^2\int_0^1 s\,\partial_i V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\\
&\qquad\qquad\qquad\quad\times\int_0^1 \partial_i V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\\
&\qquad\qquad\qquad\quad+t\int_0^1 s\,\partial_i^2 V\bigl(x+s(y-x)+\sqrt{t}\,X_0(s)\bigr)\,ds\Bigr)\Bigr]\Bigr\}\,dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} p(t, x-y) f(y)\, dy \\
&\quad \times E_0\Bigg[\exp\left(-t\int_0^1 V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\right) \\
&\qquad \times \Bigg(-\left\langle \int_0^1 \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds,\, x-y \right\rangle \\
&\qquad\quad + t^2 \bigg\langle \int_0^1 s\, \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds, \\
&\qquad\qquad \int_0^1 \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds \bigg\rangle \\
&\qquad\quad - t\int_0^1 s\, \Delta V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\Bigg)\Bigg] .
\end{aligned}
$$

Hence, by this and (4.4), we have

$$
\begin{aligned}
&[\exp(-tH), V]\, f(x) \\
&\quad = \int_{\mathbb{R}^d} p(t, x-y) f(y)\, dy \\
&\qquad \times E_0\Bigg[\exp\left(-t\int_0^1 V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\right) \\
&\qquad\quad \times \Bigg(\frac{t}{2}\int_0^1 \Delta V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds \\
&\qquad\qquad - \frac{t^2}{2}\left|\int_0^1 \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\right|^2 \\
&\qquad\qquad - \left\langle \int_0^1 \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds,\, x-y \right\rangle \\
&\qquad\qquad + t^2 \bigg\langle \int_0^1 \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds, \\
&\qquad\qquad\qquad \int_0^1 s\, \nabla V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds \bigg\rangle \\
&\qquad\qquad - t\int_0^1 s\, \Delta V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\Bigg)\Bigg],
\end{aligned}
$$

from which, together with (4.2), the desired equality follows at once.

(ii) We first suppose V is a C^2-function. Then, by (i),

$$\begin{aligned}
&\left|(V(y)-V(x))\,E_0\left[\exp\left(-t\int_0^1 V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\,ds\right)\right]\right|\\
&\quad\le \tfrac{1}{2}E_0\left[\exp\left(-t\int_0^1 V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\,ds\right)\right.\\
&\qquad\times\left\{t\int_0^1\left|\Delta V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\right|ds\right.\\
&\qquad\quad+t^2\left(\int_0^1\left|\nabla V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\right|ds\right)^2\\
&\qquad\quad\left.\left.+2\int_0^1\left|\nabla V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\right|ds\,|x-y|\right\}\right].
\end{aligned}$$

Since, by $(\mathrm{A})_2$(i),

$$|\nabla V(x)|\le C_1V(x)^{1-\delta}$$

and, by $(\mathrm{A})_2$(ii),

$$\begin{aligned}
|\Delta V(x)|&\le C_2\sqrt{d}\left\{V(x)^{(1-2\delta)_+}\big(1+\delta_{\mu,0}\big)+\delta_{\nu,0}\right\}\\
&\le C_2\sqrt{d}\left(2V(x)^{(1-2\delta)_+}+1\right),
\end{aligned}$$

it is easily seen that

$$\begin{aligned}
&\int_0^1\left|\nabla V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\right|ds\\
&\qquad\le C_1\left(\int_0^1 V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\,ds\right)^{1-\delta},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\left|\Delta V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\right|ds\\
&\qquad\le C_2\sqrt{d}\left\{2\left(\int_0^1 V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\,ds\right)^{(1-2\delta)_+}+1\right\},
\end{aligned}$$

where the Jensen inequality has been used. Hence, combining these with (3.8), we have

$$\begin{aligned}
&\left|(V(y)-V(x))\,E_0\left[\exp\left(-t\int_0^1 V\big(x+s(y-x)+\sqrt{t}\,X_0(s)\big)\,ds\right)\right]\right|\\
&\quad\le\frac{1}{2}\left\{C_2 2\sqrt{d}\,t^{1\wedge 2\delta}\left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+}+C_2\sqrt{d}\,t\right.\\
&\qquad\left.+C_1^2t^{2\delta}\left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)}+C_1 2|x-y|t^{-1+\delta}\left(\frac{1-\delta}{e}\right)^{1-\delta}\right\},
\end{aligned}$$

which is just the inequality in (ii).

Next we consider a general case. To this end, take a $\psi \in C_0^\infty(\mathbb{R}^d \to [0, \infty))$ such that $\int_{\mathbb{R}^d} \psi(x)\, dx = 1$ and set

$$V_\varepsilon(x) := \int_{\mathbb{R}^d} \psi_\varepsilon(x-y) V(y)\, dy$$

where $\psi_\varepsilon(z) := (1/\varepsilon)^d \psi(z/\varepsilon)$ $(\varepsilon > 0)$. Then V_ε is smooth and satisfies $(\mathrm{A})_2$ with the same constants as V has. Moreover $V_\varepsilon \to V$ uniformly on compact sets, as $\varepsilon \to 0$. Thus, from what was seen in the above, and then by letting $\varepsilon \to 0$, the desired inequality for a general V is obtained, and the proof is complete. □

LEMMA 4.2. *Suppose V satisfies condition* $(\mathrm{A})_2$. *Let $t > 0$ and $s \geq 0$. Then we have the following*:

(i)
$$\begin{aligned}
&[\exp(-tH), \exp(-sV)](x, y) \\
&\quad = -p(t, x-y) \int_0^s \exp(-uV(x))(V(y) - V(x)) \\
&\qquad \times E_0\Big[\exp\Big(-t\int_0^1 V\big(x + \theta(y-x) \\
&\qquad\qquad + \sqrt{t}\, X_0(\theta)\big)\, d\theta\Big)\Big] \\
&\qquad \times \exp(-(s-u)V(y))\, du.
\end{aligned}$$

(ii)
$$\begin{aligned}
&\big|[e^{-tH}, e^{-sV}](x, y)\big| \\
&\quad \leq p(t, x-y)\frac{s}{2}\bigg\{C_2 2\sqrt{d}\left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+} t^{1 \wedge 2\delta} \\
&\qquad + C_2\sqrt{d}\, t + C_1^2\left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} t^{2\delta} \\
&\qquad + C_1 2|x-y|\left(\frac{1-\delta}{e}\right)^{1-\delta} t^{-1+\delta}\bigg\}.
\end{aligned}$$

For the proof, (i) is trivial; (ii) follows immediately from (i) and Lemma 4.1.

