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LARGE DEVIATIONS FROM A KINETIC LIMIT[1]

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We study a one-dimensional particle system in which particles travel deterministically in between stochastic collisions. As the total number of particles tends to infinity, the empirical density converges to a solution of a discrete Boltzmann equation. We establish the large deviation principle for the convergence with a rate function that is given by a variational formula. Some of the properties of the rate function are discussed and a nonvariational expression for the rate function is given.

0. Introduction

It is often the case in probability theory that the convergence in a law of large numbers is exponentially fast. The exponential rate at which the convergence occurs is of great interest, especially for problems coming from statistical mechanics. For these problems, the exponential rate function is usually a familiar physical quantity (such as entropy, action functional, thermodynamic potential, etc.) that retains a great amount of valuable information about the model under study.

In a previous article, Rezakhanlou and Tarver [13] estalished a law of large numbers for some one-dimensional particle systems associated with the *discrete Boltzmann equation*. Such a law of large numbers is known as a kinetic or Boltzmann–Grad limit in the context of statistical mechanics. In this article, our goal is to establish a large deviation principle for a kinetic limit and study the corresponding rate function.

The derivation of the *full Boltzmann equation* for models with deterministic collision rules was carried out by Lanford [8] and King [7]. Lanford established the kinetic limit for short times for a model of interacting spheres with elastic collisions. King in his thesis utilizes Lanford's method to treat particle systems with collision rules based on Newton's second law. Later Illner and Pulvirenti [5, 6] showed that Lanford's restriction on time can be replaced by an assumption on the smallness of the collision rates.

The tradition of discretizing the velocity goes back to Maxwell. However, the first realistic step was taken by Broadwell, who proposed a simple model of gases with six velocities. Since Broadwell's work, discrete Boltzmann equations have been successfully used to model dilute gases and study shock waves in fluid mechanics (see, e.g., [9] and [4]).

Recently Caprino and Pulvirenti [2] have derived a discrete Boltzmann equation for particle systems on a line with four velocites. Their derivation

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is valid globally in time with no smallness condition on the initial densities. Their approach, as in [5, 6], is based on a detailed anlysis of the hierarchy equations for the correlation functions. A new approach for the derivation of Boltzmann type PDE's was proposed in [11] and [13]. This new viewpoint utilizes some probabilistic techniques which explore the Markov property of the microscopic system, the entropy bound and some microscopic bounds on the total number of collisions. The key idea behind the latter is that some well-established PDE techniques of Tartar [15] and Bony [1] have indeed microscopic counterparts that can be exploited for our purposes. Using the same general ideas, one can go beyond the macroscopic description given by the Boltzmann equation. In [12], the author derives an Ornstein–Uhlenbeck equation for the fluctuations of the system about its equilibrium states. In [11], it is shown that the dynamics of a tagged particle in the system is governed by an inhomogeneous Markov process with an infinitesimal generator that can be expressed in terms of the macroscopic densities.

The models studied in this article are continuous time Markovian particle systems with the following rules. Each particle has a label α in the set $\{1, \dots, n\}$. A particle with label α travels deterministically with velocity v_α on the circle. Two particles within a distance of order ε collide stochastically through a smooth potential with probability ε and go ahead with no collision with probability $1-\varepsilon$, where ε^{-1} is of the same order as the total number of particles N. If two particles of labels α and β collide, they gain new labels γ and δ with a rate $K(\alpha\beta, \gamma\delta)$. If f_α denotes the macroscopic density of particles with label α, then in [13] it was shown that f_α solves the discrete Boltzmann equation

$$\frac{\partial f_\alpha}{\partial t} + v_\alpha \frac{\partial f_\alpha}{\partial x} = \sum_{\beta\gamma\delta} K(\gamma\delta, \alpha\beta) f_\gamma f_\delta - K(\alpha\beta, \gamma\delta) f_\alpha f_\beta. \tag{0.1}$$

Roughly speaking, we have that as $N \to \infty$ and N/L converges to a nonzero constant,

$$\lim_{L\to\infty} P_L\left(\frac{1}{L}\sum_{i=1}^{N} \delta_{x_i(t)}(dx)\,\mathbb{1}\big(\alpha_i(t)=\alpha\big) \text{ is close to } f_\alpha(x,t)\,dx\right) = 1, \tag{0.2}$$

where $x_i(t)$ and $\alpha_i(t)$ denote the location and the label of the ith particle at time t. In other words, the empirical measure of most configurations converge weakly to a measure that is absolutely continuous with respect to the Lebesgue measure dx, with a density $f_\alpha(x,t)$ that is a solution to (0.1).

The main result of this article establishes the large deviation principle for the convergence in (0.2). Roughly speaking, we show

$$\begin{aligned} &P_L\left(\frac{1}{L}\sum_{i=1}^{N} \delta_{x_i(t)}(dx)\,\mathbb{1}\big(\alpha_i(t)=\alpha\big) \text{ is close to } g_\alpha(x,t)\,dx \text{ for } t\in[0,T]\right) \\ &\qquad \sim \exp\big(-LJ(g)+o(L)\big), \end{aligned} \tag{0.3}$$

where $J(\cdot)$ is a suitable nonnegative functional that vanishes if and only if g is a solution of (0.1). The rate function $J = J_0 + J_d$ is the sum of the *static* rate function J_0 (coming from the deviation from the initial data), and the *dynamical* rate function that for a smooth g is given by

$$J_d(g) = \sup \int_0^T \int \Bigg[\sum_{\alpha} p_\alpha D_\alpha g_\alpha - \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \times \big(\exp(p_\gamma + p_\delta - p_\alpha - p_\beta) - 1\big)\Bigg]\, dx\, dt, \tag{0.4}$$

where the supremum is over smooth functions $p = (p_1, \dots, p_n)$, and $D_\alpha g_\alpha = (\partial g_\alpha/\partial t) + v_\alpha(\partial g_\alpha/\partial x)$. For a nonsmooth g, $J_d(g)$ is defined after an integration by parts. See Section 1 for the definition J and the precise statement of (0.3).

It turns out that the function $H\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$H(g, p) = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \big(\exp(p_\gamma + p_\delta - p_\alpha - p_\beta) - 1\big) \tag{0.5}$$

is convex in the p-variable, and the rate function J_d can be expressed in terms of G, the convex conjugate of H in the p-variable. Indeed, $G\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$, and

$$J_d(g) = \int_0^T \int G(g(x, t), Dg(x, t))\, dx\, dt,$$

where $g = (g_1, \dots, g_n)$ and $Dg = (D_1 g_1, \dots, D_n g_n)$.

In previous works [10, 11, 12] and [13], some microscopic analog of Tartar's argument [15] and the duality formula played an essential role [see for example [10], formula (3.5), or [13], formula (3.12)]. We have not been able to use the same general ideas for the large deviation bounds. In fact, in this article we will initiate some new ideas which in spirit are close to those of Bony [1].

The organization of the paper is as follows. In the next section we describe our results. In Section 2 we discuss our strategy. Section 3 is devoted to an entropy bound. In Section 4 we establish some exponential bounds for the total number of collisions. Section 5 is devoted to an exponential form of Stosszahlansatz (Boltzmann's molecular chaos principle). In Sections 6 and 7 we improve the bounds of Section 4 by establishing the uniform integrability of the collision term. The large deviation upper bounds will be established in Section 8 and the lower bounds will be given in Section 9. Sections 10 and 11 are devoted to the properties of the rate function.

1. Notation and main results.

This section is devoted to the statement of our main results. We first describe the model for which the large deviation principle will be established.

We define $\mathbb{T}$ to be the interval $[-\frac{1}{2}, \frac{1}{2}]$ with the endpoints identified. Let $I = \{1, 2, \dots, n\}$; I denotes the set of labels of the n different types of particles. Each $\alpha \in I$ corresponds to a velocity $v_\alpha \in \mathbb{R}$. The state space $E = (\mathbb{T} \times I)^N$

consists of N-tuples $\mathrm{q} = (q_1, \dots, q_N)$ with $q_i = (x_i, \alpha_i)$ where x_i and α_i denote the *location* and the *label* of the ith particle, respectively.

The dynamics of $\mathrm{q}(t)$ are Markovian and are characterized by the infinitesimal generator $\mathscr{A}^{(L)} = \mathscr{A}_0 + \mathscr{A}_c$. Here $\mathscr{A}_0$ corresponds to the free motion, $\mathscr{A}_c$ describes the interaction among particles and the relation between N and L is $N = ZL$ where Z is a nonzero constant. More precisely,

$$\mathscr{A}_0 F(\mathrm{q}) = \sum_{i=1}^{N} v_{\alpha_i} \frac{\partial F}{\partial x_i}(\mathrm{q}), \tag{1.1}$$

$$\mathscr{A}_c F(\mathrm{q}) = \tfrac{1}{2} \sum_{i \neq j} V\big(L(x_i - x_j)\big) \sum_{\gamma, \delta} K(\alpha_i \alpha_j, \gamma\delta)\big(F(S_{i,j}^{\gamma,\delta}\mathrm{q}) - F(\mathrm{q})\big), \tag{1.2}$$

where $S_{i,j}^{\gamma,\delta}\mathrm{q}$ is the configuration obtained from q by changing the labels of the ith and jth particles from α_i, α_j to γ, δ, respectively, and $V: \mathbb{R} \to [0, \infty)$ is an even, continuously differentiable function of compact support with

$$\int V(z)\, dz = 1. \tag{1.3}$$

In the sequel $(-r_0, r_0)$ denotes an interval that contains the support of V.

