The Annals of Probability
2002, Vol. 30, No. 2, 933–947

LIMIT THEOREMS FOR THE NONATTRACTIVE DOMANY–KINZEL MODEL

BY MAKOTO KATORI, NORIO KONNO AND HIDEKI TANEMURA

Chuo University, Yokohama National University and Chiba University

We study the Domany–Kinzel model, which is a class of discrete time Markov processes with two parameters $(p_1, p_2) \in [0,1]^2$ and whose states are subsets of $\mathbf{Z}$, the set of integers. When $p_1 = \alpha\beta$ and $p_2 = \alpha(2\beta - \beta^2)$ with $(\alpha, \beta) \in [0,1]^2$, the process can be identified with the mixed site–bond oriented percolation model on a square lattice with the probabilities of open site α and of open bond β. For the attractive case, $0 \le p_1 \le p_2 \le 1$, the complete convergence theorem is easily obtained. On the other hand, the case $(p_1, p_2) = (1, 0)$ realizes the rule 90 cellular automaton of Wolfram in which, starting from the Bernoulli measure with density θ, the distribution converges weakly only if $\theta \in \{0, 1/2, 1\}$. Using our new construction of processes based on signed measures, we prove limit theorems which are also valid for nonattractive cases with $(p_1, p_2) \ne (1, 0)$. In particular, when $p_2 \in [0,1]$ and p_1 is close to 1, the complete convergence theorem is obtained as a corollary of the limit theorems.

1. Introduction. The Domany–Kinzel model is a two parameter family of discrete time Markov processes whose states are subsets of $\mathbf{Z}$, the set of integers, which was introduced by Domany and Kinzel (1984) and Kinzel (1985). Let $\xi_n^A \subset \mathbf{Z}$ be the state of the process with parameters $(p_1, p_2) \in [0,1]^2$ at time n which starts from $A \subset 2\mathbf{Z}$. Its evolution satisfies the following:

(i) $P(x \in \xi_{n+1}^A | \xi_n^A) = f(|\xi_n^A \cap \{x-1, x+1\}|)$;

(ii) given ξ_n^A, the events $\{x \in \xi_{n+1}^A\}$ are independent, where $f(0) = 0$, $f(1) = p_1$ and $f(2) = p_2$.

If we write $\xi(x,n) = 1$ for $x \in \xi_n^A$ and $\xi(x,n) = 0$ otherwise, each realization of the process is identified with a configuration $\xi \in \{0,1\}^{\mathbf{S}} = X$ with $\mathbf{S} = \{s = (x,n) \in \mathbf{Z} \times \mathbf{Z}_+ : x + n = \text{even}\}$, where $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. As special cases the Domany–Kinzel model is equivalent to the oriented bond percolation model ($p_1 = p,\ p_2 = 2p - p^2$) and the oriented site percolation model ($p_1 = p_2 = p$) on a square lattice. The two-dimensional mixed site–bond oriented percolation model with α the probability of an open site and with β the probability of an open bond corresponds to the case of $p_1 = \alpha\beta$ and $p_2 = \alpha(2\beta - \beta^2)$. When $(p_1, p_2) = (1, 0)$, Wolfram's (1983, 1984) rule 90 cellular automaton is realized. See Durrett [(1988), pages 90–98] for details.

Received April 2000; revised January 2001.
AMS 2000 subject classifications. 60K35, 82B43, 82C22.
Key words and phrases. The Domany–Kinzel model, nonattractive process, limit theorem, complete convergence theorem.

For any $p_1, p_2 \in [0,1]$ and $A \subset 2\mathbf{Z}$, $\lim_{n\to\infty} P(\xi_{2n}^A \neq \varnothing)$ exists, since $\varnothing$ is an absorbing set. Let $Y = \{A \subset 2\mathbf{Z} : 0 < |A| < \infty\}$, where $|A|$ is the cardinality of A, and we write the connectedness from $A \subset 2\mathbf{Z}$ to $B \subset 2\mathbf{Z}$ as

$$\sigma(A, B) = \lim_{n\to\infty} P(\xi_{2n}^A \cap B \neq \varnothing)$$

if the right-hand side exists.

When $0 \le p_1 \le p_2 \le 1$, this process has the following good property called *attractiveness*: if $\xi_n^A \subset \xi_n^B$, then we can guarantee that $\xi_{n+1}^A \subset \xi_{n+1}^B$ by using an appropriate coupling. For the attractive case, it is easy to prove the following:

(i) If $A \subset 2\mathbf{Z}$, $B \in Y$, then $\sigma(A, B)$ exists. In particular, $\sigma(2\mathbf{Z}, B)$ exists.

(ii) Let $\mathbf{0}$ (resp. $\mathbf{1}$) denote the configuration $\eta(x) = 0$ (resp. $= 1$) for any $x \in 2\mathbf{Z}$. For any $A \subset 2\mathbf{Z}$,

$$P(\xi_n^A \in \cdot) \Rightarrow \big(1 - P(\Omega_\infty^A)\big)\delta_{\mathbf{0}} + P(\Omega_\infty^A)\mu_\infty \qquad \text{as } n \to \infty,$$

where $\Rightarrow$ means weak convergence, $\Omega_\infty^A = \{\xi_n^A \neq \varnothing \text{ for any } n \ge 0\}$, $\delta_{\mathbf{0}}$ is the point mass on the configuration $\mathbf{0}$, and a limit μ_∞ is a stationary distribution of the process $\xi_{2n}^{2\mathbf{Z}}$. This complete convergence theorem can be obtained by similar arguments for the lemma in Griffeath (1978) [see also Durrett (1988), Section 5c] which treated a continuous time version. It should be remarked that the complete convergence theorem is equivalent to the equality

$$\sigma(A, B) = \sigma(A, 2\mathbf{Z})\sigma(2\mathbf{Z}, B) \qquad \text{for any } A, B \in Y.$$

It is easy to see that the process with $p_1 \in [0, 1/2]$ and $p_2 \in [0,1]$ starting from a finite set dies out. That is,

$$\sigma(A, B) = 0 \qquad \text{if } p_1 \in [0, 1/2],\ p_2 \in [0,1],\ A \in Y,\ B \subset 2\mathbf{Z}. \tag{1.1}$$

It is concluded by comparison with a branching process.

The purpose of the present paper is to prove limit theorems which are valid also for the nonattractive cases except Wolfram's rule 90 cellular automaton $(p_1, p_2) = (1, 0)$. For this purpose we introduce $\sigma(\nu, B)$ for a probability distribution ν on X and $B \in Y$ defined by

$$\sigma(\nu, B) = \lim_{n\to\infty} P(\xi_{2n}^\nu \cap B \neq \varnothing),$$

if the right-hand side exists, where ξ_n^ν is the process with initial distribution ν. We first prove the following lemma.

LEMMA 1. *We assume that $(p_1, p_2) \in [0,1]^2$ with $(p_1, p_2) \neq (1, 0)$ and $p_2 < 2p_1$. Let ν_θ be the Bernoulli measure with $\theta \in (0, 1]$ and $B \in Y$. Then we have*

$$\sigma(\nu_\theta, B) = \begin{cases} \displaystyle\sum_{D \subset B, D \neq \varnothing} \alpha^{|D|}(1-\alpha)^{|B\setminus D|}\sigma(D, 2\mathbf{Z}), & \\ & \text{if } p_2 \neq 0 \text{ or } 0 < \theta < 1, \\ 0, & \text{if } p_2 = 0 \text{ and } \theta = 1, \end{cases} \tag{1.2}$$

where $\alpha = p_1^2/(2p_1 - p_2)$.