CLAIM 4.4.

$$\begin{aligned}
I_2 &\leq p(t, x-y) \\
&\quad \times \mathrm{const}\left\{\frac{C_2}{n}(t^{1+1\wedge 2\delta} + t^2) + \frac{1}{n}\big(C_1 t^\delta(|x-y| + t^{1/2}) + C_1^2 t^{1+2\delta}\big)\right\}.
\end{aligned}$$

Here const *depends only on δ and d.*

PROOF. By Lemma 4.2 and Proposition 4.1, this estimate is obtained in the following way:

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} \left| [e^{-(t/2n)V}, e^{-((n-1)/n)tH}](x, z) \right| p\left(\frac{t}{n}, z-y\right) dz \\
&\le \int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right) p\left(\frac{t}{n}, z-y\right) dz \\
&\quad \times \frac{t}{4n} \left\{ C_2 2\sqrt{d} \left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+} \left(\frac{n-1}{n}t\right)^{1\wedge 2\delta} + C_2\sqrt{d}\,\frac{n-1}{n}t \right. \\
&\qquad + C_1^2 \left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} \left(\frac{n-1}{n}t\right)^{2\delta} \\
&\qquad \left. + C_1 2|x-z| \left(\frac{1-\delta}{e}\right)^{1-\delta} \left(\frac{n-1}{n}t\right)^{-1+\delta} \right\} \\
&= \frac{t}{4n} \left\{ C_2 2\sqrt{d} \left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+} \left(\frac{n-1}{n}t\right)^{1\wedge 2\delta} p(t, x-y) \right. \\
&\qquad + C_2\sqrt{d}\,\frac{n-1}{n}t\,p(t, x-y) \\
&\qquad + C_1^2 \left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} \left(\frac{n-1}{n}t\right)^{2\delta} p(t, x-y) \\
&\qquad + C_1 2 \left(\frac{1-\delta}{e}\right)^{1-\delta} \left(\frac{n-1}{n}t\right)^{-1+\delta} \\
&\qquad \left. \times \int_{\mathbb{R}^d} p\left(\frac{n-1}{n}t, x-z\right) p\left(\frac{t}{n}, z-y\right) |x-z|\, dz \right\} \\
&= \frac{t}{4n} p(t, x-y) \left\{ C_2 2\sqrt{d} \left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+} \left(\frac{n-1}{n}t\right)^{1\wedge 2\delta} \right. \\
&\qquad + C_2\sqrt{d}\,\frac{n-1}{n}t + C_1^2 \left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} \left(\frac{n-1}{n}t\right)^{2\delta} \\
&\qquad + C_1 2 \left(\frac{1-\delta}{e}\right)^{1-\delta} \left(\frac{n-1}{n}t\right)^{-1+\delta} \\
&\qquad \left. \times \int_{\mathbb{R}^d} p(1, z) \left| \sqrt{\frac{1}{n}\frac{n-1}{n}t}\, z + \frac{n-1}{n}(x-y) \right| dz \right\} \\
&\le p(t, x-y) \mathrm{const} \left\{ \frac{C_2}{n} (t^{1+1\wedge 2\delta} + t^2) \right. \\
&\qquad \left. + \frac{1}{n} \left(C_1 t^{\delta} \left(|x-y| + t^{1/2} \right) + C_1^2 t^{1+2\delta} \right) \right\},
\end{aligned}
$$

where const depends only on δ and d. □

4.6. *Proof for* $n \geq 2$. Recall (4.1). By Claims 4.2, 4.3 and 4.4,

$$\begin{aligned}
&\left|\left(G\left(\frac{t}{n}\right)^n - e^{-tH}\right)(x, y)\right| \\
&\quad \leq p(t, x-y) \\
&\qquad \times \text{const}\Bigg\{C_1\left(\frac{1}{n}\right)^{1/2+1/2\wedge\delta} t^{\delta}\big(|x-y| + t^{1/2}\big) \\
&\qquad\qquad + C_1^2\left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{2\delta}\big(|x-y|^2 + t\big) \\
&\qquad\qquad + \max\left\{C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}, \left(C_2\left(\frac{1}{n}\right)^{1\wedge 2\delta}\right)^2\right\} \\
&\qquad\qquad \times \Bigg(t^2 + \sum_{j=1}^{2}\Big[t^{j(1\wedge 2\delta)}\big(|x-y|^{2j} + t^j + |x-y|^{j(2+\mu)} \\
&\qquad\qquad\qquad + t^{j(1+\mu/2)}\big) + t^j\big(|x-y|^{j(2+\nu)} + t^{j(1+\nu/2)}\big)\Big]\Bigg)\Bigg\}.
\end{aligned}$$

Here const depends only on δ, μ, ν and d.

5. Proof of Theorem 1.3. Condition $(\mathrm{V})_2$ implies condition $(\mathrm{A})_2$ with $\delta = 1 \wedge 1/\rho$, $C_1 = c_1 c^{-(1-1\wedge 1/\rho)}$, $C_2 = c_2 2^{(\rho-3)_+}(\frac{1}{2}c^{-(1-2(1\wedge 1/\rho))_+} \vee 1)$, $\mu = 0$ and $\nu = (\rho - 2)_+$. Hence from Theorems 1.1 and 1.2, the latter half of Theorem 1.3 follows.