CONVENTION 1.1. In (1.2), $x_i - x_j$ is defined to be the signed distance between x_i and x_j. Hence the argument of V belongs to $\mathbb{R}$. In (1.1) and several places below, the function F is regarded as a periodic function of period one in each x_i variable. In the sequel, for every $z \in \mathbb{R}$, the sum $x_i + z$ is defined periodically so that $x_i + z$ is regarded as a point in $\mathbb{T}$.

Our assumptions on K are:

(1.4) (i) $K(\alpha\beta, \gamma\delta) \geq 0$;
(1.4) (ii) $K(\alpha\beta, \gamma\delta) = K(\beta\alpha, \gamma\delta) = K(\alpha\beta, \delta\gamma)$;
(1.4) (iii) $K(\alpha\beta, \gamma\delta) = 0$ if $v_\alpha = v_\beta$;
(1.4) (iv) $K(\alpha\beta, \gamma\delta) = 0$ if $\{v_\alpha, v_\beta\} = \{v_\gamma, v_\delta\}$;
(1.4) (v) $K(\alpha\beta, \gamma\delta) = 0$ if $v_\alpha + v_\beta \neq v_\gamma + v_\delta$;
(1.4) (vi) There exists $\Lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_\alpha > 0$, such that for all $\alpha, \beta, \gamma, \delta \in I$, $K(\alpha\beta, \gamma\delta)\lambda_\alpha\lambda_\beta = K(\gamma\delta, \alpha\beta)\lambda_\gamma\lambda_\delta$.

Since we are thinking of K as a collision rate, K is necessarily positive; (ii) states that the collision rates depend upon the labels only and are independent of the particle numbers; (iii) implies that only particles of different velocities can collide; (iv) ensures that a collision always results in a change in velocities; (v) means that a microscopic *conservation of momentum* holds; (vi) states that Λ is a *Maxwellian* [i.e., Λ is an equilibrium solution of our discrete equation (0.1)]. A consequence of our assumptions on K is the following:

$$\text{if } K(\alpha\beta, \gamma\delta) \neq 0 \text{ then } v_\alpha \neq v_\gamma, v_\delta \text{ and } v_\beta \neq v_\gamma, v_\delta. \tag{1.5}$$

We use the last assumption on K to help determine an invariant measure for our process $\mathrm{q}(t)$. Given $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_\alpha > 0$, we define a measure ν_Λ on E by

$$\begin{aligned}\int F(\mathsf{q})\nu_\Lambda(d\mathsf{q})\\ = \int\cdots\int \sum_{\alpha_1=1}^{n}\cdots\sum_{\alpha_N=1}^{n} F(x_1,\alpha_1,\dots,x_N,\alpha_N)\lambda_{\alpha_1}\dots\lambda_{\alpha_N}\,dx_1\dots dx_N,\end{aligned} \tag{1.6}$$

where dx denotes the Lebesgue measure on $\mathbb{T}$. It is not hard to see that ν_Λ is invariant with respect to $\mathscr{A}_0$. Moreover, if Λ satisfies the *Maxwell* conditions (1.4)(vi), then ν_Λ is also invariant with respect to $\mathscr{A}_c$.

Initially, particles are located on $\mathbb{T}$ independently and with probability density $(1/Z)f_\alpha^0(x)$ where Z is a normalizing constant.

NOTATION 1.2. Let μ_L be a sequence of *probability* measures on E, and let $f^0=(f_1^0,\dots,f_n^0)$: $\mathbb{T}\to[0,\infty)^n$ be a nonnegative bounded measurable function. We then write $\mu_L\sim f^0$ if for every continuous function F,

$$\begin{aligned}\int F(\mathsf{q})\mu_L(d\mathsf{q}) = \frac{1}{Z^N}\int\cdots\int\sum_{\alpha_1\dots\alpha_N} F(x_1,\alpha_1,\dots,x_N,\alpha_N)\\ \times f_{\alpha_1}^0(x_1)\dots f_{\alpha_N}^0(x_N)\,dx_1\dots dx_N\end{aligned} \tag{1.7}$$

where $Z=\int\sum_\alpha f_\alpha^0(x)\,dx$. The relationship between L and N is

$$L=\frac{N}{Z}. \tag{1.8}$$

Given f^0 as in Notation 1.2, we define $f=(f_1,\dots,f_n)$: $\mathbb{T}\times[0,+\infty)\to[0,\infty)^n$ to be the unique solution to the initial value problem

$$\begin{aligned}\frac{\partial f_\alpha}{\partial t}+v_\alpha\frac{\partial f_\alpha}{\partial x} &= Q_\alpha(f,f), \qquad \alpha\in I,\\ f_\alpha(x,0) &= f_\alpha^0(x), \qquad \alpha\in I,\end{aligned} \tag{1.9}$$

where

$$Q_\alpha(f,f):=\sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta)f_\gamma f_\delta - K(\alpha\beta,\gamma\delta)f_\alpha f_\beta.$$

Where there is no danger of confusion we write $Q_\alpha(x,t)$ for $Q_\alpha(f,f)(x,t)$.

A solution to (1.9) is understood in the following sense:

1. $f_\alpha\in C([0,T],L^1(\mathbb{T}))$;
2. $f_\beta f_\gamma\in L^1([0,T]\times\mathbb{T})$ for every positive T and whenever $v_\beta\neq v_\gamma$;
3. For every t and $\alpha\in I$, and almost all x,

$$f_\alpha(x,t)=f_\alpha^0(x-v_\alpha t)+\int_0^t Q_\alpha(x-(t-s)v_\alpha,s)\,ds. \tag{1.10}$$

Let $\mathrm{q}(t) = (x_1(t), \alpha_1(t), \ldots, x_N(t), \alpha_N(t))$ denote the process generated by $\mathscr{A}^{(L)}$ with $\mathrm{q}(0)$ distributed according to $\mu_L(d\mathrm{q})$. Let P_L and E_L denote the probability and expectation with respect to the process $\mathrm{q}(\cdot)$. For each trajectory $\mathrm{q}(\cdot)$, we define the empirical density

$$m_\alpha(t, dx) = \frac{1}{L}\sum_{i=1}^{N} \delta_{x_i(t)}(dx)\,\mathbb{1}\big(\alpha_i(t) = \alpha\big), \tag{1.11}$$

which is a random measure on $\mathbb{T}$, for every $t \in [0, T]$. We take a version of $\mathrm{q}(t)$ that is right continuous and has left limits. As a result, $m_\alpha(t, dx)$ is weakly right continuous with left limits. If $M_n(\mathbb{T})$ denotes the space of vector measures $m = (m_1, \ldots, m_n)$ with $\sum_{\alpha=1}^n m_\alpha(\mathbb{T}) = Z$, we can regard $m(t, dx) = (m_1(t, dx), \ldots, m_n(t, dx))$ as an element of the Skorohod space $\mathscr{D} := D([0, T), M_n(\mathbb{T}))$ where $M_n(\mathbb{T})$ is endowed with the topology of weak convergence. The transformation $\mathrm{q} \mapsto m$ with m defined by (1.11) and q distributed according to P_L, induces a probability measure $\mathscr{P}_L$ on $\mathscr{D}$. The main result of [13] in our setting asserts the theorem.

THEOREM 1.3. *For every continuous function r: $[0, T] \times \mathbb{T} \to \mathbb{R}$, every $\alpha \in I$, and each positive δ,*

$$\lim_{L\to\infty} \mathscr{P}_L\left(\left|\int_0^T \int r(x, t) m_\alpha(t, dx)\,dt - \int_0^T \int r(x, t) f_\alpha(x, t)\,dx\,dt\right| > \delta\right) = 0.$$

To prepare for the statement of the main result, we start with the definition of the large deviation rate function J: $\mathscr{D} \to [0, +\infty]$.