Remark that (1.1) implies $\sigma(\nu_\theta, B) = 0$, $B \in Y$ if $p_2 \ge 2p_1$ with $(p_1, p_2) \ne (\frac{1}{2}, 1)$. When $(p_1, p_2) = (\frac{1}{2}, 1)$, the model is the discrete time voter model and $\sigma(\nu_\theta, B) = \theta$ if $B \in Y$. In Theorem 1 of Katori, Konno and Tanemura (2000), (1.2) for $\sigma(2\mathbf{Z}, B)$ with $p_1 \in [0, 1)$, $p_2 \in (0, 1]$ was given. The present lemma is an extension of it which includes the interesting cases where $p_2 = 0$ or $p_1 = 1$. In the proof of Lemma 1, we use the new construction of the process using a signed measure with $\alpha = p_1^2/(2p_1 - p_2)$ and $\beta = 2 - p_2/p_1$ which was introduced in Katori, Konno and Tanemura (2000). From this lemma we can immediately get the next limit theorem. [The standard argument can be found in Durrett (1988), page 71.]

PROPOSITION 2. *We assume that* $(p_1, p_2) \in [0, 1]^2$ *with* $(p_1, p_2) \ne (1, 0)$ *and* $p_2 < 2p_1$. *Then we have*

$$P(\xi_{2n}^{\nu_\theta} \in \cdot) \Rightarrow \mu_\infty \qquad \text{as } n \to \infty,$$

where μ_∞ *is the translation invariant probability measure such that*

$$\mu_\infty(\xi \cap B \ne \varnothing) = \sum_{D \subset B, D \ne \varnothing} \alpha^{|D|}(1-\alpha)^{|B \setminus D|}\sigma(D, 2\mathbf{Z}),$$

for any $B \in Y$.

We note that $P(\Omega_\infty^{\{0\}}) = \sigma(\{0\}, 2\mathbf{Z}) = 0$ (resp. $= 1$) is equivalent to $P(\Omega_\infty^A) = \sigma(A, 2\mathbf{Z}) = 0$ (resp. $= 1$) for any $A \in Y$. It is obvious that if $P(\Omega_\infty^{\{0\}}) = 0$, then $\mu_\infty = \delta_{\mathbf{0}}$. From this corollary, we obtain the following interesting result, since $\sigma(D, 2\mathbf{Z}) = 1$ for any $D \in Y$ if $p_1 = 1$.

COROLLARY 3. *When* $p_1 = 1$ *and* $p_2 \in (0, 1]$, μ_∞ *is the Bernoulli measure* ν_α, *where* $\alpha = \frac{1}{2 - p_2}$.

We should remark that when $(p_1, p_2) = (1, 0)$, that is, in the case of rule 90 of Wolfram's cellular automaton, Miyamoto (1979) and Lind (1984) proved that, starting from the Bernoulli measure ν_θ, the distribution converges weakly only if $\theta \in \{0, 1/2, 1\}$.

Proposition 2 can be generalized as follows. Let $\mathbf{N} = \{1, 2, 3, \ldots\}$, $X = \{0, 1\}^{2\mathbf{Z}}$ and $\mathcal{P}(X)$ be the collection of probability measures on X. We introduce the following conditions (C.1) and (C.2) for $\nu \in \mathcal{P}(X)$:

(C.1) For any $\varepsilon > 0$ there exists $k \in 2\mathbf{N}$ such that

$$\nu(\xi \cap [x-k, x+k] = \varnothing) \le \varepsilon \qquad \text{for any } x \in 2\mathbf{Z}.$$

(C.2) For any $\varepsilon > 0$ there exists $k \in 2\mathbf{N}$ such that

$$\nu(\xi^c \cap [x-k, x+k] = \varnothing) \le \varepsilon \qquad \text{for any } x \in 2\mathbf{Z}.$$

THEOREM 4. (i) *Suppose that* $p_1 \in [0,1], p_2 \in (0,1]$ *and* $p_2 < 2p_1$. *If* ν *satisfies* (C.1), *then*

$$P(\xi_{2n}^{\nu} \in \cdot) \Rightarrow \mu_\infty \qquad \text{as } n \to \infty.$$

(ii) *Suppose that* $p_1 \in [0,1)$, $p_2 = 0$. *If* ν *satisfies* (C.1) *and* (C.2), *then*

$$P(\xi_{2n}^{\nu} \in \cdot) \Rightarrow \mu_\infty \qquad \text{as } n \to \infty.$$

Let $\mathcal{S}(X) = \{\nu \in \mathcal{P}(X) : \nu \text{ is translation invariant}\}$. We remark that $\nu \in \mathcal{S}(X)$ with $\nu(\{\mathbf{0}\}) = 0$ [resp. $\nu(\{\mathbf{1}\}) = 0$] satisfies (C.1) [resp. (C.2)]. Then we have the following corollary of Theorem 4.

COROLLARY 5. (i) *Suppose that* $\nu \in \mathcal{S}(X)$. *If* $p_1 \in [0,1], p_2 \in (0,1]$ *and* $p_2 < 2p_1$, *then*

$$P(\xi_{2n}^{\nu} \in \cdot) \Rightarrow \nu(\{\mathbf{0}\})\delta_{\mathbf{0}} + \bigl(1 - \nu(\{\mathbf{0}\})\bigr)\mu_\infty \qquad \text{as } n \to \infty.$$

Also, *if* $P(\Omega_\infty^{\{0\}}) > 0$, *then* $\mu_\infty(\{\mathbf{0}\}) = 0$.

(ii) *Suppose that* $\nu \in \mathcal{S}(X)$. *If* $p_1 \in [0,1)$, $p_2 = 0$, *then*

$$P(\xi_{2n}^{\nu} \in \cdot) \Rightarrow \nu(\{\mathbf{0},\mathbf{1}\})\delta_{\mathbf{0}} + \bigl(1 - \nu(\{\mathbf{0},\mathbf{1}\})\bigr)\mu_\infty \qquad \text{as } n \to \infty.$$

Also, *if* $P(\Omega_\infty^{\{0\}}) > 0$, *then* $\mu_\infty(\{\mathbf{0},\mathbf{1}\}) = 0$.

We also obtain the following complete convergence theorem.

THEOREM 6. *There exists* $\widehat{p}_1 \in (0,1)$ *such that*, *for any* $A \subset 2\mathbf{Z}$,

$$P(\xi_{2n}^{A} \in \cdot) \Rightarrow \bigl(1 - P(\Omega_\infty^A)\bigr)\delta_{\mathbf{0}} + P(\Omega_\infty^A)\mu_\infty \qquad \text{as } n \to \infty,$$

when $p_1 \in [\widehat{p}_1, 1]$ *and* $p_2 \in [0,1]$, *but* $(p_1,p_2) \neq (1,0)$.