In the following, we assume condition $(\mathrm{V})_1$. That is, $V\colon \mathbb{R}^d \to [0, \infty)$ is a C^1-function such that (i) $V(z) \geq c\langle z\rangle^{\rho}$ and (ii) $|\nabla V(z)| \leq c_1\langle z\rangle^{(\rho-1)_+}$ for some $0 \leq \rho < \infty$, $0 < c < \infty$ and $0 \leq c_1 < \infty$.

Let us adopt an idea from [2]. Take a $\psi \in C_0^{\infty}(\mathbb{R}^d \to [0, \infty))$ such that

$$\operatorname{supp}\psi \subset \{x \in \mathbb{R}^d; |x| < 1\},$$

$$\int_{\mathbb{R}^d}\psi(x)\,dx = 1.$$

For $0 < \varepsilon \leq \frac{1}{4}$, set

$$V_{\varepsilon}(x) := \left(\frac{1}{\varepsilon\langle x\rangle^{\eta}}\right)^d \int_{\mathbb{R}^d}\psi\left(\frac{x-y}{\varepsilon\langle x\rangle^{\eta}}\right)V(y)\,dy. \tag{5.1}$$

Here

$$\eta := ((\rho-1)\vee 0)\wedge 1 = \begin{cases} 0, & 0 \leq \rho < 1, \\ \rho - 1, & 1 \leq \rho < 2, \\ 1, & \rho \geq 2. \end{cases}$$

Clearly, V_{ε} is a smooth function, and it satisfies the following.

LEMMA 5.1. (i) $V_\varepsilon(x) \geq \tilde{c}\langle x\rangle^\rho$, *where* $\tilde{c} = c/4^\rho$.

(ii) $|V_\varepsilon(x) - V(x)| \leq \tilde{C}\varepsilon\langle x\rangle^{(\rho-1)_+ + \eta}$, *where* $\tilde{C} = c_1(\frac{5}{4})^{(\rho-1)_+}$.

(iii) $|\nabla V_\varepsilon(x)| \leq \tilde{c}_1\langle x\rangle^{(\rho-1)_+}$, *where* $\tilde{c}_1 = c_1(\frac{5}{4})^{\rho\vee 1}$.

(iv) $|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \leq (1/\varepsilon)\tilde{c}_2\{\langle x\rangle^{(\rho-2\lambda)_+} + |x-y|^{(\rho-2\lambda)_+}\}|x-y|$, *where* $\lambda := \frac{1}{2}(1+\eta)$ *and* $\tilde{c}_2 = c_1(\frac{5}{4})^{(\rho-1)_+}2^{(\rho-3)_+}(5d/16+2)$.

PROOF. In [2], this lemma is presented without proof. To be complete we here give the proof. First of all, we note that for $0 < \varepsilon \leq \frac{1}{4}$ and $|z| \leq 1$,

$$\frac{1}{4}\langle x\rangle \leq \langle x - \varepsilon\langle x\rangle^\eta z\rangle \leq \frac{5}{4}\langle x\rangle. \tag{5.2}$$

Indeed, the first half of (5.2) is seen in the following way:

$$\begin{aligned}
\langle x - \varepsilon\langle x\rangle^\eta z\rangle &\geq \tfrac{1}{2}\big(1 + |x - \varepsilon\langle x\rangle^\eta z|\big)\\
&\geq \tfrac{1}{2}\big(1 + |x| - \varepsilon\langle x\rangle^\eta|z|\big)\\
&\geq \tfrac{1}{2}\big(1 + |x| - \tfrac{1}{4}\langle x\rangle^\eta\big)\\
&\geq \tfrac{1}{2}\big\{1 + |x| - \tfrac{1}{4}(1 + |x|^\eta)\big\}\\
&= \tfrac{1}{2}\big\{\tfrac{3}{4} + |x| - \tfrac{1}{4}|x|^\eta\big\}\\
&\geq \tfrac{1}{2}\big\{\tfrac{3}{4} + |x| - \tfrac{1}{4}(1 - \eta + \eta|x|)\big\}\\
&= \tfrac{1}{2}\big\{\tfrac{1}{2} + \tfrac{\eta}{4} + (1 - \tfrac{\eta}{4})|x|\big\}\\
&\geq \tfrac{1}{4}\{1 + |x|\} \geq \tfrac{1}{4}\langle x\rangle
\end{aligned}$$

where we have used the inequality $t^\eta \leq 1 - \eta + \eta t$ $(t \geq 0)$. The second half of (5.2) is as follows:

$$\begin{aligned}
\langle x - \varepsilon\langle x\rangle^\eta z\rangle &\leq \langle x\rangle + \varepsilon\langle x\rangle^\eta|z|\\
&\leq \langle x\rangle + \tfrac{1}{4}\langle x\rangle^\eta\\
&\leq \langle x\rangle + \tfrac{1}{4}\langle x\rangle \leq \tfrac{5}{4}\langle x\rangle.
\end{aligned}$$

We also note that

$$V_\varepsilon(x) = \int_{|z|\leq 1}\psi(z)V(x - \varepsilon\langle x\rangle^\eta z)\,dz. \tag{5.3}$$

By this, condition $(\mathrm{V})_1$(i) and (5.2),

$$V_\varepsilon(x) \geq \int_{|z|\leq 1}\psi(z)c\langle x - \varepsilon\langle x\rangle^\eta z\rangle^\rho\,dz \geq \frac{c}{4^\rho}\langle x\rangle^\rho,$$

which shows (i). By (5.3) and the C^1-smoothness of V,

$$\begin{aligned}
V_\varepsilon(x) - V(x) &= \int_{|z|\leq 1}\psi(z)\big(V(x - \varepsilon\langle x\rangle^\eta z) - V(x)\big)\,dz\\
&= -\varepsilon\langle x\rangle^\eta\int_{|z|\leq 1}\psi(z)\,dz\int_0^1\big\langle\nabla V(x - \theta\varepsilon\langle x\rangle^\eta z), z\big\rangle\,d\theta,
\end{aligned} \tag{5.4}$$

$$\nabla V_\varepsilon(x) = \int_{|z|\le 1} \psi(z)\, dz \left\{\nabla V(x-\varepsilon\langle x\rangle^\eta z) - \varepsilon\eta\langle x\rangle^{\eta-2} x \left\langle \nabla V(x-\varepsilon\langle x\rangle^\eta z), z\right\rangle\right\}. \tag{5.5}$$