DEFINITION 1.4. We say that a measurable function g_α is *α-differentiable* if there exists an L^1-function $D_\alpha g_\alpha$ such that for any smooth function $r(x, t)$ with support in $\mathbb{T} \times (0, T)$,

$$\int_0^T \int g_\alpha D_\alpha r\,dx\,dt = -\int_0^T \int r D_\alpha g_\alpha\,dx\,dt. \tag{1.12}$$

We set $J(m) = +\infty$ unless $m_\alpha(t, dx) = g_\alpha(x, t)\,dx$, g_α is α-differentiable for every α, $\int \sum_\alpha g_\alpha(x, 0)\,dx = Z$ and for any pair (α, β) with $v_\alpha \neq v_\beta$,

$$\int_0^T \int g_\alpha g_\beta\,dx\,dt < \infty.$$

Note that g_α: $[0, T] \to L^1(\mathbb{T})$ is weakly right continuous in t with left limits because $m \in \mathscr{D}$. We then define

$$J_0(m) = \int \sum_\alpha \left(g_\alpha(x, 0) \log \frac{g_\alpha(x, 0)}{f_\alpha^0(x)} - g_\alpha(x, 0) + f_\alpha^0(x)\right) dx, \tag{1.13}$$

$$
\begin{aligned}
J_d(m) &= \sup_p J_d(m;p),\\
J_d(m;p) &= \int_0^T \int p(x,t) Dg(x,t)\,dx\,dt\\
&\quad - \int_0^T \int \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g_\alpha(x,t) g_\beta(x,t)\\
&\qquad \times \big(\exp(p_\gamma(x,t)+p_\delta(x,t)\\
&\qquad\quad - p_\alpha(x,t) - p_\beta(x,t)) - 1\big)\,dx\,dt,
\end{aligned} \tag{1.14}
$$

where the supremum is over bounded measurable functions $p = (p_1, \dots, p_n)$: $\mathbb{T} \times [0, T] \to \mathbb{R}^n$,

$$
Dg = (D_1 g_1, \dots, D_n g_n), \tag{1.15}
$$

and $a \cdot b$ denotes the inner product of $a, b \in \mathbb{R}^n$. Finally, we define $J(m) = J_0(m) + J_d(m)$, and when there is no danger of confusion we write $J(g)$, $J_0(g)$ and $J_d(g)$ for $J(m)$, $J_0(m)$ and $J_d(m)$, respectively.

If there exists a bounded measurable function $\hat{p}$ so that g satisfies

$$
\begin{aligned}
D_\alpha g_\alpha = \sum_{\beta\gamma\delta} & K(\gamma\delta, \alpha\beta) \exp(\hat{p}_\alpha + \hat{p}_\beta - \hat{p}_\gamma - \hat{p}_\delta) g_\gamma g_\delta\\
& - K(\alpha\beta, \gamma\delta) \exp(\hat{p}_\gamma + \hat{p}_\delta - \hat{p}_\alpha - \hat{p}_\beta) g_\alpha g_\beta,
\end{aligned} \tag{1.16}
$$

in the distributional sense, then the rate function is equal to (see Proposition 11.7)

$$
J(m) = \int_0^T \int \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g_\alpha g_\beta \psi(\hat{p}_\gamma + \hat{p}_\delta - \hat{p}_\alpha - \hat{p}_\beta)\,dx\,dt, \tag{1.17}
$$

where $\psi(z) = e^z(z-1) + 1$. Let $\mathscr{B}$ (resp. $\mathscr{C}$) denote the set of measures $m(t,dx) = g(x,t)\,dx$ for which (1.16) holds for a bounded measurable (resp. smooth) $\hat{p}$.

The main result of this paper is Theorem 1.5.

THEOREM 1.5. *For every open set $\mathscr{G} \subseteq \mathscr{D}$ and compact set $\mathscr{F} \subseteq \mathscr{D}$ we have*

$$
\limsup_{L\to\infty} \frac{1}{L} \log \mathscr{P}_L(\mathscr{F}) \le - \inf_{m\in\mathscr{F}} J(m), \tag{1.18}
$$

$$
\liminf_{L\to\infty} \frac{1}{L} \log \mathscr{P}_L(\mathscr{G}) \ge - \inf_{m\in\mathscr{G}\cap\mathscr{B}} J(m). \tag{1.19}
$$

The proof of (1.18) will be given in Section 8. In Section 9 we establish (1.19) but with $\mathscr{B}$ replaced with $\mathscr{C}$. In the final section we will establish various properties of J that would eventually lead to the statement

$$
\inf_{m\in\mathscr{G}\cap\mathscr{B}} J(m) = \inf_{m\in\mathscr{G}\cap\mathscr{C}} J(m) \tag{1.20}
$$

for every open subset $\mathscr{G} \subseteq \mathscr{D}$.

The large deviation bound (1.18) for noncompact closed sets $\mathscr{F}$ remains open. We also *conjecture* that

$$\inf_{m\in\mathscr{G}\cap\mathscr{B}} J(m) = \inf_{m\in\mathscr{G}} J(m).$$

REMARK 1.6. The condition (1.4)(v) can be replaced by a weaker condition. With some slight changes in the proofs, one can establish the results of this paper for a larger class of systems. Basically we need to replace (1.4)(v) with a condition that would provide us with a Lyapunov functional similar to (2.1) of the next section. For example, we may assume that there exists a vector $(b_1, \dots, b_n)$ such that

$$\text{(1.4) (v)}' \qquad K(\alpha\beta, \gamma\delta) = 0 \quad \text{if } b_\alpha + b_\beta \neq b_\gamma + b_\delta,$$

$$\text{(1.4) (iv)}'' \qquad (v_\alpha - v_\beta)(b_\alpha - b_\beta) \geq 0 \quad \text{and} \quad b_\alpha \neq b_\beta \text{ if } v_\alpha \neq v_\beta.$$

We end this section with an example of a model for which the new condition is satisfied.

EXAMPLE 1.7. (The left–right model). There exists a decomposition $I = I_1 \cup I_2$ such that $v_\alpha > 0$ for $\alpha \in I_1$ and $v_\alpha < 0$ for $\alpha \in I_2$, and if $K(\alpha\beta, \gamma\delta) \neq 0$, then

$$(\alpha, \beta), (\gamma, \delta) \in (I_1 \times I_2) \cup (I_2 \times I_1).$$

One can readily verify (1.4)(iv)$'$ and (1.4)(iv)$''$ by choosing $b_\alpha = 1$ if $\alpha \in I_1$ and $b_\alpha = 0$ otherwise.

2. Sketch of proofs. Two well-known methods for the existence of solutions to the Boltzmann equation (1.9) are due to [1] and [15]. In [10], a combination of these methods was employed to establish the kinetic limit for some particle systems on one-dimensional lattices. The arguments of [13], however, were largely modeled upon Tartar's method [15]. As will be apparent below, we have adopted a new approach that is essentially different from [13] and in spirit is close to one of the arguments used in Section 5 of [10]. The reason behind our change of strategy is that although the arguments of [13] and most of [10] were successful for the kinetic limit, we have not been able to utilize them for purposes of the present article. However, it is worth mentioning that our approach in this article seems to be strictly one-dimensional, whereas it might be possible to carry out some of the arguments of [13] for multidimensional models.

Let f be a solution to (1.9). We define

$$\text{(2.1)} \qquad X(t) = \int_{\mathbb{T}}\int_{\mathbb{T}} \sum_{\alpha,\beta} (v_\alpha - v_\beta) f_\alpha(x,t) f_\beta(y,t) \xi(x-y)\, dx\, dy,$$

where $\xi(z)$ is a periodic function of period one that is defined to be $z - \frac{1}{2}$ for $z \in [0,1)$. By conservation of mass and momentum, the integrand is invariant with respect to the collision term. From this and after some integration by parts, we obtain

$$\frac{d}{dt}X(t) = -\int_{\mathbb{T}} \sum_{\alpha,\beta} (v_\alpha - v_\beta)^2 f_\alpha(x,t) f_\beta(x,t)\, dx + \int_{\mathbb{T}}\int_{\mathbb{T}} \sum_{\alpha,\beta} (v_\alpha - v_\beta)^2 f_\alpha(x,t) f_\beta(y,t)\, dx\, dy.$$

This in turn implies

$$\int_0^\infty \int_{\mathbb{T}} \sum_{\alpha,\beta} (v_\alpha - v_\beta)^2 f_\alpha(x,t) f_\beta(x,t)\, dx\, dt \le \text{const.}\left(\int \sum_\alpha f_\alpha^0(x)\, dx\right)^2. \tag{2.2}$$

A microscopic version of the above argument would lead to an exponential bound on the collision term of the form

$$\sup_L \frac{1}{cL} \log E_L \exp\left[c \int_0^{T_L} \sum_{i\ne j} (v_{\alpha_i(t)} - v_{\alpha_j(t)})^2 V(L(x_i(t) - x_j(t)))\, dt\right] < \infty, \tag{2.3}$$

where T_L is a suitable sequence of *stopping times* and c is any constant. As we will see in Section 4, the sequence T_L will be chosen so that the probability of $T_L \ne T$ is super exponentially small.

REMARK 2.1. Note that if instead of the conservation of momentum, we have the conditions (1.4)$(v)'$ and (1.4)$(v)''$, then the factor $v_\alpha - v_\beta$ in (2.1) is replaced with $b_\alpha - b_\beta$.

The functional $X(t)$ is known as *Bony's Lyapunov functional* and it was employed in [10] to obtain a uniform (nonexponential) bound on the collision term. In the microscopic model studied in [10], particles travel as independent random walks, and whenever two particles occupy the same site, they collide stochastically. The derivative of the microscopic analog of $X(t)$ in [10] was a sum of a nonpositive term and an error term where the error term comes from the randomness of the free motion part of the dynamics. In the model studied in this paper the *error term* comes from the fact that the particles can collide without being at the same location. To treat the error term, we will appeal to some *entropy bounds* that will be discussed in Section 3. The entropy bound (3.3) of Section 3 should be regarded as the microscopic (exponential) analog of the entropy bound

$$\sup_{0\le t\le T} \int \sum_\alpha f_\alpha \log^+ f_\alpha(x,t)\, dx < \infty. \tag{2.4}$$

In Section 4 we use (3.3) and Bony's Lyapunov functional to establish (2.3). It turns out that we can do better than (2.3). As we will see in Theorem 4.1, we can afford to substitute for c in (2.3) a sequence that diverges like $\log\log L$.