We conjecture that the complete convergence theorem holds for any $(p_1,p_2) \in [0,1]^2$ except $(p_1,p_2) = (1,0)$. In attractive particle systems, the block construction arguments have been used to prove the complete convergence theorem; see Durrett [(1984), Section 9] and Durrett [(1988), Section 5b]. One of the essential properties used in the proofs is that if $P(\Omega_\infty^0) > 0$, then the probability $P(\Omega_\infty^A)$ is close to 1 for any sufficiently large initial set A. In general it is unknown whether the property holds for nonattractive systems. Here we can show that it holds for the nonattractive Domany–Kinzel model.

PROPOSITION 7. (i) *Suppose that* $p_1 \in [0,1]$, $p_2 \in (0,1]$ *and* $P(\Omega_\infty^{\{0\}}) > 0$. *Then*

$$\lim_{|A| \to \infty} P(\Omega_\infty^A) = 1. \tag{1.3}$$

(ii) *Suppose that* $p_1 \in [0, 1)$, $p_2 = 0$ *and* $P(\Omega_\infty^{\{0\}}) > 0$. *Then*

$$\lim_{|\partial A| \to \infty} P(\Omega_\infty^A) = 1, \tag{1.4}$$

where $\partial A = (A + 1) \triangle (A - 1)$ *for* $A \subset 2\mathbf{Z}$.

The paper is organized as follows. Section 2 is devoted to the proof of Lemma 1. The proof of Theorem 4 is given in Section 3. We prove Theorem 6 and Proposition 7 in Section 4.

2. Proof of Lemma 1. First we introduce these spaces:

$$\mathbf{S} = \{s = (x, n) \in \mathbf{Z} \times \mathbf{Z}_+ : x + n = \text{even}\},$$
$$\mathbf{B} = \{b = ((x, n), (x + 1, n + 1)), ((x, n), (x - 1, n + 1)) : (x, n) \in \mathbf{S}\},$$
$$\mathcal{X}(\mathbf{S}) = \{0, 1\}^{\mathbf{S}}, \qquad \mathcal{X}(\mathbf{B}) = \{0, 1\}^{\mathbf{B}}, \qquad \mathcal{X} = \mathcal{X}(\mathbf{S}) \times \mathcal{X}(\mathbf{B}),$$

where $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. For given $\zeta = (\zeta_1, \zeta_2) \in \mathcal{X}$, we say that $s = (y, n + k) \in \mathbf{S}$ can be reached from $s' = (x, n) \in \mathbf{S}$ and write $s' \to s$, if there exists a sequence $s_0, s_1, \ldots, s_k$ of members of $\mathbf{S}$ such that $s' = s_0$, $s = s_k$ and $\zeta_1(s_i) = 1$, $i = 0, 1, \ldots, k$, $\zeta_2((s_i, s_{i+1})) = 1$, $i = 0, 1, \ldots, k - 1$. We also say that $G \subset \mathbf{S}$ can be reached from $G' \subset \mathbf{S}$ and write $G' \to G$ (resp. $G' \nrightarrow G$), if there exist $s \in G$ and $s' \in G'$ such that $s' \to s$ (resp. if not). Furthermore we define

$$\mathbf{S}^{(N)} = \{s = (x, n) \in \mathbf{S} : |x|, n \le N\},$$
$$\mathbf{B}^{(N)} = \{(s, s') \in \mathbf{B} : s, s' \in \mathbf{S}^{(N)}\}$$

and let $\mathcal{F}^{(N)}$ be the σ-field generated by the events of configurations depending on $\mathbf{S}^{(N)}$ and $\mathbf{B}^{(N)}$.

For given $\alpha, \beta \in \mathbf{R}$, we introduce the signed measure $m^{(N)}$ on $(\mathcal{X}, \mathcal{F}^{(N)})$ defined by

$$m^{(N)}(\Lambda) = \alpha^{k_1}(1 - \alpha)^{j_1} \beta^{k_2}(1 - \beta)^{j_2},$$

for any cylinder set

$$\Lambda = \{(\zeta_1, \zeta_2) \in \mathcal{X} : \zeta_1(s_i) = 1,\ i = 1, 2, \ldots, k_1,\ \zeta_1(s_i') = 0,\ i = 1, 2, \ldots, j_1,$$
$$\zeta_2(b_i) = 1,\ i = 1, 2, \ldots, k_2,\ \zeta_2(b_i') = 0,\ i = 1, 2, \ldots, j_2\},$$

where $s_1, \ldots, s_{k_1}, s_1', \ldots, s_{j_1}'$ are distinct elements of $\mathbf{S}^{(N)}$ and $b_1, \ldots, b_{k_2}$, $b_1', \ldots, b_{j_2}'$ are distinct elements of $\mathbf{B}^{(N)}$. We define the conditional signed measures on $(\mathcal{X}, \mathcal{F}^{(N)})$ as follows:

$$m_k^{(N)}(\cdot) = m^{(N)}(\cdot | \zeta_1(s) = 1,\ s \in \mathbf{S}_k^{(N)}),$$
$$m_{k,j}^{(N)}(\cdot) = m^{(N)}(\cdot | \zeta_1(s) = 1,\ s \in \mathbf{S}_k^{(N)} \cup \mathbf{S}_j^{(N)}),$$

where $\mathbf{S}_k^{(N)} = \{(x,n) \in \mathbf{S}^{(N)} : n = k\}$. We should remark that $\mathcal{F}^{(N)} \subset \mathcal{F}^{(N+1)}$ and, for any $\Lambda \in \mathcal{F}^{(N)}$,

$$m^{(N)}(\Lambda) = m^{(N+1)}(\Lambda),$$
$$m_k^{(N)}(\Lambda) = m_k^{(N+1)}(\Lambda),$$
$$m_{k,j}^{(N)}(\Lambda) = m_{k,j}^{(N+1)}(\Lambda).$$

From this consistency property, there exist the unique real-valued additive functions m, m_k and $m_{k,j}$ on $\bigcup_{N=1}^{\infty} \mathcal{F}^{(N)}$ such that, for any $\Lambda \in \mathcal{F}^{(N)}$,

$$m(\Lambda) = m^{(N)}(\Lambda),$$
$$m_k(\Lambda) = m_k^{(N)}(\Lambda),$$
$$m_{k,j}(\Lambda) = m_{k,j}^{(N)}(\Lambda).$$

See Figure 1. In this paper, we take $\alpha = p_1^2/(2p_1 - p_2)$ and $\beta = 2 - p_2/p_1$.

For $A, B \subset \mathbf{Z}$, $k, j \in \mathbf{Z}_+$ with $A \times \{k\}, B \times \{j\} \subset \mathbf{S}$ we write $A \times \{k\} \rightrightarrows B \times \{j\}$ if $A \times \{k\} \to (x, j)$ for any $x \in B$ and $A \times \{k\} \nrightarrow B^c \times \{j\}$. Then the observation shown by Figure 2 and the Markov property of the Domany–Kinzel model give

$$P(\xi_{n+1}^A = B | \xi_n^A = D) = m_n(D \times \{n\} \rightrightarrows B \times \{n+1\})$$

and

$$P(\xi_n^A = B) = m_0(A \times \{0\} \rightrightarrows B \times \{n\}),$$

where B is finite. From the above equation, the following equations can be quickly derived:

$$P(\xi_n^A \ni y) = m_0\big(A \times \{0\} \to (y, n)\big), \tag{2.1}$$

$$P(\xi_n^A \cap B \neq \varnothing) = m_0(A \times \{0\} \to B \times \{n\}). \tag{2.2}$$

If $p_2 < 2p_1$ and $p_2 > 2p_1 - p_1^2$, then $\alpha > 1$ and $\beta \in (0,1)$. If $p_2 \le 2p_1 - p_1^2$ and $p_2 \ge p_1$, then $\alpha, \beta \in [0,1]$. This case corresponds to the mixed site–bond oriented percolation with α the probability of an open site and with β the probability of an open bond, where $p_1 = \alpha\beta$ and $p_2 = \alpha(2\beta - \beta^2)$. That is why we choose $\alpha = p_1^2/(2p_1 - p_2)$ and $\beta = 2 - p_2/p_1$ in our construction. Moreover, if $p_2 < p_1$, then $\alpha \in (0,1)$ and $\beta \in (1,2]$.