Hence, by $(\mathrm{V})_1$(ii) and (5.2),

$$\begin{aligned}
|V_\varepsilon(x) - V(x)| &\le \varepsilon\langle x\rangle^\eta \int_{|z|\le 1} \psi(z)\, dz \int_0^1 c_1 \langle x - \theta\varepsilon\langle x\rangle^\eta z\rangle^{(\rho-1)_+}\, d\theta \\
&\le \varepsilon\langle x\rangle^\eta c_1\left(\tfrac{5}{4}\langle x\rangle\right)^{(\rho-1)_+} = c_1\left(\tfrac{5}{4}\right)^{(\rho-1)_+}\varepsilon\langle x\rangle^{(\rho-1)_+ +\eta}, \\
|\nabla V_\varepsilon(x)| &\le \int_{|z|\le 1} \psi(z)\left(1+\tfrac{1}{4}\langle x\rangle^{\eta-1}\right)\left|\nabla V(x-\varepsilon\langle x\rangle^\eta z)\right| dz \\
&\le \int_{|z|\le 1} \psi(z)\tfrac{5}{4} c_1 \langle x - \varepsilon\langle x\rangle^\eta z\rangle^{(\rho-1)_+}\, dz \\
&\le c_1 \tfrac{5}{4}\left(\tfrac{5}{4}\langle x\rangle\right)^{(\rho-1)_+} = c_1\left(\tfrac{5}{4}\right)^{\rho\vee 1}\langle x\rangle^{(\rho-1)_+},
\end{aligned}$$

which shows (i) and (ii), respectively.

It remains to show (iv). First note that $(\rho-2\lambda)_+ = (\rho-2)_+ = (\rho-1)_+ - \eta$. Equation (5.5) is rewritten as

$$\nabla V_\varepsilon(x) = \left(\frac{1}{\varepsilon\langle x\rangle^\eta}\right)^d \times \int_{\mathbb{R}^d} \psi\left(\frac{x-y}{\varepsilon\langle x\rangle^\eta}\right)\left\{\nabla V(y) - \varepsilon\eta\langle x\rangle^{\eta-2}x\left\langle \nabla V(y), \frac{x-y}{\varepsilon\langle x\rangle^\eta}\right\rangle\right\} dy.$$

Differentiating with respect to x_i, we see

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\nabla V_\varepsilon(x) \\
&\quad = (-\eta d)\langle x\rangle^{-2} x_i \int_{\mathbb{R}^d} \psi(z)\left\{\nabla V(x-\varepsilon\langle x\rangle^\eta z) - \varepsilon\eta\langle x\rangle^{\eta-2}x\left\langle \nabla V(x-\varepsilon\langle x\rangle^\eta z), z\right\rangle\right\} dz \\
&\qquad + \int_{\mathbb{R}^d} \langle \nabla\psi(z), \delta_i(\varepsilon\langle x\rangle^\eta)^{-1} - \eta\langle x\rangle^{-2}x_i z\rangle \\
&\qquad\quad \times\left\{\nabla V(x-\varepsilon\langle x\rangle^\eta z) - \varepsilon\eta\langle x\rangle^{\eta-2}x\left\langle \nabla V(x-\varepsilon\langle x\rangle^\eta z), z\right\rangle\right\} dz \\
&\qquad - \varepsilon\eta\int_{\mathbb{R}^d} \psi(z)\left\{\langle x\rangle^{\eta-2}\left(\delta_i + (\eta-2)\langle x\rangle^{-2}x_i x\right)\right. \\
&\qquad\quad \times\left\langle \nabla V(x-\varepsilon\langle x\rangle^\eta z), z\right\rangle + \langle x\rangle^{\eta-2}x\langle \nabla V(x-\varepsilon\langle x\rangle^\eta z), \\
&\qquad\quad \left.\delta_i(\varepsilon\langle x\rangle^\eta)^{-1} - \eta\langle x\rangle^{-2}x_i z\rangle\right\} dz,
\end{aligned}$$

where $\delta_i = (0, \dots, \overset{i}{1}, \dots, 0) \in \mathbb{R}^d$, and hence, for $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
&\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \nabla V_\varepsilon(x) \xi_i \\
&\quad = (-\eta d) \langle x \rangle^{-2} \langle x, \xi \rangle \int_{\mathbb{R}^d} \psi(z) \Big\{ \nabla V(x - \varepsilon \langle x \rangle^\eta z) \\
&\qquad\qquad - \varepsilon \eta \langle x \rangle^{\eta - 2} x \big\langle \nabla V(x - \varepsilon \langle x \rangle^\eta z), z \big\rangle \Big\} \, dz \\
&\qquad + \int_{\mathbb{R}^d} \left(\frac{1}{\varepsilon \langle x \rangle^\eta} \langle \nabla \psi(z), \xi \rangle - \eta \frac{1}{\langle x \rangle^2} \langle \nabla \psi(z), z \rangle \langle x, \xi \rangle \right) \\
&\qquad\quad \times \Big(\nabla V(x - \varepsilon \langle x \rangle^\eta z) - \varepsilon \eta \langle x \rangle^{\eta - 2} x \big\langle \nabla V(x - \varepsilon \langle x \rangle^\eta z), z \big\rangle \Big) \, dz \\
&\qquad - \varepsilon \eta \int_{\mathbb{R}^d} \psi(z) \bigg\{ \langle x \rangle^{\eta - 2} \big(\xi - 2 \langle x \rangle^{-2} \langle x, \xi \rangle x \big) \big\langle \nabla V(x - \varepsilon \langle x \rangle^\eta z), z \big\rangle \\
&\qquad\qquad + \langle x \rangle^{\eta - 2} x \frac{1}{\varepsilon \langle x \rangle^\eta} \big\langle \nabla V(x - \varepsilon \langle x \rangle^\eta z), \xi \big\rangle \bigg\} \, dz.
\end{aligned}$$