We next sketch another argument for the solutions to (1.9) that will be carried out microscopically in Section 5.

Let f be a solution to (1.9). Set $V_\varepsilon(z) = \varepsilon^{-1}V(\varepsilon^{-1}z)$ and $f_{\alpha,\varepsilon} = f_\alpha * V_\varepsilon$. To show that for every smooth function $r(x)$,

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}}(f_{\alpha,\varepsilon}f_{\beta,\varepsilon} - f_\alpha f_\beta)r\,dx\,dt = 0,$$

it suffices to prove

$$\lim_{\varepsilon\to 0}\sup_{|z|\le\varepsilon}\int_0^T\int_{\mathbb{T}} f_\alpha(x,t)(f_\beta(x+z,t) - f_\beta(x,t))r(x)\,dx\,dt = 0. \tag{2.5}$$

Fix a pair (α,β) with $v_\alpha \ne v_\beta$, and define

$$Y(z;t) = \int_{\mathbb{T}}\int_{\mathbb{T}} f_\alpha(x,t)f_\beta(y+z,t)\xi(x-y)r(x)\,dx\,dy. \tag{2.6}$$

Then after a differentiation and an integration by parts,

$$\begin{aligned}\frac{dY}{dt} &= -(v_\alpha - v_\beta)\int f_\alpha(x,t)f_\beta(x+z,t)r(x)\,dx\\ &\quad + (v_\alpha - v_\beta)\int\int f_\alpha(x,t)f_\beta(y+z,t)r(x)\,dx\,dy\\ &\quad + \int\int v_\alpha f_\alpha(x,t)f_\beta(y+z,t)\xi(x-y)r'(x)\,dx\,dy\\ &\quad + \int\int Q_\alpha(x,t)f_\beta(y+z,t)\xi(x-y)r(x)\,dx\,dy\\ &\quad + \int\int f_\alpha(x,t)Q_\beta(y+z,t)\xi(x-y)r(x)\,dx\,dy\\ &= \sum_{j=1}^5 \Omega_j(z,t).\end{aligned} \tag{2.7}$$

To prove (2.5), it suffices to show

$$\lim_{\varepsilon\to 0}\sup_{|z|\le\varepsilon}\left|\int_0^T(\Omega_j(z,t) - \Omega_j(0,t))\,dt\right| = 0 \tag{2.8}$$

for $j = 2,3,4,5$ and

$$\lim_{\varepsilon\to 0}\sup_{|z|\le\varepsilon}\sup_{0\le t\le T}\left|Y(z,t) - Y(0,t)\right| = 0. \tag{2.9}$$

As an example, we sketch the proof of (2.8) for $j = 5$. Suppose $0 \le z \le \varepsilon$. Then it is not hard to see that

$$\begin{aligned}&\left|\int_0^T(\Omega_3(z,t) - \Omega_3(0,t))\,dt\right|\\ &\qquad \le \int_0^T\int\left|Q_\alpha(x,t)r(x)\right|\int_x^{x+z} f_\beta(y,t)\,dy\,dx\,dt.\end{aligned} \tag{2.10}$$

The entropy bound (2.4) implies (see Lemma 10.8)

$$\int_x^{x+z} f_\beta(y,t)\,dy \le \text{const.}|\log z|^{-1}. \tag{2.11}$$

This and (2.2) imply that the right-hand side of (2.10) is bounded by a multiple constant of $|\log\varepsilon|^{-1}$, proving (2.8) in the case of $j = 5$. Similar arguments would treat (2.9) and (2.8) in the remaining cases. See Section 5 for more details.

We now turn to the uniform integrability of the collision term. Let (α, β, γ) be three labels for which $v_\alpha, v_\beta, v_\gamma$ are distinct. We have

$$\int \Gamma_1\left(\int_0^T f_\alpha f_\beta(x + v_\gamma t, t)\,dt\right) dx < \infty, \tag{2.12}$$

where $\Gamma_1(z) = z(\log^+ z)^b$ for some $0 < b < 1$. Let $U(x)$ denote the argument of the function Γ_1 in (2.12). It turns out that (2.12) is followed by

$$\int U(x)\,\mathbb{1}(U(x) > l)\,dx \le \text{const.}(\log l)^{-1}.$$

Let $A = \{x\colon U(x) > l\}$. By the collision bound (2.2), we have that $|A|$, the Lebesgue measure of A, is bounded above by a constant multiple of l^{-1}. Hence it suffices to show

$$\sup_{|A|\le\delta} \int U(x)\,\mathbb{1}_A(x) \le \text{const.}|\log\delta|^{-1}. \tag{2.13}$$

For this we define

$$\begin{aligned} Z(t) = \int\int f_\alpha(x,t) f_\beta(y,t)\xi(x-y) \\ \times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dx\,dy \end{aligned} \tag{2.14}$$

with $H(z) = \mathbb{1}_A(z)$. The integrand of Z is chosen in such a way that dZ/dt is a sum of four terms with one term of the form (2.13) (without supremum), and the other terms of order $|\log\delta|^{-1}$. As was demonstrated in Section 8 of [10], (2.12) can be used to yield the uniform integrability of the collision

$$\int_{\mathbb{T}}\int_0^T \Gamma_1(f_\alpha f_\beta)\,dx\,dt < \infty. \tag{2.15}$$

We omit the details and refer the reader to Sections 6 and 7 for microscopic calculations similar to the above argument.

Morally speaking, the large deviation rate function $J(g)$ tells us at what price the profile g can be reached. We know that $J(g) = 0$ if and only if g is a solution to the Boltzmann equation (1.9). If $J(g) > 0$, we may regard g as an atypical profile for the microscopic model. The reader may wonder how our macroscopic arguments for f, a solution to (1.9), are relevant when we are interested in atypical profiles! The point is that for the large deviation principle we are only interested in atypical profiles with $J(g) < \infty$, and it turns out that such g will solve a perturbed Boltzmann equation (1.16) for a suitable $\hat{p}$. This new PDE may be regarded as a Boltzmann equation with a new jump rate

$$\hat{K}(\alpha\beta, \gamma\delta; x, t) = \hat{K}(\alpha\beta, \gamma\delta) := K(\alpha\beta, \gamma\delta)\exp(\hat{p}_\gamma + \hat{p}_\delta - \hat{p}_\alpha - \hat{p}_\beta). \tag{2.16}$$

When $\hat{p}$ is bounded, with a minor modification, all the above macroscopic arguments can be repeated, in spite of temporal and spatial dependence of $\hat{K}$. For example, (1.4) holds for $\hat{K}$, which in particular implies the conservation of momentum. Hence (2.2) holds for such g. In general, $\hat{p}$ is not bounded. Nonetheless a condition of the form $J(g) < \infty$ implies (2.2) and (2.12) (see Section 10). As we will see in Theorem 10.6, (2.15) is also true but with $\Gamma_2(z) = z(\log^+ \log^+ z)^b$ instead of Γ_1.

It turns out that there is a price of two logarithms to pay as we go from the macroscopic bounds of Section 10 to the microscopic bounds of Sections 6 and 7. For the microscopic analogs of (2.12) and (2.15), we are forced to replace Γ_1 with

$$\Gamma_3(z) = z(\log^+ \log^+ \log^+ z)^b \quad \text{and} \quad \Gamma_4(z) = z(\log^+ \log^+ \log^+ \log^+ z)^b, \tag{2.17}$$

respectively. The additional logarithms are apparently related to the fact that a collision between two particles x_i and x_j can occur when $|x_i - x_j|$ is of order $O(L^{-1})$ (as opposed to $x_i = x_j$). Because of this we can only afford to choose c in (2.2) that grows like $\log\log L$ and not faster!

3. Entropy bound.

In this section we recall an entropy bound from [13] and discuss some of its consequences.

For each set $R \subseteq \mathbb{T}$, set

$$\mathscr{N}(R, \mathrm{q}) = \#\{i\colon x_i \in R\}. \tag{3.1}$$

If $R = (a, b]$, we simply write $\mathscr{N}(a, b, \mathrm{q})$ for $\mathscr{N}(R, \mathrm{q})$. Set $\phi(z) = z \log z - z + 1$. Define

$$\Phi(\mathrm{q}) = \frac{1}{L} \sum_{i=0}^{L-1} \phi\left(\mathscr{N}\left(-\frac{1}{2} + \frac{i}{L}, -\frac{1}{2} + \frac{i+1}{L}, \mathrm{q}\right)\right). \tag{3.2}$$

Recall that f^0 is bounded. Fix $\theta_0 \in (0, 1)$. In Section 2 of [13] we showed one following lemma.