For a fixed even nonnegative number k, we introduce the map r_k from $\mathbf{S}$ to $\mathbf{S}$ defined by

$$r_k(x,n) = \begin{cases} (x, k-n), & n = 0, 1, \dots, k, \\ (x, n), & \text{otherwise}, \end{cases}$$

and the map R_k from x to x defined by

$$R_k\zeta = \big((R_k\zeta)_1, (R_k\zeta)_2\big),$$

where $(R_k\zeta)_1(s) = \zeta_1(r_k s)$ and $(R_k\zeta)_2((s,s')) = \zeta_2((r_k s', r_k s))$. Note that m is R_k-invariant. To prove Lemma 1 we use the following lemma.

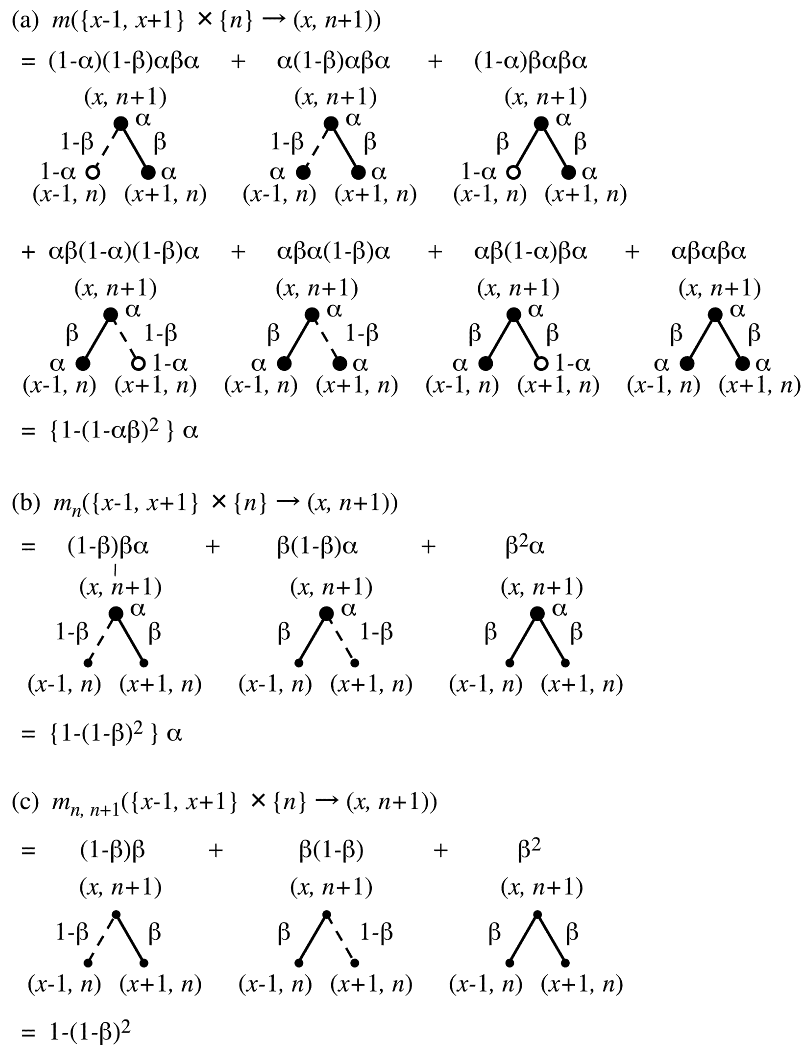

FIG. 1.

LEMMA 8. *Suppose that* $p_1, p_2 \in [0,1]$ *with* $(p_1, p_2) \neq (1,0)$. *Then, for any positive integer* ℓ *and* $A \subset 2\mathbf{Z}$, *we have*

$$\lim_{n\to\infty} P(1 \le |\xi_n^A| \le \ell, \Omega_\infty^A) = 0.$$

When $p_1 \neq 1$, the lemma was proved in Katori, Konno and Tanemura [(2000), Lemma 4]. When $p_1 = 1$, the lemma is derived from Lemma 10, which is given in Section 4.

Now we prove Lemma 1. Suppose that n is even. Let ν be a probability measure on X and let A_ν be a random variable with distribtion ν which is independent

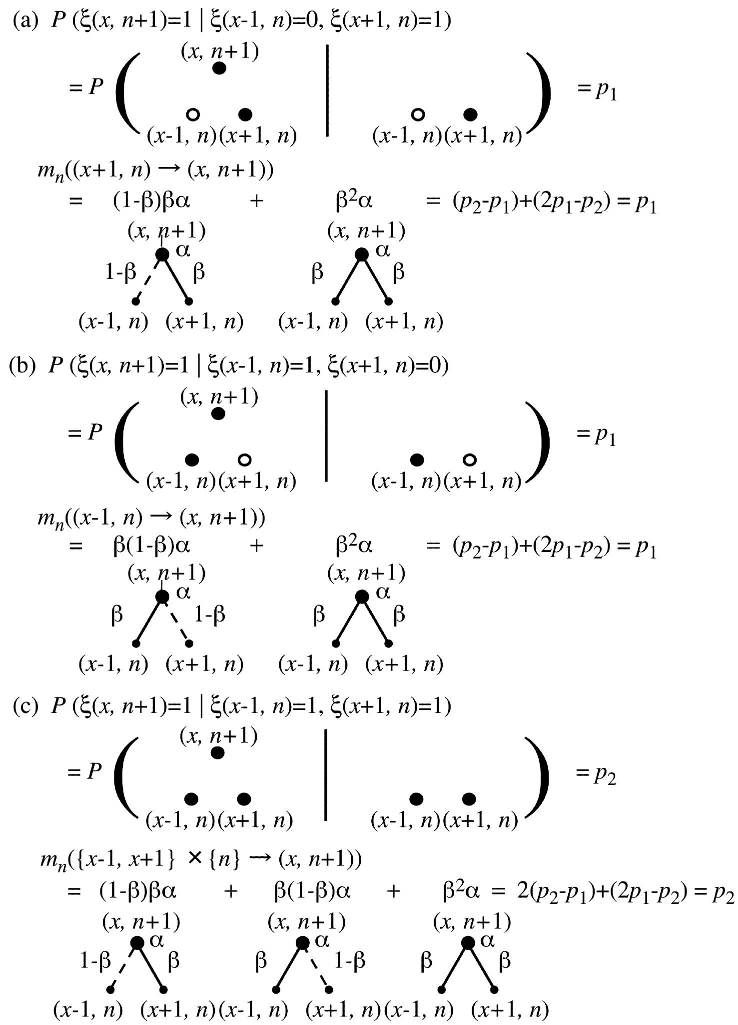

FIG. 2.