By $(\mathrm{V})_1$(ii) and (5.2),

$$\begin{aligned}
&\left| \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \nabla V_\varepsilon(x) \xi_i \right| \\
&\quad \le d \langle x \rangle^{-1} \big(1 + \tfrac{1}{4} \langle x \rangle^{\eta - 1} \big) |\xi| \int_{\mathbb{R}^d} \psi(z) c_1 \langle x - \varepsilon \langle x \rangle^\eta z \rangle^{(\rho - 1)_+} \, dz \\
&\qquad + \big(1 + \tfrac{1}{4} \langle x \rangle^{\eta - 1} \big) \left(\frac{1}{\varepsilon \langle x \rangle^\eta} + \frac{1}{\langle x \rangle} \right) |\xi| \\
&\qquad \times \int_{\mathbb{R}^d} |\nabla \psi(z)| c_1 \langle x - \varepsilon \langle x \rangle^\eta z \rangle^{(\rho - 1)_+} \, dz \\
&\qquad + \varepsilon \langle x \rangle^{\eta - 1} \left(3 \frac{1}{\langle x \rangle} + \frac{1}{\varepsilon (x)^\eta} \right) |\xi| \int_{\mathbb{R}^d} \psi(z) c_1 \langle x - \varepsilon \langle x \rangle^\eta z \rangle^{(\rho - 1)_+} \, dz \\
&\quad \le d \frac{1}{4} \frac{5}{4} \frac{1}{\varepsilon} \langle x \rangle^{-\eta} |\xi| c_1 \left(\frac{5}{4} \langle x \rangle \right)^{(\rho - 1)_+} \\
&\qquad + \left(\frac{5}{4} \right)^2 \frac{1}{\varepsilon} \langle x \rangle^{-\eta} |\xi| \int_{\mathbb{R}^d} |\nabla \psi(z)| \, dz \, c_1 \left(\frac{5}{4} \langle x \rangle \right)^{(\rho - 1)_+} \\
&\qquad + \frac{1}{4} \frac{7}{4} \frac{1}{\varepsilon} \langle x \rangle^{-\eta} |\xi| c_1 \left(\frac{5}{4} \langle x \rangle \right)^{(\rho - 1)_+} \\
&\quad = \frac{1}{\varepsilon} c_1 \left(\frac{5}{4} \right)^{(\rho - 1)_+} \left(\frac{5d}{16} + 2 \right) |\xi| \langle x \rangle^{(\rho - 1)_+ - \eta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \\
&\quad \le \int_0^1 \left| \sum_{i=1}^d \partial_i \nabla V_\varepsilon(\theta x + (1-\theta)y)(x_i - y_i) \right| d\theta \\
&\quad \le \int_0^1 \frac{1}{\varepsilon} c_1 \left(\frac{5}{4}\right)^{(\rho-1)_+} \left(\frac{5d}{16} + 2\right) |x-y| \langle \theta x + (1-\theta) y \rangle^{(\rho - 2\lambda)_+} \, d\theta \\
&\quad \le \frac{1}{\varepsilon} c_1 \left(\frac{5}{4}\right)^{(\rho-1)_+} \left(\frac{5d}{16} + 2\right) (\langle x \rangle + |x-y|)^{(\rho-2\lambda)_+} |x-y| \\
&\quad \le \frac{1}{\varepsilon} c_1 \left(\frac{5}{4}\right)^{(\rho-1)_+} \left(\frac{5d}{16} + 2\right) 2^{((\rho-2\lambda)_+ - 1)_+} \\
&\qquad \times \left(\langle x \rangle^{(\rho-2\lambda)_+} + |x-y|^{(\rho-2\lambda)_+}\right) |x-y|
\end{aligned}$$

and we have (iv). □

As a consequence of Lemma 5.1, it is easily seen that V_ε satisfies condition $(A)_2$. That is,

$$(A)_{2,\varepsilon} \qquad |\nabla V_\varepsilon(x)| \le \tilde{C}_1 V_\varepsilon(x)^{1 - 1\wedge\lambda/\rho}$$

$$|\nabla V_\varepsilon(x) - \nabla V_\varepsilon(y)| \le \frac{1}{\varepsilon} \tilde{C}_2 \left\{ V_\varepsilon(x)^{(1 - 2(1\wedge\lambda/\rho))_+} + |x-y|^{(\rho-2)_+} \right\} |x-y|,$$

where $\tilde{C}_1 = \tilde{c}_1 c^{-(1 - 1\wedge\lambda/\rho)}$ and $\tilde{C}_2 = \tilde{c}_2 (c^{-(1-2(1\wedge\lambda/\rho))_+} \vee 1)$.

Now let

$$\begin{aligned}
H_\varepsilon &:= -\frac{\Delta}{2} + V_\varepsilon \\
K_\varepsilon(t) &:= e^{-(t/2)V_\varepsilon} e^{(t/2)\Delta} e^{-(t/2)V_\varepsilon} \\
G_\varepsilon(t) &:= e^{-tV_\varepsilon} e^{(t/2)\Delta}
\end{aligned}$$

and let us denote the integral kernels of e^{-tH_ε}, $K_\varepsilon(t/n)^n$ and $G_\varepsilon(t/n)^n$ by $e^{-tH_\varepsilon}(x,y)$, $K_\varepsilon(t/n)^n(x,y)$ and $G_\varepsilon(t/n)^n(x,y)$, respectively. Then, by Theorems 1.1 and 1.2, we have the following.