LEMMA 3.1. *There exists a constant $C_0(T)$ such that for every L,*

$$E_L \exp\Big[L\theta_0 \sup_{0 \le t \le T} \Phi(\mathrm{q}(t))\Big] \le \exp(C_0(T)L). \tag{3.3}$$

An easy consequence of (3.3) and Hölder inequality is

$$E_L \exp\Big[L\theta\theta_0 \sup_{0 \le t \le T} \Phi(\mathrm{q}(t))\Big] \le \exp(C_0(T)\theta L), \tag{3.4}$$

for every $\theta \in [0, 1]$.

Let $h(\delta) = |1 + \log \delta|^{-1}$ if $\delta < 1$; $h(\delta) = 1$ otherwise. Let $\zeta\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative function of compact support. For each measurable ρ, define

$$\rho_L(x) = \int L\zeta(L(x - z))\rho(z)\, dz.$$

The proof of the following lemma can be found in the Section 5 of [13].

LEMMA 3.2. *There exists a constant $\hat{C}_0(\zeta)$ such that for every nonnegative ρ,*

$$\frac{1}{N}\sum_{i=1}^{N}\rho_L(x_i) \le \hat{C}_0(\zeta)\|\rho\|_{L^\infty} h(\|\rho\|_{L^1})(1+\Phi(\mathrm{q})). \tag{3.5}$$

If we choose a ζ that is identically 1 in the interval [0,1], Lemma 3.2 yields Lemma 3.3.

LEMMA 3.3. *There exists a constant $\hat{C}_0$ such that for every nonnegative measurable function ρ,*

$$\int \mathscr{N}\left(z, z+\frac{1}{L}, \mathrm{q}\right)\rho(z)\,dz \le \hat{C}_0\|\rho\|_{L^\infty} h(\|\rho\|_{L^1})(1+\Phi(\mathrm{q})). \tag{3.6}$$

In particular, for every interval $(a, b]$,

$$\frac{1}{L}\mathscr{N}(a, b, \mathrm{q}) \le \hat{C}_0 h(b-a)(1+\Phi(\mathrm{q})). \tag{3.7}$$

Note that (3.7) follows from (3.6) by choosing $\rho(x)$ to be the indicator function of the interval $(a, b]$.

4. Exponential bounds on the collision term.

Recall that the support of V is contained in the interval $(-r_0, r_0)$. We take $W\colon \mathbb{R} \to \mathbb{R}$ to be an odd function with $W' = V$, $W(z) = -\frac{1}{2}$ if $z \le -r_0$, and $W(z) = \frac{1}{2}$ if $z \ge r_0$. We then take a twice continuously differentiable periodic odd function $W_L\colon \mathbb{R} \to \mathbb{R}$ of period one such that $|W_L(z)| \le \frac{1}{2}$, $W_L(z) = W(Lz)$ for $z \in [-r_0/L, r_0/L]$, and $|W_L'(z)| \le 2$, $|W_L''(z)| \le 3$ for $z \in [r_0/L, 1-r_0/L]$. Such W_L exists if we choose L sufficiently large. As a result, we can write

$$W_L'(z) = LV(Lz) + R_L(z) \tag{4.1}$$

for $z \in [-r_0/L, 1-r_0/L]$, where R_L is a continuously differentiable periodic function with $|R_L| \le 2$, $|R_L'| \le 3$.

In the sequel P^{q} and E^{q} denote the probability and the expectation of the process $\mathrm{q}(t)$ with $\mathrm{q}(0) = \mathrm{q}$. Define

$$\begin{aligned} A_L(\mathrm{q}, z) &= \sum_{i,j} V\big(L(x_i - x_j + z)\big)(v_{\alpha_i} - v_{\alpha_j})^2, \\ A_L(\mathrm{q}) &= A_L(\mathrm{q}, 0), \\ \bar{v} &= \max_{\alpha} |v_\alpha|. \end{aligned} \tag{4.2}$$

The main result of this section is the following exponential bound on the total number of collisions.

THEOREM 4.1. *There exist a sequence of stopping times $T_L \in [0, T]$ and four positive constants $\eta_1 = \eta_1(T)$, $\hat{\eta}_1 = \hat{\eta}_1(T)$, $C_2 = C_2(T)$ and $\hat{C}_2 = \hat{C}_2(T)$*

such that

$$\lim_{L\to\infty}\frac{1}{L}\log P_L(T_L \neq T) = -\infty, \tag{4.3}$$

and

(a) *for every positive c,*

$$C_1(c,T) := \sup_L \sup_{z\in\mathbb{T}} \frac{1}{L}\log E_L \exp\int_0^{T_L} cA_L(\mathrm{q}(t),z)\,dt < \infty; \tag{4.4}$$

(b) *if $c \le \eta_1 \log\log L$, then*

$$E_L \exp\int_0^{T_L} cA_L(\mathrm{q}(t))\,dt \le \exp(C_2 cL); \tag{4.5}$$

(c) *if $c \le \hat{\eta}_1 \log\log\log L$ then*

$$\sup_{z\in\mathbb{T}} E_L \exp\int_0^{T_L} cA_L(\mathrm{q}(t),z)\,dt \le \exp(\hat{C}_2 L + \hat{C}_2 c e^{16\bar{v}Zc} L). \tag{4.6}$$

PROOF.

Step 1. Clearly, part (c) implies (4.4). So we only establish parts (b) and (c). Recall the function W_L defined right before (4.1), and define

$$F(\mathrm{q}) = \frac{4c}{L}\sum_{i,j} W_L(x_i - x_j + z)(v_{\alpha_j} - v_{\alpha_i}),$$

$$G(y) = \frac{1}{2L}\sum_k W_L(x_k - y + z) + W_L(x_k - y - z).$$

Since W_L is odd,

$$F(\mathrm{q}) = \frac{4c}{L}\sum_{i,j}\big(W_L(x_i - x_j + z) + W_L(x_i - x_j - z)\big)v_{\alpha_j} = 8c\sum_j G(x_j)v_{\alpha_j}.$$

It is well known that the process

$$M_t = \exp\left(F(\mathrm{q}(t)) - F(\mathrm{q}(0)) - \int_0^t e^{-F}\mathscr{A}^{(L)}e^F(\mathrm{q}(s))\,ds\right)$$

is a martingale for $t \in [0,T]$. Hence

$$E^{\mathrm{q}} M_\tau = 1, \tag{4.7}$$

for every stopping time τ. On the other hand,

$$
\begin{aligned}
\Omega_1(\mathrm{q}) &:= e^{-F}\mathscr{A}_0 e^F(\mathrm{q}) = \mathscr{A}_0 F(\mathrm{q}) \\
&= -4c\sum_{i,\,j} V\big(L(x_i - x_j + z)\big)(v_{\alpha_i} - v_{\alpha_j})^2 \\
&\qquad - \frac{4c}{L}\sum_{i,\,j} R_L(x_i - x_j + z)(v_{\alpha_i} - v_{\alpha_j})^2 \\
&:= \Omega_{11}(\mathrm{q}) + \Omega_{12}(\mathrm{q}), \\
\Omega_2(\mathrm{q}) &:= e^{-F}\mathscr{A}_c e^F(\mathrm{q}) \\
&= \tfrac{1}{2}\sum_{i,\,j}\sum_{\gamma,\,\delta} K\big(\alpha_i\alpha_j, \gamma\delta\big) V\big(L(x_i - x_j)\big) \\
&\qquad\qquad \times \big[\exp\big[8c(G(x_i) - G(x_j))(v_\gamma - v_{\alpha_i})\big] - 1\big],
\end{aligned} \tag{4.8}
$$

where for Ω_1 we have used (4.1) and for Ω_2 we have used the conservation of momentum (1.4)(v):

$$
\begin{aligned}
F\big(S_{ij}^{\gamma\delta}\mathrm{q}\big) - F(\mathrm{q}) &= 8cG(x_i)(v_\gamma - v_{\alpha_i}) + 8cG(x_j)(v_\delta - v_{\alpha_j}) \\
&= 8c\big(G(x_i) - G(x_j)\big)(v_\gamma - v_{\alpha_i}).
\end{aligned} \tag{4.9}
$$

We rewrite (4.7) as

$$
E_L^{\mathrm{q}} \exp\bigg[F\big(\mathrm{q}(\tau)\big) - F\big(\mathrm{q}(0)\big) - \int_0^\tau \Omega_1\big(\mathrm{q}(s)\big)\,ds - \int_0^\tau \Omega_2\big(\mathrm{q}(s)\big)\,ds\bigg] = 1. \tag{4.10}
$$

Step 2. Applying the Schwarz inequality in the form $E_L^{\mathrm{q}} B^{1/2} \le (E_L^{\mathrm{q}} AB)^{1/2} \cdot (E_L^{\mathrm{q}} A^{-1})^{1/2}$ to (4.10) yields

$$
\begin{aligned}
&E_L^{\mathrm{q}} \exp\bigg(-\tfrac{1}{2}\int_0^\tau \Omega_{11}\big(\mathrm{q}(s)\big) + \Omega_2\big(\mathrm{q}(s)\big)\,ds\bigg) \\
&\qquad \le \bigg(E_L^{\mathrm{q}} \exp\big(F(\mathrm{q}(0)) - F(\mathrm{q}(\tau))\big) + \int_0^\tau \Omega_{12}(\mathrm{q}(s))\,ds\bigg)^{1/2}.
\end{aligned} \tag{4.11}
$$

Since $|W_L| \le \frac{1}{2}$, $|R_L| \le 2$, for some constant C_2 the right-hand side of (4.11) is bounded above by

$$
\exp\bigg((2c\bar{v} + 16cT\bar{v}^2)\frac{N^2}{L}\bigg) \le \exp(C_2 cL) \tag{4.12}
$$

so long as $\tau \in [0, T]$. We would like to choose τ sufficiently small so that Ω_2 can only cancel at most one-half of Ω_{11}. For this we need to obtain a suitable upper bound on Ω_2.