of ξ_n^D, $D \subset 2\mathbf{Z}$. Then from (2.2) we can show that

$$
\begin{aligned}
P(\xi_n^\nu \cap B \neq \varnothing) &= \int_X \nu(d\eta) P(\xi_n^\eta \cap B \neq \varnothing) \\
&= \int_X \nu(d\eta) m_0(\eta \times \{0\} \to B \times \{n\}) \\
&= \int_X \nu(d\eta) \sum_{D \subset B, D \neq \varnothing} m_{0,n}(\eta \times \{0\} \to D \times \{n\}) \alpha^{|D|}(1-\alpha)^{|B \setminus D|} \\
&= \int_X \nu(d\eta) \sum_{D \subset B, D \neq \varnothing} m_{0,n}(D \times \{0\} \to \eta \times \{n\}) \alpha^{|D|}(1-\alpha)^{|B \setminus D|}
\end{aligned}
$$

and

$$\begin{aligned}
&m_{0,n}(D \times \{0\} \to \eta \times \{n\}) \\
&\quad = \sum_{0<|C|<\infty} m_0(D \times \{0\} \rightrightarrows C \times \{n-1\}) m_{n-1,n}(C \times \{n-1\} \to \eta \times \{n\}) \\
&\quad = \sum_{0<|C|<\infty} m_0(D \times \{0\} \rightrightarrows C \times \{n-1\}) \bigl[1 - (1-\beta)^{|\eta \cap (C+1)| + |\eta \cap (C-1)|}\bigr] \\
&\quad = P(\xi_{n-1}^D \neq \varnothing) - E\bigl[(1-\beta)^{|\eta \cap (\xi_{n-1}^D + 1)| + |\eta \cap (\xi_{n-1}^D - 1)|};\ \xi_{n-1}^D \neq \varnothing\bigr].
\end{aligned}$$

Then we have

$$\begin{aligned}
P(\xi_n^\nu \cap B \neq \varnothing) &= \sum_{D \subset B, D \neq \varnothing} P(\xi_{n-1}^D \neq \varnothing) \alpha^{|D|} (1-\alpha)^{|B \setminus D|} \\
&\quad - \sum_{D \subset B, D \neq \varnothing} E\bigl[(1-\beta)^{|A_\nu \cap (\xi_{n-1}^D + 1)| + |A_\nu \cap (\xi_{n-1}^D - 1)|};\ \xi_{n-1}^D \neq \varnothing\bigr] \\
&\qquad \times \alpha^{|D|} (1-\alpha)^{|B \setminus D|}.
\end{aligned} \tag{2.3}$$

Then, to prove

$$\sigma(\nu, B) = \sum_{D \subset B, D \neq \varnothing} \sigma(D, 2\mathbf{Z}) \alpha^{|D|} (1-\alpha)^{|B \setminus D|}$$

for $B \in Y$, it is enough to show that, for any $D \subset B$ with $D \neq \varnothing$,

$$\lim_{n \to \infty} E\bigl[(1-\beta)^{|A_\nu \cap (\xi_{n-1}^D + 1)| + |A_\nu \cap (\xi_{n-1}^D - 1)|};\ \xi_{n-1}^D \neq \varnothing\bigr] = 0. \tag{2.4}$$

We show (2.4) for $\nu = \nu_\theta$ to prove Lemma 1. We set $A_\theta = A_{\nu_\theta}$. If $p_1 \in [0, 1]$ and $p_2 \in (0, 1]$, we see that $1 - \beta \in (-1, 1)$. So (2.4) is derived from Lemma 8. If $p_1 \in [0, 1)$ and $p_2 = 0$, then $1 - \beta = -1$. Since

$$|A_\theta \cap (\xi_{n-1}^D + 1)| + |A_\theta \cap (\xi_{n-1}^D - 1)| = |A_\theta \cap \partial \xi_{n-1}^D| \quad \mod 2$$

and ξ_{n-1}^D and A_θ are independent, we have

$$\begin{aligned}
E\bigl[(-1)^{|A_\theta \cap (\xi_{n-1}^D + 1)| + |A_\theta \cap (\xi_{n-1}^D - 1)|};\ \xi_{n-1}^D \neq \varnothing\bigr] &= E\bigl[(-1)^{|A_\theta \cap \partial \xi_{n-1}^D|};\ \xi_{n-1}^D \neq \varnothing\bigr] \\
&= E\bigl[(1-2\theta)^{|\partial \xi_{n-1}^D|};\ \xi_{n-1}^D \neq \varnothing\bigr].
\end{aligned}$$

Noting that $1 - 2\theta \in (-1, 1)$ for $\theta \in (0, 1)$, and that $\partial \xi_{n-1}^D \supset \xi_n^D$, we obtain (2.4) from Lemma 8.

3. Proof of Theorem 4. In this section we show equation (2.4) under condition (C.1) if $p_2 \neq 0$, and under conditions (C.1) and (C.2) if $p_2 = 0$. Then, we obtain Theorem 4. Since $P(\xi_{n+k} \in \cdot) = P(\xi_n^{\widehat{\xi}_k^\nu} \in \cdot)$, $k \in 2\mathbf{N}$, it is enough to show that for any $\varepsilon > 0$ there exists $k \in 2\mathbf{N}$ such that

$$\lim_{n\to\infty} E\big[(1-\beta)^{|\widehat{\xi}_k^\nu \cap (\xi_{n-1}^D + 1)| + |\widehat{\xi}_k^\nu \cap (\xi_{n-1}^D - 1)|};\ \xi_{n-1}^D \neq \varnothing\big] \le \varepsilon, \tag{3.1}$$

for any $D \in Y$, where $\widehat{\xi}_k^\nu$ is an independent copy of ξ_k^ν.

First we consider the case $p_2 \neq 0$. By (C.1), for any $\delta \in (0,1)$ there exists $k = k(\delta) \in 2\mathbf{N}$ so that

$$\nu(\eta \cap [x-k, x+k] = \varnothing) \le \delta \qquad \text{for any } x \in 2\mathbf{Z}.$$

If $\eta \cap [x-k, x+k] \neq \varnothing$, then $P(\widehat{\xi}_k^\eta(x) = 1) \ge (p_1 \wedge p_2)^k$. Put $\gamma = 1 - (p_1 \wedge p_2)^k$. Then

$$\nu\big(\eta : P\big(\widehat{\xi}_k^\eta(x) = 0\big) > \gamma\big) = \nu\big(\eta : P\big(\widehat{\xi}_k^\eta(x) = 1\big) \le 1 - \gamma\big) \le \delta, \qquad x \in 2\mathbf{Z}.$$

Put $h_k(\zeta) = P(\widehat{\xi}_k^\nu \cap \zeta = \varnothing)$ for $\zeta \subset 2\mathbf{Z}$ with $|\zeta| < \infty$. If ζ satisfies $\triangle(\zeta) = \min_{x,y\in\zeta, x\neq y} |x-y| \ge 2k$, then

$$\begin{aligned} h_k(\zeta) &= \int_X \nu(d\eta) E\bigg[\prod_{x\in\zeta} \big(1 - \widehat{\xi}_k^\eta(x)\big)\bigg] \\ &= \int_X \nu(d\eta) \prod_{x\in\zeta} P\big(\widehat{\xi}_k^\eta(x) = 0\big). \end{aligned}$$

Here we refer to Lemma 9.13 in Harris (1976).