CLAIM 5.1.

$$\begin{aligned}
&\left| K_\varepsilon\left(\frac{t}{n}\right)^n (x,y) - e^{-tH_\varepsilon}(x,y) \right| \\
&\quad \le p(t, x-y) \mathrm{const} \left\{ \tilde{C}_1^2 t^{1+2(1\wedge\lambda/\rho)} \left(\frac{1}{n}\right)^{2(1\wedge\lambda/\rho)} \right. \\
&\qquad\qquad \left. + \max\left\{ \frac{\tilde{C}_2}{\varepsilon} \left(\frac{1}{n}\right)^{1\wedge 2(\lambda/\rho)}, \left(\frac{\tilde{C}_2}{\varepsilon} \left(\frac{1}{n}\right)^{1\wedge 2(\lambda/\rho)} \right)^2 \right\} \right.
\end{aligned}$$

$$\times \sum_{j=1}^{2} \Big[t^{j(1 \wedge 2(\lambda/\rho))}\big(|x-y|^{2j} + t^j\big) + t^j\big(|x-y|^{j(2\vee\rho)} + t^{j(2\vee\rho)/2}\big)\Big]\Bigg\},$$

$$\begin{aligned}
&\left| G_\varepsilon\left(\frac{t}{n}\right)^n (x,y) - e^{-tH_\varepsilon}(x,y)\right| \\
&\quad \le p(t, x-y)\mathrm{const}\Bigg\{\left(\frac{1}{n}\right)^{1\wedge 2(\lambda/\rho)} \max\left\{\tilde{C}_1, \tilde{C}_1^2\right\} \\
&\qquad\qquad \times \sum_{j=1}^{2} t^{j(1\wedge\lambda/\rho)}\big(|x-y|^j + t^{j/2}\big) \\
&\qquad\qquad + \max\left\{\frac{\tilde{C}_2}{\varepsilon}\left(\frac{1}{n}\right)^{1\wedge 2(\lambda/\rho)}, \left(\frac{\tilde{C}_2}{\varepsilon}\left(\frac{1}{n}\right)^{1\wedge 2(\lambda/\rho)}\right)^2\right\} \\
&\qquad\qquad \times \Bigg(t^2 + \sum_{j=1}^{2}\Big[t^{j(1\wedge 2(\lambda/\rho))}\big(|x-y|^{2j} + t^j\big) \\
&\qquad\qquad\qquad + t^j\big(|x-y|^{j(2\vee\rho)} + t^{j(2\vee\rho)/2}\big)\Big]\Bigg)\Bigg\}.
\end{aligned}$$

Here const *depends only on* ρ *and* d (*and hence it is independent of* ε).

CLAIM 5.2.

$$\begin{aligned}
\left|e^{-tH}(x,y) - e^{-tH_\varepsilon}(x,y)\right| &\le p(t, x-y)\mathrm{const}\,\varepsilon t^{2/((\rho\wedge 2)\vee 1)-1}, \\
\left| K\left(\frac{t}{n}\right)^n (x,y) - K_\varepsilon\left(\frac{t}{n}\right)^n (x,y)\right| &\le p(t, x-y)\mathrm{const}\,\varepsilon t^{2/((\rho\wedge 2)\vee 1)-1}, \\
\left| G\left(\frac{t}{n}\right)^n (x,y) - G_\varepsilon\left(\frac{t}{n}\right)^n (x,y)\right| &\le p(t, x-y)\mathrm{const}\,\varepsilon t^{2/((\rho\wedge 2)\vee 1)-1}.
\end{aligned}$$

Here const *depends only on* $\tilde{c}$, $\tilde{C}$ *and* ρ, *that is,* c, c_1 *and* ρ, *so that it is independent of* $\varepsilon > 0$.

PROOF. By Proposition 2.1,

$$\begin{aligned}
&\exp(-tH)(x,y) - \exp(-tH_\varepsilon)(x,y) \\
&\quad = p(t, x-y)\, E_0\Big[\exp\Big(-t\int_0^1 V\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\Big) \\
&\qquad\qquad - \exp\Big(-t\int_0^1 V_\varepsilon\big(x + s(y-x) + \sqrt{t}\, X_0(s)\big)\, ds\Big)\Big],
\end{aligned}$$

$$K\left(\frac{t}{n}\right)^n(x,y) - K_\varepsilon\left(\frac{t}{n}\right)^n(x,y)$$
$$= p(t, x-y)\,E_0\Bigg[\exp\bigg(-\frac{t}{2}\int_0^1 \Big(V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)$$
$$+ V\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\Big)\,ds\bigg)$$
$$-\exp\bigg(-\frac{t}{2}\int_0^1 \Big(V_\varepsilon\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)$$
$$+ V_\varepsilon\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\Big)\,ds\bigg)\Bigg],$$

$$G\left(\frac{t}{n}\right)^n(x,y) - G_\varepsilon\left(\frac{t}{n}\right)^n(x,y)$$
$$= p(t, x-y)\,E_0\Bigg[\exp\bigg(-t\int_0^1 V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\,ds\bigg)$$
$$-\exp\bigg(-t\int_0^1 V_\varepsilon\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\,ds\bigg)\Bigg].$$