Step 3. To bound Ω_2, first observe that $\|G\|_\infty \le N/2L = Z/2$, and as a result,

$$
\big|8c\big(G(x_i) - G(x_j)\big)(v_\gamma - v_{\alpha_i})\big| \le 16\bar{v}cZ. \tag{4.13}
$$

Further, the inequality $|A| \le B$ implies $|e^A - 1| \le e^B|A|$. Therefore, (4.13) implies

$$\left|\exp\left[8c(G(x_i) - G(x_j))(v_\gamma - v_{\alpha_i})\right] - 1\right| \le 16\bar{v}ce^{16\bar{v}cZ}\left|G(x_i) - G(x_j)\right|.$$

This in turn implies

$$\left|\Omega_2(\mathrm{q})\right| \le c_1 ce^{16\bar{v}cZ}\sum_{i,j} V\big(L(x_i - x_j)\big)\left|G(x_i) - G(x_j)\right|(v_{\alpha_i} - v_{\alpha_j})^2 \tag{4.14}$$

for some constant c_1. Notice that $V(L(x_i - x_j)) \ne 0$ only if $x_i \in (x_j - r_0/L, x_j + r_0/L)$. Since W'_L is bounded by two on the interval $[r_0/L, 1 - r_0/L]$, we have $|W_L(x_k - a) - W_L(x_k - b)| \le 2(r_0/L)$ whenever $b \in (a - r_0/L, a + r_0/L)$ and $x_k \notin (a - 2r_0/L, a + 2r_0/L)$. Therefore,

$$\begin{aligned}
\left|G(x_i) - G(x_j)\right| &\le \left|\frac{1}{2L}\sum_k\left[W\big(L(x_k - x_i + z)\big) - W\big(L(x_k - x_j + z)\big)\right]\right| \\
&\qquad + \left|\frac{1}{2L}\sum_k\left[W\big(L(x_k - x_i - z)\big) - W\big(L(x_k - x_j - z)\big)\right]\right| \\
&\le \frac{1}{L}\|W\|_\infty \mathscr{N}\left(x_i + z - \frac{2r_0}{L}, x_i + z + \frac{2r_0}{L}, \mathrm{q}\right) \\
&\qquad + \frac{1}{L}\|W\|_\infty \mathscr{N}\left(x_i - z - \frac{2r_0}{L}, x_i - z + \frac{2r_0}{L}, \mathrm{q}\right) + 2\frac{r_0}{L}\frac{N}{L} \\
&\le \frac{1}{L}\sup_a \mathscr{N}\left(a - \frac{2r_0}{L}, a + \frac{2r_0}{L}, \mathrm{q}\right) + \frac{2r_0 Z}{L},
\end{aligned}$$

whenever $V(L(x_i - x_j)) \ne 0$. Using this for (4.14) yields

$$\begin{aligned}
\left|\Omega_2(\mathrm{q})\right| &\le c_1 ce^{16\bar{v}cZ}\sum_{i,j} V\big(L(x_i - x_j)\big) \\
&\qquad \times \left[\frac{1}{L}\sup_a \mathscr{N}\left(a - \frac{2r_0}{L}, a + \frac{2r_0}{L}, \mathrm{q}\right) \right. \\
&\qquad\qquad \left. + \frac{2r_0 Z}{L}\right](v_{\alpha_i} - v_{\alpha_j})^2,
\end{aligned}$$

$$\begin{aligned}
\left|\Omega_2(\mathrm{q})\right| &\le c_1 ce^{16\bar{v}cZ}\left[\hat{C}_0 h\left(\frac{4r_0}{L}\right)\big(1 + \Phi(\mathrm{q})\big) + \frac{2r_0 Z}{L}\right] \\
&\qquad \times \sum_{i,j} V(L(x_i - x_j))(v_{\alpha_i} - v_{\alpha_j})^2,
\end{aligned}$$

where for the last inequality we applied (3.7). This, (4.11) and (4.12) imply

$$\begin{aligned} E_L^{\mathrm{q}} \exp\biggl\{ 2c \int_0^\tau \sum_{i,j} V\bigl(L(x_i - x_j + z)\bigr)(v_{\alpha_i} - v_{\beta_j})^2\, ds \\ - cC_2(c)\biggl[\frac{1}{L} \sup_a \sup_{0\le t\le T} \mathscr{N}\biggl(a - \frac{2r_0}{L}, a + \frac{2r_0}{L}, \mathrm{q}(\mathrm{t})\biggr) + \frac{r_0 Z}{L}\biggr] \\ \times \int_0^\tau A_L(\mathrm{q}(s))\, ds \biggr\} \\ \le \exp(C_2 cL), \end{aligned} \tag{4.15}$$

where $C_2(c)$ is a constant multiple of $e^{16\bar{v}cZ}$. In the case of $z = 0$ we may write

$$\begin{aligned} E_L^{\mathrm{q}} \exp\biggl\{\biggl[2c - cC_2(c) h\biggl(\frac{4r_0}{L}\biggr)\Bigl(1 + \sup_{0\le s\le T} \Phi(\mathrm{q}(s))\Bigr)\biggr] \int_0^\tau A_L(\mathrm{q}(s))\, ds\biggr\} \\ \le \exp(C_2 cL). \end{aligned} \tag{4.16}$$

Step 4. We choose η_1 so that for $c(L) = \eta_1 \log\log L$,

$$\lim_{L\to\infty} C_2\bigl(c(L)\bigr) h\biggl(\frac{4r_0}{L}\biggr) = 0. \tag{4.17}$$

We then define T_L by

$$T_L = \inf\biggl\{t \in [0, T]\colon C_2(c(L)) h\biggl(\frac{4r_0}{L}\biggr)\Bigl(1 + \sup_{0\le s\le t} \Phi(\mathrm{q}(t))\Bigr) \ge 1\biggr\}. \tag{4.18}$$

For such T_L, the inequality (4.16) implies (4.5). On the other hand, by the Chebyshev inequality,

$$\begin{aligned} &P_L(T_L \ne T) \\ &\quad = P_L\biggl(C_2(c(L)) h\biggl(\frac{4r_0}{L}\biggr)\biggl(1 + \sup_{0\le s\le T} \Phi(\mathrm{q}(s))\biggr) \ge 1\biggr) \\ &\quad \le E_L^{\mathrm{q}} \exp\Bigl(\theta_0 L\Bigl(1 + \sup_{0\le s\le t} \Phi(\mathrm{q}(s))\Bigr)\Bigr) \exp\biggl(-\frac{\theta_0 L}{C_2(c(L)) h(4r_0/L)}\biggr) \\ &\quad \le \exp\biggl(\theta_0 L + C_0(T) L - \frac{\theta_0 L}{C_2(c(L)) h(4r_0/L)}\biggr), \end{aligned} \tag{4.19}$$

where for the last inequality we used (3.3). Evidently, (4.19) and (4.17) imply (4.5).

Final Step. From (4.15) and the Schwarz inequality $E_L B^{1/2} \le (E_L AB)^{1/2} \cdot (E_L A^{-1})^{1/2}$, we obtain

$$\begin{aligned} &E_L \exp\biggl(c \int_0^{T_L} A_L(\mathrm{q}(s), z)\, ds\biggr) \\ &\quad \le \exp\biggl(\frac{1}{2} C_2 cL\biggr)\biggl(E_L \exp\biggl(cC_2(c)\biggl(\frac{N}{L} + \frac{r_0 Z}{L}\biggr) \int_0^t A_L(\mathrm{q}(s))\, ds\biggr)\biggr)^{1/2}. \end{aligned} \tag{4.20}$$

Now we apply (4.4) to conclude that if

$$cC_2(c)\left(Z + \frac{r_0 Z}{L}\right) \le \eta_1 \log\log L,$$

then the right-hand side of (4.20) is bounded above by the right-hand side of (4.6). This completes the proof of part (c). □

5. Molecular chaos. The main result of this section asserts that if ζ_ε is an approximation to identity, the empirical density $m_\alpha(t, dx)$ can be replaced with $m_\alpha * \zeta_\varepsilon$, with an error that has a *superexponentially* small probability distribution.