LEMMA 9 (Harris). *Let $X_1, X_2, \dots, X_k$ be random variables with $0 \le X_i \le 1$ and $P(X_i > \gamma) \le \varepsilon$ for any $i \in \{1, 2, \dots, k\}$. Then we have*

$$E[X_1 X_2 \cdots X_k] \le \varepsilon + \gamma^k.$$

Applying Lemma 9 implies that if $\triangle(\zeta) \ge 2k$, then

$$h_k(\zeta) \le \delta + \gamma^{|\zeta|}.$$

From the fact that, for $\zeta \subset 2\mathbf{Z}$ with $|\zeta| < \infty$, $\max\{l \ge 1 : \{y_1, y_2, \dots, y_l\} \subset \zeta,\ y_i + 2k \le y_{i+1}\ (i = 1, 2, \dots, l-1)\}$ is bounded from below by $|\zeta|/k$, we see that

$$P(\widehat{\xi}_k^\nu \cap \zeta = \varnothing) \le \delta + \gamma^{|\zeta|/k}.$$

Let $\ell \in \mathbf{N}$ and $\zeta_i \subset 2\mathbf{Z}$ with $|\zeta_i| < \infty$ $(i = 1, 2, \dots, \ell)$ satisfying $\zeta = \bigcup_{i=1}^\ell \zeta_i$ and $\zeta_i \cap \zeta_j = \varnothing (i \neq j)$. Then

$$P(|\widehat{\xi}_k^\nu \cap \zeta| < \ell) \le \ell\delta + \sum_{i=1}^{\ell} \gamma^{|\zeta_i|/k}. \tag{3.2}$$

Since $1-\beta \in (-1, 1)$ if $p_2 \neq 0$, for any $\varepsilon > 0$ we can take $\ell \in \mathbf{N}$ with $(1-\beta)^\ell \le \frac{\varepsilon}{2}$ and then take $k(\delta)$ such that $\ell\delta \le \frac{\varepsilon}{2}$. Then,

$$\lim_{|\zeta|\to\infty} E\big[(1-\beta)^{|\widehat{\xi}_k^\nu \cap \zeta|}\big] \le \varepsilon. \tag{3.3}$$

Combining this with Lemma 8 gives (3.1).

Next, we consider the case $p_2 = 0$ and $p_1 \in (0, 1)$. In this case $\beta = 2$ and (3.1) is rewritten as

$$\lim_{n\to\infty} E\big[(-1)^{|\widehat{\xi}_k^\nu \cap \partial \xi_{n-1}^D|};\ \xi_{n-1}^D \neq \varnothing\big] \le \varepsilon. \tag{3.4}$$

By (C.1) and (C.2), for any $\delta \in (0, 1)$ there exists $k = k(\delta) \in 2\mathbf{N}$ so that

$$\nu\big(\eta(y) = \eta(y+2),\ y \in [x-k, x+k-2] \cap 2\mathbf{Z}\big) \le \delta, \qquad x \in 2\mathbf{Z}.$$

If $\eta(y) \neq \eta(y+2)$ for some $y \in [x-k, x+k-2] \cap 2\mathbf{Z}$, then $P(\widehat{\xi}_k^\eta(x) = 1) \ge p_1^k(1-p_1)^{2k}$. Put $\gamma = 1 - p_1^k(1-p_1)^{2k}$. Then

$$\nu\big(\eta : P\big(\widehat{\xi}_k^\eta(x) = 0\big) > \gamma\big) = \nu\big(\eta : P\big(\widehat{\xi}_k^\eta(x) = 1\big) \le 1-\gamma\big) \le \delta, \qquad x \in 2\mathbf{Z}.$$

Using the same argument as in the case of $p_2 \neq 0$, we obtain (3.2) in the present case. Since $\widehat{\xi}_k^\nu \subset \partial\widehat{\xi}_{k-1}^\nu$, we have

$$P(|\partial\widehat{\xi}_{k-1}^\nu \cap \zeta| < \ell) \le \ell\delta + \sum_{i=1}^{\ell} \gamma^{|\zeta_i|/k}. \tag{3.5}$$

By the Markov property we have

$$\begin{aligned}
E\big[(-1)^{|\widehat{\xi}_k^\nu \cap \zeta|}\big] &= E\bigg[\prod_{x\in\zeta} (-1)^{|\widehat{\xi}_k^\nu(x)|}\bigg] \\
&= \sum_{S\subset(\zeta\pm 1)} E\bigg[\prod_{x\in\zeta} (-1)^{|\widehat{\xi}_k^\nu(x)|} \,\bigg|\, \widehat{\xi}_{k-1}^\nu \cap (\zeta \pm 1) = S\bigg] \\
&\qquad\qquad \times P\big(\widehat{\xi}_{k-1}^\nu \cap (\zeta \pm 1) = S\big) \\
&= \sum_{S\subset(\zeta\pm 1)} (1-2p_1)^{|\partial S \cap \zeta|} P\big(\widehat{\xi}_{k-1}^\nu \cap (\zeta \pm 1) = S\big) \\
&= \sum_{j=0}^{\infty} (1-2p_1)^j P(|\partial\widehat{\xi}_{k-1}^\nu \cap \zeta| = j),
\end{aligned}$$

where $\zeta \pm 1 = (\zeta + 1) \cup (\zeta - 1)$. Then

$$E\big[(-1)^{|\widehat{\xi}_k^\nu \cap \zeta|}\big] \le P(|\partial\widehat{\xi}_{k-1}^\nu \cap \zeta| < \ell) + (1-2p_1)^\ell. \tag{3.6}$$

From (3.5) and (3.6), for any $\varepsilon > 0$ we can take $\ell \in \mathbf{N}$ and $k(\delta) \in 2\mathbf{N}$ such that

$$\lim_{|\zeta|\to\infty} E\big[(-1)^{|\widehat{\xi}_k^\nu \cap \zeta|}\big] \le \varepsilon. \tag{3.7}$$

From Lemma 8 and the fact that $\partial\xi_{n-1}^D \supset \xi_n^D$ we have

$$\lim_{n\to\infty} P(|\partial\xi_{n-1}^D| \le \ell;\, \Omega_\infty^D) = 0, \qquad D \in Y. \tag{3.8}$$

Combining (3.7) and (3.8), we have the desired conclusion (3.4).