Here we use a simple formula:

$$e^{-a} - e^{-b} = \int_0^1 (b-a)e^{-\theta a}e^{-(1-\theta)b}\,d\theta.$$

Then

$$\left|\exp(-tH)(x,y) - \exp(-tH_\varepsilon)(x,y)\right|$$
$$\le p(t, x-y)\,E_0\Bigg[\int_0^1 d\theta\, t\int_0^1 ds\,\Big|V_\varepsilon\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)$$
$$-V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\Big|$$
$$\times\exp\bigg(-t\theta\int_0^1 V\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds\bigg)$$
$$\times\exp\bigg(-t(1-\theta)\int_0^1 V_\varepsilon\big(x + s(y-x) + \sqrt{t}\,X_0(s)\big)\,ds\bigg)\Bigg],$$

$$\left|K\left(\frac{t}{n}\right)^n(x,y) - K_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right|$$
$$\le p(t, x-y)\,E_0\Bigg[\int_0^1 d\theta\,\frac{t}{2}\int_0^1 ds\,\Big(\Big|V_\varepsilon\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)$$
$$-V\big(x + s_n^-(y-x) + \sqrt{t}\,X_0(s_n^-)\big)\Big|$$
$$+\Big|V_\varepsilon\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)$$
$$-V\big(x + s_n^+(y-x) + \sqrt{t}\,X_0(s_n^+)\big)\Big|\Big)$$

$$\times \exp\biggl(-\frac{t}{2}\theta\int_0^1 \Bigl(V\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)$$

$$+V\bigl(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\bigr)\Bigr)\,ds\biggr)$$

$$\times \exp\biggl(-\frac{t}{2}(1-\theta)\int_0^1 \Bigl(V_\varepsilon\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)$$

$$+V_\varepsilon\bigl(x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\bigr)\Bigr)\,ds\biggr)\biggr],$$

$$\left|G\left(\frac{t}{n}\right)^n(x,y)-G_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right|$$

$$\leq p(t,x-y)\,E_0\biggl[\int_0^1 d\theta\, t\int_0^1 ds\,\Bigl|V_\varepsilon\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)$$

$$-V\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)\Bigr|$$

$$\times\exp\biggl(-t\theta\int_0^1 V\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)\,ds\biggr)$$

$$\times\exp\biggl(-t(1-\theta)\int_0^1 V_\varepsilon\bigl(x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr)\,ds\biggr)\biggr].$$

By Lemma 5.1(i) and 5.1(ii), these are dominated as follows:

$$\bigl|\exp(-tH)(x,y)-\exp(-tH_\varepsilon)(x,y)\bigr|$$

$$\leq p(t,x-y)\,E_0\biggl[\int_0^1 d\theta\, t\int_0^1 \tilde{C}\varepsilon\bigl\langle x+s(y-x)+\sqrt{t}\,X_0(s)\bigr\rangle^{(\rho-1)_++\eta}\,ds$$

$$\times\exp\biggl(-t\tilde{c}\int_0^1\bigl\langle x+s(y-x)+\sqrt{t}\,X_0(s)\bigr\rangle^{\rho}\,ds\biggr)\biggr]$$

$$=p(t,x-y)\tilde{C}t\varepsilon E_0\biggl[\int_0^1\bigl\langle x+s(y-x)+\sqrt{t}\,X_0(s)\bigr\rangle^{(\rho-1)_++\eta}\,ds$$

$$\times\exp\biggl(-t\tilde{c}\int_0^1\bigl\langle x+s(y-x)+\sqrt{t}\,X_0(s)\bigr\rangle^{\rho}\,ds\biggr)\biggr],$$

$$\left|K\left(\frac{t}{n}\right)^n(x,y)-K_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right|$$

$$\leq p(t,x-y)$$

$$\times E_0\biggl[\int_0^1 d\theta\,\frac{t}{2}\int_0^1\Bigl(\tilde{C}\varepsilon\bigl\langle x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\bigr\rangle^{(\rho-1)_++\eta}$$

$$+\tilde{C}\varepsilon\bigl\langle x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\bigr\rangle^{(\rho-1)_++\eta}\Bigr)\,ds$$

$$\times\exp\bigg(-\frac{t}{2}\tilde{c}\int_0^1\Big(\Big\langle x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\Big\rangle^{\rho}$$

$$+\Big\langle x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\Big\rangle^{\rho}\Big)\,ds\bigg)\bigg]$$

$$\le p(t,x-y)\tilde{C}t\varepsilon$$

$$\times\frac{1}{2}E_0\bigg[\int_0^1\Big\langle x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\Big\rangle^{(\rho-1)_++\eta}\,ds$$

$$\times\exp\bigg(-\frac{t}{2}\tilde{c}\int_0^1\Big\langle x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\Big\rangle^{\rho}\,ds\bigg)$$

$$+\int_0^1\Big\langle x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\Big\rangle^{(\rho-1)_++\eta}\,ds$$

$$\times\exp\bigg(-\frac{t}{2}\tilde{c}\int_0^1\Big\langle x+s_n^+(y-x)+\sqrt{t}\,X_0(s_n^+)\Big\rangle^{\rho}\,ds\bigg)\bigg],$$

$$\left|G\left(\frac{t}{n}\right)^n(x,y)-G_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right|$$

$$\le p(t,x-y)\tilde{C}t\varepsilon E_0\bigg[\int_0^1\Big\langle x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\Big\rangle^{(\rho-1)_++\eta}\,ds$$

$$\times\exp\bigg(-t\tilde{c}\int_0^1\Big\langle x+s_n^-(y-x)+\sqrt{t}\,X_0(s_n^-)\Big\rangle^{\rho}\,ds\bigg)\bigg].$$

Therefore, noting that for a measurable function $f\colon[0,1]\to[0,\infty)$,

$$\int_0^1 f(s)^{(\rho-1)_++\eta}\,ds\,\exp\left(-t\tilde{c}\int_0^1 f(s)^{\rho}\,ds\right)$$

$$\le\begin{cases}\left(\dfrac{(\rho-1)_++\eta}{\rho e}\right)^{((\rho-1)_++\eta)/\rho}(t\tilde{c})^{-((\rho-1)_++\eta)/\rho}, & \text{if } \rho>0,\\ 1, & \text{if } \rho=0,\end{cases}$$

we have

$$\left|e^{-tH}(x,y)-e^{-tH_\varepsilon}(x,y)\right|$$

$$\le\begin{cases}p(t,x-y)\tilde{C}\varepsilon t^{2/((\rho\wedge 2)\vee 1)-1}\\ \qquad\times\left(\dfrac{(\rho-1)_++\eta}{\rho e\tilde{c}}\right)^{((\rho-1)_++\eta)/\rho}, & \text{if } \rho>0,\\ p(t,x-y)\tilde{C}t\varepsilon, & \text{if } \rho=0,\end{cases}$$