For the sake of definiteness, we pick a nonnegative smooth function ζ of compact support with $\int \zeta(z)\,dz = 1$, and we set $\zeta_\varepsilon(z) = \varepsilon^{-1}\zeta(\varepsilon^{-1}z)$. Define

$$m_{\alpha,\varepsilon}(x,t) = m_{\alpha,\varepsilon}(x,t;\mathsf{q}) = \frac{1}{L}\sum_{i=1}^{N} \zeta_\varepsilon\big(x - x_i(t)\big)\,\mathbb{1}\big(\alpha_i(t) = \alpha\big). \tag{5.1}$$

THEOREM 5.1. *For every pair (α, β) with $v_\alpha \neq v_\beta$, and any smooth function H,*

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{L\to\infty} \\
&\quad\times\Bigg\{\frac{1}{L}\log E_L \exp\Bigg[\int_0^{T_L} \sum_{i\neq j} V\big(L(x_i(t) - x_j(t))\big) \\
&\qquad\qquad\times \mathbb{1}\big(\alpha_i(t) = \alpha, \alpha_j(t) = \beta\big) H(x_i(t), t)\,dt \\
&\qquad\qquad - L\int_0^T\!\!\int m_{\alpha,\varepsilon}(x,t) m_{\beta,\varepsilon}(x,t) H(x,t)\,dx\,dt\Bigg]\Bigg\} \le 0.
\end{aligned} \tag{5.2}$$

The main ingredient for the proof of (5.2) is the following lemma.

LEMMA 5.2. *Let (α, β) and H be as in Theorem 5.1. Then there are constants C_4 and C_5 (depending only on T and H) such that for every L and ε,*

$$\begin{aligned}
&\sup_{|z_1|,|z_2|\le\varepsilon} E_L \exp\Bigg\{\int_0^{T_L} \sum_{i,j} V(L(x_i(t) - x_j(t) + z_1 - z_2)) H(x_i(t) + z_1, t) \\
&\qquad\qquad - V(L(x_i(t) - x_j(t))) H(x_i(t), t)\,dt\Bigg\} \\
&\quad\le C_4 \exp\big[C_5 L (h(L^{-1} + \varepsilon))^{1/2}\big].
\end{aligned} \tag{5.3}$$

We first demonstrate how (5.3) implies (5.2).

PROOF OF THEOREM 5.1. Without loss of generality we assume that the support of ζ is contained in $[-1, 1]$. As a result, the support of ζ_ε is contained in $[-\varepsilon, \varepsilon]$. We integrate the term with expectation in (5.3) against $\zeta_\varepsilon(z_1)\zeta_\varepsilon(z_2)$. If $X(z_1, z_2)$ denotes the exponent in (5.3), we have

$$
\begin{aligned}
&E_L \int\int \exp(X(z_1, z_2))\zeta_\varepsilon(z_1)\zeta_\varepsilon(z_2)\, dz_1\, dz_2 \\
&\quad \le C_4 \exp\bigl[C_5 L\bigl(h(L^{-1}+\varepsilon)\bigr)^{1/2}\bigr].
\end{aligned}
\tag{5.4}
$$

Write

$$
Y(t, \mathsf{q}) = \sum_{i\ne j} V\bigl(L(x_i - x_j)\bigr)H(x_i, t)\,\mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta).
\tag{5.5}
$$

Then by Jensen's inequality the right-hand side of (5.4) dominates

$$
\begin{aligned}
&E_L \exp\biggl(\int\int X(z_1, z_2)\zeta_\varepsilon(z_1)\zeta_\varepsilon(z_2)\, dz_2\, dz_2\biggr) \\
&\quad = E_L \exp\biggl\{\int_0^{T_L} \int\int \sum_{i,j} V\bigl(L(x_i(t) - x_j(t) + z_1 - z_2)\bigr) H\bigl(x_i(t) + z_1, t\bigr) \\
&\qquad\qquad \times \mathbb{1}\bigl(\alpha_i(t) = \alpha, \alpha_j(t) = \beta\bigr)\zeta_\varepsilon(z_1)\zeta_\varepsilon(z_2)\, dz_1\, dz_2\, dt \\
&\qquad\qquad - \int_0^{T_L} Y(t, \mathsf{q}(t))\, dt\biggr\} \\
&\quad = E_L \exp\biggl\{\int_0^{T_L} \sum_{i,j} \int\int V\bigl(L(z_1 - z_2)\bigr)H(z_1, t) \\
&\qquad\qquad \times \mathbb{1}\bigl(\alpha_i(t) = \alpha, \alpha_j(t) = \beta\bigr) \\
&\qquad\qquad \times \zeta_\varepsilon\bigl(z_1 - x_i(t)\bigr)\zeta_\varepsilon\bigl(z_2 - x_j(t)\bigr)\, dz_1\, dz_2\, dt \\
&\qquad\qquad - \int_0^{T_L} Y\bigl(t, \mathsf{q}(t)\bigr)\, dt\biggr\} \\
&\quad = E_L \exp\biggl\{L\int_0^{T_L} \int\int LV\bigl(L(z_1 - z_2)\bigr) \\
&\qquad\qquad \times H(z_1, t) m_{\alpha,\varepsilon}(z_1) m_{\beta,\varepsilon}(z_2)\, dz_1\, dz_2\, dt \\
&\qquad\qquad - \int_0^{T_L} Y(t, \mathsf{q}_t)\, dt\biggr\}.
\end{aligned}
\tag{5.6}
$$

Since (5.6) is bounded by the right-hand side of (5.4), we obtain

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{L\to\infty} \\
&\quad \times \biggl\{\frac{1}{L}\log E_L \exp\biggl\{L\int_0^{T_L}\int\int LV\bigl(L(z_1 - z_2)\bigr)H(z_1, t) \\
&\qquad\qquad \times m_{\alpha,\varepsilon}(z_1, t) m_{\beta,\varepsilon}(z_2, t)\, dz_1\, dz_2\, dt \\
&\qquad\qquad - \int_0^{T_L} Y\bigl(t, \mathsf{q}(t)\bigr)\, dt\biggr\}\biggr\} \le 0.
\end{aligned}
\tag{5.7}
$$

Since the function $H(z_1,t)m_{\alpha,\varepsilon}(z_1,t)m_{\beta,\varepsilon}(z_2,t)$ is continuous and uniformly bounded in a z_1-variable, we have

$$\begin{aligned}\biggl|\int\int LV(L(z_1-z_2))H(z_1,t)m_{\alpha,\varepsilon}(z_1,t)m_{\beta,\varepsilon}(z_2,t)\,dz_1\,dz_2\\ -\int H(z,t)m_{\alpha,\varepsilon}(z)m_{\beta,\varepsilon}(z)\,dz\biggr|\le a_L(\varepsilon),\end{aligned}\tag{5.8}$$

where $a_L(\varepsilon)$ is a nonrandom constant with $\lim_{L\to\infty}a_L(\varepsilon)=0$. This and (5.7) imply (5.2) because $L\to\infty$ before $\varepsilon\to 0$. $\square$

We end this section with the proof of (5.3). A review of (2.5)–(2.11) should motivate some of the steps of the following proof.

PROOF OF LEMMA 5.3.

Step 1. Recall the function W_L that was defined in the beginning of the previous section. Fix a pair (α,β) with $v_\alpha\ne v_\beta$ and a pair (z_1,z_2) with $|z_1|,|z_2|<\varepsilon$. Set $z_3=z_2-z_1$ and define $F(t,\mathrm{q})$ to be

$$\begin{aligned}\frac{1}{L}\sum_{i,j}\bigl[W_L(x_i-x_j+z_3)H(x_i+z_1,t)-W_L(x_i-x_j)H(x_i,t)\bigr]\\ \times\mathbb{1}(\alpha_i=\alpha,\alpha_j=\beta).\end{aligned}\tag{5.9}$$

Using exponential martingales, we obtain

$$\begin{aligned}E_L\exp\biggl(F\bigl(T_L,\mathrm{q}(T_L)\bigr)-F\bigl(0,\mathrm{q}(0)\bigr)-\int_0^{T_L}\frac{\partial F}{\partial t}\bigl(t,\mathrm{q}(t)\bigr)\,dt\\ -\int_0^{T_L}e^{-F}\mathscr{A}^{(L)}e^F\bigl(t,\mathrm{q}(t)\bigr)\,dt\biggr)=1.\end{aligned}\tag{5.10}$$

In the succeeding steps we will study various terms that appeared in the exponent. In the sequel, $c,c_1,c_2\dots$ denote constants whose values may change from line to line.

Step 2. We start with $\Omega_1=F(T_L,\mathrm{q}(T_L))=\Omega_{11}+\Omega_{12}$, where

$$\begin{aligned}\Omega_{11}&=\frac{1}{L}\sum_{i,j}W_L(x_i-x_j+z_3)\bigl(H(x_i+z_1,T_L)-H(x_i,T_L)\bigr)\\ &\qquad\times\mathbb{1}(\alpha_i=\alpha,\alpha_j=\beta),\\ \Omega_{12}&=\frac{1}{L}\sum_{i,j}\bigl(W_L(x_i-x_j+z_3)-W_L(x_i-x_j)\bigr)H\bigl(x_i,T_L\bigr)\\ &\qquad\times\mathbb{1}(\alpha_i=\alpha,\alpha_j=\beta).\end{aligned}\tag{5.11}$$

By smoothness of H, we clearly have

$$|\Omega_{11}|\le c_1\frac{N^2}{L}|z_1|\le c_1LZ^2\varepsilon.\tag{5.12}$$

Recall that the support of V is contained in $[-r_0, r_0]$ and that $\mathbb{T}$ is the interval $[-\frac{1}{2}, \frac{1}{2}]$ with the endpoints identified. Recall that W'_L is bounded by two, outside the interval $[-r_0/L, r_0/L]$. We then write

$$\Omega_{12} = \Omega_{121} + \Omega_{122} + \Omega_{123}, \tag{5.13}$$

where Ω_{121} is obtained from Ω_{12} by restricting the summation to i, j with either $L(x_i - x_j) \in [-r_0, r_0]$ or $L(x_i - x_j + z_3) \in [-r_0, r_0]$, Ω_{122} is obtained by restricting the summation to the case of either $L(x_i - x_j), L(x_i - x_j + z_3) \leq -r_0$ or $L(x_i - x_j), L(x_i - x_j + z_3) \geq r_0$ and Ω_{123} is obtained by restricting the summation to the remaining cases.