4. Proofs of Theorem 6 and Proposition 7. We consider a collection of random variables $\{w(x,n) : (x,n) \in \mathbf{S}\}$ with values in $\{0,1\}$ having the following property: if any sequence (x_j, n_j), $1 \le j \le \ell$, satisfies $|x_i - x_j| > 4$ whenever both $i \ne j$ and $n_i = n_j$, then $P(w(x_j, n_j) = 1,\ \text{for } 1 \le j \le \ell) = q^\ell$ with $q \in [0,1]$. Let $A \subset 2\mathbf{Z}$ and

$$W_k^A = \{z : \text{there is an open path from } (y,0) \text{ to } (z,k) \text{ for some } y \in A\}.$$

This is called a 2-dependent oriented site percolation. The following result can be obtained by a slight modification of argument in Durrett and Neuhauser [(1991), Appendix] for 1-dependent oriented site percolation. [See also Bramson and Neuhauser (1994), Lemma 2.3.] For any $\delta > 0$, there exists $\widehat{q}(\delta) \in [0,1]$ such that if $q \in [\widehat{q}(\delta), 1]$, then

$$\liminf_{n\to\infty} \frac{|W_n^A|}{n} > 1 - \delta \qquad \text{a.s. on } \Omega_\infty^{A,W},$$

where $\Omega_\infty^{A,W} = \bigcap_{n=1}^\infty \{W_n^A \ne \varnothing\}$.

Now we prove Theorem 6. When $p_2 = 0$, Bramson and Neuhauser (1994) developed block construction method and compared the process with the 2-dependent oriented site percolation. Their technique and argument can be extended to the case $p_2 \ne 0$. Then we have the following.

LEMMA 10. *For any $\delta > 0$, there exists $\widehat{p_1}(\delta) \in (0,1)$ such that if $p_1 \in (\widehat{p_1}(\delta), 1]$ and $p_2 \in [0,1]$ with $(p_1, p_2) \ne (1,0)$, then there exists $k \in 2\mathbf{N}$ so that*

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \sharp\{x \in 2k\mathbf{Z} \cap [-kn, kn) : \xi_{nk}^A \cap [x-k, x+k) \ne \varnothing\} \\ &\qquad > 1 - \delta \qquad \text{a.s. on } \Omega_\infty^A, \end{aligned} \tag{4.1}$$

for any $A \subset 2\mathbf{Z}$.

A sufficient condition for the proof of Theorem 6 is

$$\lim_{n\to\infty} P(\xi_{2n}^A \cap B \ne \varnothing) = \mu_\infty(\xi \cap B \ne \varnothing) P(\Omega_\infty^A), \qquad B \in Y.$$

Since $P(\xi_{2n} \in \cdot) = P(\xi_n^{\widehat{\xi}_n^A} \in \cdot)$, by the same way we obtained (2.3) we have

$$\begin{aligned} P(\xi_{2n}^A \cap B \ne \varnothing) &= P\Big(\xi_n^{\widehat{\xi}_n^A} \cap B \ne \varnothing\Big) \\ &= E\Big[m_0(\widehat{\xi}_n^A \times \{0\} \to B \times \{n\});\ \widehat{\xi}_n^A \ne \varnothing\Big] \end{aligned}$$

$$= \sum_{D\subset B, D\neq\varnothing} P(\xi_{n-1}^D \neq \varnothing)\alpha^{|D|}(1-\alpha)^{|B\setminus D|} P(\widehat{\xi}_n^A \neq \varnothing)$$
$$- \sum_{D\subset B, D\neq\varnothing} E\Big[(1-\beta)^{|\widehat{\xi}_n^A\cap(\xi_{n-1}^D+1)|+|\widehat{\xi}_n^A\cap(\xi_{n-1}^D-1)|};$$
$$\widehat{\xi}_n^A \neq \varnothing,\ \xi_{n-1}^D \neq \varnothing\Big]\alpha^{|D|}(1-\alpha)^{|B\setminus D|}.$$

Then it is sufficient to show that

$$\lim_{n\to\infty} E\Big[(1-\beta)^{|\widehat{\xi}_n^A\cap(\xi_{n-1}^D+1)|+|\widehat{\xi}_n^A\cap(\xi_{n-1}^D-1)|};\ \widehat{\xi}_n^A \neq \varnothing,\ \xi_{n-1}^D \neq \varnothing\Big] = 0, \tag{4.2}$$

for $D \subset B$. By Lemma 10 if $p_1 \in [\widehat{p_1}(\frac{2}{3}), 1]$ and $p_2 \in [0, 1]$ with $(p_1, p_2) \neq (1, 0)$, then

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n}\sharp\{x \in 2k\mathbf{Z}\cap[-kn, kn) : {}& [x-k, x+k)\cap\widehat{\xi}_{nk}^A \neq \varnothing,\\ & [x-k, x+k)\cap\xi_{nk}^D \neq \varnothing\}\\ > \frac{1}{3} \quad \text{a.s. on } \widehat{\Omega}_\infty^A \cap \Omega_\infty^D,&\end{aligned} \tag{4.3}$$

where $\widehat{\Omega}_\infty^A = \bigcap_{n=1}^\infty \{\widehat{\xi}_n^A \neq \varnothing\}$. Suppose that $\eta, \zeta \subset 2\mathbf{Z}$ satisfy $[x_i - k, x_i + k) \cap \eta \neq \varnothing$ and $[x_i - k, x_i + k) \cap \zeta \neq \varnothing$ for some $x_i \in 2k\mathbf{Z}$, $i = 1, 2, \ldots, m$. Then

$$P\big(\widehat{\xi}_k^\eta \cap (\xi_{k-1}^\zeta + 1) \ni x_i\big) \geq (p_1 \wedge p_2)^{2k-1}, \qquad i = 1, 2, \ldots, m, \tag{4.4}$$

and

$$\big\{\widehat{\xi}_k^\eta \cap (\xi_{k-1}^\zeta + 1) \ni x_i\big\}, \qquad i = 1, 2, \ldots, m, \text{ are independent.} \tag{4.5}$$

From (4.3), (4.4) and (4.5) we see that

$$\lim_{n\to\infty} P\big(|\widehat{\xi}_k^A \cap (\xi_{k-1}^D + 1)| \leq \ell,\ \widehat{\Omega}_\infty^A \cap \Omega_\infty^D\big) = 0,$$

for any $\ell \in \mathbf{N}$. Hence we have (4.2) when $p_2 \neq 0$.

When $p_2 = 0$ and $p_1 \in (0, 1)$, (4.2) is rewritten as

$$\lim_{n\to\infty} E\big[(-1)^{|\widehat{\xi}_n^A \cap \partial\xi_{n-1}^D|};\ \widehat{\xi}_n^A \neq \varnothing,\ \xi_{n-1}^D \neq \varnothing\big] = 0. \tag{4.6}$$

Suppose that $\eta, \zeta \subset 2\mathbf{Z}$ satisfy $(x_i - k, x_i + k) \cap \partial\eta \neq \varnothing$ and $(x_i - k, x_i + k) \cap \partial\zeta \neq \varnothing$ for some $x_i \in 2k\mathbf{Z}$, $i = 1, 2, \ldots, m$. Then

$$P\big(\partial\widehat{\xi}_{k-1}^\eta \cap \partial\xi_{k-1}^\zeta \ni x_i\big) \geq p_1^{2k-2}, \qquad i = 1, 2, \ldots, m, \tag{4.7}$$

and

$$\big\{\partial\widehat{\xi}_{k-1}^\eta \cap \partial\xi_{k-1}^\zeta \ni x_i\big\}, \qquad i = 1, 2, \ldots, m, \text{ are independent.} \tag{4.8}$$

From (4.3), (4.7) and (4.8) we see that

$$\lim_{n\to\infty} P\big(|\partial\widehat{\xi}_{n-1}^A \cap \partial\xi_{n-1}^D| \leq \ell,\ \widehat{\Omega}_\infty^A \cap \Omega_\infty^D\big) = 0, \tag{4.9}$$

for any $\ell \in \mathbf{N}$. By the same procedure used to get (3.6), we have

$$E\Big[(-1)^{|\widehat{\xi}_n^A \cap \partial \xi_{n-1}^D|};\ \widehat{\xi}_n^A \neq \varnothing,\ \partial\xi_{n-1}^D \neq \varnothing\Big]$$
$$\leq P\big(|\partial\widehat{\xi}_{n-1}^A \cap \partial\xi_{n-1}^D| < \ell,\ \widehat{\xi}_n^A \neq \varnothing,\ \partial\xi_{n-1}^D \neq \varnothing\big) + (1-2p_1)^\ell.$$

Hence we obtain (4.6) from (4.9).