$$\left| K\left(\frac{t}{n}\right)^n (x,y) - K_\varepsilon\left(\frac{t}{n}\right)^n (x,y)\right|$$

$$\le \begin{cases} p(t, x-y)\tilde{C}\varepsilon\left(\dfrac{t}{2}\right)^{2/((\rho\wedge 2)\vee 1)-1} \\ \qquad \times 2\left(\dfrac{(\rho-1)_+ + \eta}{\rho e\tilde{c}}\right)^{((\rho-1)_+ + \eta)/\rho}, & \text{if } \rho > 0, \\ p(t, x-y)\tilde{C}t\varepsilon, & \text{if } \rho = 0, \end{cases}$$

$$\left| G\left(\frac{t}{n}\right)^n (x,y) - G_\varepsilon\left(\frac{t}{n}\right)^n (x,y)\right|$$

$$\le \begin{cases} p(t, x-y)\tilde{C}\varepsilon t^{2/((\rho\wedge 2)\vee 1)-1}\left(\dfrac{(\rho-1)_+ + \eta}{\rho e\tilde{c}}\right)^{((\rho-1)_+ + \eta)/\rho}, & \text{if } \rho > 0, \\ p(t, x-y)\tilde{C}t\varepsilon, & \text{if } \rho = 0, \end{cases}$$

which completes the proof. □

PROOF OF THEOREM 1.3. Let

$$\varepsilon := (1/n)^{1/2\wedge\lambda/\rho} = (1/n)^{1/2\wedge 1/\rho} = (1/n)^{1/(2\vee\rho)}$$

where $n \ge 2^{2(2\vee\rho)}$. Then $(1/\varepsilon)(1/n)^{1\wedge 2(\lambda/\rho)} = (1/n)^{1/(2\vee\rho)}$. By Claims 5.1 and 5.2,

$$\left| K_\varepsilon\left(\frac{t}{n}\right)^n (x,y) - e^{-tH_\varepsilon}(x,y)\right|$$

$$\le p(t, x-y)\,\text{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)}$$

$$\times\Biggl\{t^{1+2(1\wedge\lambda/\rho)} + \sum_{j=1}^{2}\Bigl[t^{j2/(2\vee\rho)}\bigl(|x-y|^{2j} + t^j\bigr)$$

$$+ t^j\bigl(|x-y|^{j(2\vee\rho)} + t^{j(2\vee\rho)/2}\bigr)\Bigr]\Biggr\},$$

$$\left| G_\varepsilon\left(\frac{t}{n}\right)^n (x,y) - e^{-tH_\varepsilon}(x,y)\right|$$

$$\le p(t, x-y)\,\text{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)}$$

$$\times\Biggl\{t^2 + \sum_{j=1}^{2} t^{j(1\wedge\lambda/\rho)}\bigl(|x-y|^j + t^{j/2}\bigr)$$

$$+ \sum_{j=1}^{2}\Bigl[t^{j2/(2\vee\rho)}\bigl(|x-y|^{2j} + t^j\bigr) + t^j\bigl(|x-y|^{j(2\vee\rho)} + t^{j(2\vee\rho)/2}\bigr)\Bigr]\Biggr\},$$

$$\left|e^{-tH}(x,y)-e^{-tH_\varepsilon}(x,y)\right| \le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} t^{2/((\rho\wedge 2)\vee 1)-1},$$

$$\left|K\left(\frac{t}{n}\right)^n(x,y)-K_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right| \le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} t^{2/((\rho\wedge 2)\vee 1)-1},$$

$$\left|G\left(\frac{t}{n}\right)^n(x,y)-G_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right| \le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)} t^{2/((\rho\wedge 2)\vee 1)-1},$$

where const depends only on c, c_1, ρ and d. Consequently we have

$$\begin{aligned}
&\left|K\left(\frac{t}{n}\right)^n(x,y)-e^{-tH}(x,y)\right|\\
&\quad\le\left|K\left(\frac{t}{n}\right)^n(x,y)-K_\varepsilon\left(\frac{t}{n}\right)^n(x,y)\right|+\left|K_\varepsilon\left(\frac{t}{n}\right)^n(x,y)-e^{-tH_\varepsilon}(x,y)\right|\\
&\qquad+\left|e^{-tH_\varepsilon}(x,y)-e^{-tH}(x,y)\right|\\
&\quad\le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)}\\
&\qquad\times\Bigg\{t^{2/((\rho\wedge 2)\vee 1)-1}+t^{1+2(1\wedge\lambda/\rho)}\\
&\qquad\quad+\sum_{j=1}^{2}\left[t^{j2/(2\vee\rho)}\left(|x-y|^{2j}+t^j\right)+t^j\left(|x-y|^{j(2\vee\rho)}+t^{j(2\vee\rho)/2}\right)\right]\Bigg\},\\
&\left|G\left(\frac{t}{n}\right)^n(x,y)-e^{-tH}(x,y)\right|\\
&\quad\le p(t,x-y)\mathrm{const}\left(\frac{1}{n}\right)^{1/(2\vee\rho)}\\
&\qquad\times\Bigg\{t^{2/((\rho\wedge 2)\vee 1)-1}+t^2+\sum_{j=1}^{2}t^{j(1\wedge\lambda/\rho)}\left(|x-y|^j+t^{j/2}\right)\\
&\qquad\quad+\sum_{j=1}^{2}\left[t^{j2/(2\vee\rho)}\left(|x-y|^{2j}+t^j\right)+t^j\left(|x-y|^{j(2\vee\rho)}+t^{j(2\vee\rho)/2}\right)\right]\Bigg\}
\end{aligned}$$

and the proof is complete. □

Acknowledgment. The author thanks Professor T. Ichinose who stimulated my interest in this problem and kindly instructed me on it.

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