Next we prove Proposition 7:

$$\begin{aligned}
P(\xi_{2n}^A \cap 2\mathbf{Z} \neq \varnothing) &= m_0(A \times \{0\} \to 2\mathbf{Z} \times \{2n\}) \\
&= \int_X \nu_\alpha(d\eta) m_{0,2n}(A \times \{0\} \to \eta \times \{2n\}) \\
&= \int_X \nu_\alpha(d\eta) m_{0,2n}(\eta \times \{0\} \to A \times \{2n\}) \\
&= \int_X \nu_\alpha(d\eta) \sum_{D \subset (A \pm 1), D \neq \varnothing} m_0(\eta \times \{0\} \rightrightarrows D \times \{2n-1\}) \\
&\qquad\qquad \times m_{2n-1,2n}(D \times \{2n-1\} \to A \times \{2n\}) \\
&= P\big(\xi_{2n-1}^{A_\alpha} \cap (A \pm 1) \neq \varnothing\big) \\
&\quad - E\Big[(1-\beta)^{|\xi_{2n-1}^{A_\alpha} \cap (A+1)| + |\xi_{2n-1}^{A_\alpha} \cap (A-1)|};\ \xi_{2n-1}^{A_\alpha} \cap (A \pm 1) \neq \varnothing\Big].
\end{aligned}$$

Taking $n \to \infty$, by Lemmas 1 and 2, we have

$$\begin{aligned}
\sigma(A, 2\mathbf{Z}) &= \mu_\infty\big(\eta : (\eta - 1) \cap (A \pm 1) \neq \varnothing\big) \\
&\quad - \int_{(\eta-1)\cap(A\pm1)\neq\varnothing} \mu_\infty(d\eta)(1-\beta)^{|(\eta-1)\cap(A+1)| + |(\eta-1)\cap(A-1)|},
\end{aligned}$$

where we used the fact that

$$\lim_{n\to\infty} P(\xi_{2n-1}^{A_\alpha} \cap B \neq \varnothing) = \mu_\infty\big(\eta : (\eta-1) \cap B \neq \varnothing\big), \qquad B \in Y.$$

It is obvious that

$$\lim_{|A|\to\infty} \mu_\infty\big(\eta : (\eta-1) \cap (A+1) \neq \varnothing\big) = \lim_{|A|\to\infty} \mu_\infty\big(\eta : \eta \cap (A+2) \neq \varnothing\big) = 1,$$
$$\lim_{|A|\to\infty} \mu_\infty\big(\eta : (\eta-1) \cap (A-1) \neq \varnothing\big) = \lim_{|A|\to\infty} \mu_\infty(\eta : \eta \cap A \neq \varnothing) = 1.$$

Then, to prove Proposition 7 it is sufficient to show that

$$\lim_{|A|\to\infty} \int_X \mu_\infty(d\eta)(1-\beta)^{|\eta\cap(A+2)| + |\eta \cap A|} = 0, \tag{4.10}$$

for the case of $p_2 \neq 0$, and

$$\lim_{|\partial A|\to\infty} \int_X \mu_\infty(d\eta)(-1)^{|\eta \cap \partial(A+1)|} = 0, \tag{4.11}$$

for the case of $p_2 = 0$. Note that μ_∞ is an invariant probability distribution satisfying (C.1) and (C.2). Then we have

$$\int_X \mu_\infty(d\eta)(1-\beta)^{|\eta\cap(A+2)|+|\eta\cap A|} = E\big[(1-\beta)^{|\xi_k^{\mu_\infty}\cap(A+2)|+|\xi_k^{\mu_\infty}\cap A|}\big]$$

and

$$\int_X \mu_\infty(d\eta)(-1)^{|\eta\cap\partial(A+1)|} = E\big[(-1)^{|\xi_k^{\mu_\infty}\cap\partial(A+1)|}\big],$$

for any $k \in 2\mathbf{Z}$, and (4.2) and (4.3) are derived from (3.3) and (3.7), respectively.

REFERENCES

BRAMSON, M. and NEUHAUSER, C. (1994). Survival of one-dimensional cellular automata under random perturbations. *Ann. Probab.* **22** 244–263.

DOMANY, E. and KINZEL, W. (1984). Equivalence of cellular automata to Ising models and directed percolation. *Phys. Rev. Lett.* **53** 311–314.

DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040.

DURRETT, R. (1988). *Lecture Notes on Particle Systems and Percolation.* Wadsworth, Belmont, CA.

DURRETT, R. and NEUHAUSER, C. (1991). Epidemics with recover in $D = 2$. *Ann. Appl. Probab.* **1** 189–206.

GRIFFEATH, D. (1978). Limit theorems for nonergodic set-valued Markov processes. *Ann. Probab.* **6** 379–387.

HARRIS, T. E. (1976). On a class of set-valued processes. *Ann. Probab.* **4** 175–194.

KATORI, M., KONNO, N. and TANEMURA, H. (2000). Survival probabilities for discrete-time models in one dimension. *J. Statist. Phys.* **99** 603–612.

KINZEL, W. (1985). Phase transitions of cellular automata. *Z. Phys. B* **58** 229–244.

LIND, D. A. (1984). Applications of ergodic theory and sofic systems to cellular automata. *Phys. D* **10** 36–44.

MIYAMOTO, M. (1979). An equilibrium state for a one-dimensional life game. *J. Math. Kyoto Univ.* **19** 525–540.

WOLFRAM, S. (1983). Statistical mechanics of cellular automata. *Rev. Modern Phys.* **55** 601–644.

WOLFRAM, S. (1984). Universality and complexity in cellular automata. *Phys. D* **10** 1–35.

M. KATORI
DEPARTMENT OF PHYSICS
FACULTY OF SCIENCE AND ENGINEERING
CHUO UNIVERSITY
KASUGA, BUNKYO-KU, TOKYO 112-8551
JAPAN
E-MAIL: katori@phys.chuo-u.ac.jp

N. KONNO
DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING
YOKOHAMA NATIONAL UNIVERSITY
HODOGAYA-KU, YOKOHAMA 240-8501
JAPAN
E-MAIL: norio@mathlab.sci.ynu.ac.jp

H. TANEMURA
DEPARTMENT OF MATHEMATICS AND INFORMATICS
FACULTY OF SCIENCE
CHIBA UNIVERSITY
YAYOI, INAGE-KU, CHIBA 263-8522
JAPAN
E-MAIL: tanemura@math.s.chiba-u.ac.jp