Step 3. If either $L(x_i - x_j)$ or $L(x_i - x_j + z_3)$ belongs to $[-r_0, r_0]$, then either $x_i \in [x_j - r_0/L, x_j + r_0/L]$ or $x_i \in [x_j - z_3 - r_0/L, x_j - z_3 + r_0/L]$. As a result,

$$|\Omega_{121}| \leq Z\|W_L\|_\infty \|H\|_\infty \sup_a \mathscr{N}\left(a - \frac{r_0}{L}, a + \frac{r_0}{L}, \mathrm{q}(T_L)\right). \tag{5.14}$$

We now turn to Ω_{122}. If both $L(x_i - x_j)$ and $L(x_i - x_j + z_3)$ are outside $[-r_0, r_0]$ and on the same side, then $|W_L(x_i - x_j + z_3) - W_L(x_i - x_j)| \leq 2|z_3|$. Hence

$$|\Omega_{122}| \leq 2\varepsilon \|H\|_\infty \frac{N^2}{L} = 4\varepsilon Z^2 \|H\|_\infty L. \tag{5.15}$$

For Ω_{123}, assume for example $L(x_i - x_j) < -r_0$ but $L(x_i - x_j + z_3) > r_0$. This implies $z_3 > 2r_0/L$ and

$$x_i \in \left(x_j + \frac{r_0}{L} - z_3, x_j - \frac{r_0}{L}\right).$$

Hence

$$|\Omega_{123}| \leq 2\|W_L\|_\infty \|H\|_\infty \sup_a \mathscr{N}\left(a, a + 2\varepsilon - \frac{2r_0}{L}, \mathrm{q}(T_L)\right). \tag{5.16}$$

Combining (5.13)–(5.16) yields

$$|\Omega_{12}| \leq c \sup_a \mathscr{N}\left(a, a + 2\varepsilon + \frac{2r_0}{L}, \mathrm{q}(T_L)\right)$$

for some constant c. From this and (5.12) we conclude

$$|\Omega_1| \leq cL\varepsilon + c \sup_a \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(T_L)\right)$$

for some constant c. The term $F(0, \mathrm{q}(0))$ can be treated likewise. Thus

$$\begin{aligned} &\left|F(T_L, \mathrm{q}(T_L)) - F(0, \mathrm{q}(0))\right| \\ &\qquad \leq cL\varepsilon + c \sup_a \sup_{0 \leq t \leq T} \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\right). \end{aligned} \tag{5.17}$$

Step 4. Next we study another term in the exponent of (5.10):

$$\begin{aligned}\frac{\partial F}{\partial t}(t,\mathrm{q}) = \frac{1}{L}\sum_{i,j}\Big[&W_L(x_i - x_j + z_3)\frac{\partial H}{\partial t}(x_i + z_1, t)\\ &- W_L(x_i - x_j)\frac{\partial H}{\partial t}(x_i, t)\Big]\\ &\times \mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta).\end{aligned}\tag{5.18}$$

This term can be treated in the same way we established (5.17). The only difference is that H is replaced with $\partial H/\partial t$. As a result

$$\begin{aligned}&\left|\int_0^{T_L} \frac{\partial F}{\partial t}\,(t, \mathrm{q}(t))\,dt\right|\\ &\quad\le cL\varepsilon + c\sup_a \sup_{0\le t\le T} \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\right)\end{aligned}\tag{5.19}$$

for some constant c.

Step 5. For the last integral in (5.10), we write

$$e^{-F}\mathscr{A}^{(L)}e^F = \mathscr{A}_0 F + e^{-F}\mathscr{A}_c e^F =: \Omega_2 + \Omega_3.\tag{5.20}$$

A straightforward calculation yields that $\Omega_2 = \mathscr{A}_0 F$ is equal to

$$\begin{aligned}&(v_\alpha - v_\beta)\sum_{i,j}\big[V(L(x_i - x_j + z_3))H(x_i + z_1, t)\\ &\qquad\qquad - V(L(x_i - x_j))H(x_i, t)\big]\mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta)\\ &\quad + \frac{v_\alpha - v_\beta}{L}\sum_{i,j}\big[R_L(x_i - x_j + z_3)H(x_i + z_1, t)\\ &\qquad\qquad - R_L(x_i - x_j)H(x_i, t)\big]\mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta)\\ &\quad + \frac{v_\alpha}{L}\sum_{i,j}\Big[W_L(x_i - x_j + z_3)\frac{\partial H}{\partial x}(x_i + z_1, t)\\ &\qquad\qquad - W_L(x_i - x_j)\frac{\partial H}{\partial x}(x_i, t)\Big]\mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta)\\ &=: \Omega_{21} + \Omega_{22} + \Omega_{23},\end{aligned}\tag{5.21}$$

where R_L is defined by (4.1). Note that Ω_{21} is a constant multiple of the integrand in (5.3). The term Ω_{23} is nothing other than (5.18) with $\partial H/\partial x$ instead of $\partial H/\partial t$. Hence we can bound it as in (5.19):

$$\left|\int_0^{T_L}\Omega_{23}\,dt\right| \le cL\varepsilon + \sup_a \sup_{0\le t\le T}\mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\right).\tag{5.22}$$

On the other hand, since $|R_L'| \le 3$ and H is smooth, it is not hard to show that

$$\left|\int_0^{T_L}\Omega_{22}\,dt\right| \le cL\varepsilon.\tag{5.23}$$

Step 6. We now turn to $e^{-F}\mathscr{A}_c e^F$:

$$\begin{aligned}\Omega_3 &= e^{-F}\mathscr{A}_c e^F(t,\mathrm{q})\\ &= \tfrac{1}{2}\sum_{i,j}\sum_{\gamma\delta} K(\alpha_i\alpha_j,\gamma\delta)V\big(L(x_i-x_j)\big)\\ &\qquad\times\big[\exp\big(F(t,S_{ij}^{\gamma\delta}\mathrm{q})-F(t,\mathrm{q})\big)-1\big],\end{aligned}\tag{5.24}$$

where

$$\begin{aligned}&F\big(t,S_{ij}^{\gamma\delta}\mathrm{q}\big)-F(t,\mathrm{q})\\ &\quad=\frac{1}{L}\big[W_L(x_i-x_j+z_3)H(x_i+z_1,t)-W_L(x_i-x_j)H(x_i,t)\big]\\ &\qquad\times\big(\mathbb{1}(\gamma=\alpha,\delta=\beta)-\mathbb{1}(\alpha_i=\alpha,\alpha_j=\beta)\big)\\ &\qquad+\frac{1}{L}\sum_{k\neq i,j}\big[W_L(x_i-x_k+z_3)H(x_i+z_1,t)\\ &\qquad\qquad-W_L(x_i-x_k)H(x_i,t)\big]\\ &\qquad\qquad\times\big(\mathbb{1}(\gamma=\alpha,\alpha_k=\beta)-\mathbb{1}(\alpha_i=\alpha,\alpha_k=\beta)\big)\\ &\qquad+\frac{1}{L}\sum_{k\neq i,j}\big[W_L(x_k-x_j+z_3)H(x_k+z_1,t)\\ &\qquad\qquad-W_L(x_k-x_j)H(x_k,t)\big]\\ &\qquad\qquad\times\big(\mathbb{1}(\alpha_k=\alpha,\delta=\beta)-\mathbb{1}(\alpha_k=\alpha,\alpha_j=\beta)\big)\\ &\quad=R_1+R_2+R_3.\end{aligned}\tag{5.25}$$

First observe that the right-hand side of (5.25) is bounded:

$$\big|F(t,S_{ij}^{\gamma\delta}\mathrm{q})-F(t,\mathrm{q})\big|\le 3\frac{N}{L}\|H\|_\infty=3Z\|H\|_\infty.$$

Since the inequality $|A|\le B$ implies $|e^A-1|\le e^B|A|$, we have

$$\begin{aligned}|\Omega_3|&\le\tfrac{1}{2}\sum_{i\neq j}\sum_{\gamma\delta}K\big(\alpha_i\alpha_j,\gamma\delta\big)V\big(L(x_i-x_j)\big)\exp\big(3Z\|H\|_\infty\big)\\ &\qquad\times\big|F\big(t,S_{ij}^{\gamma\delta}\mathrm{q}\big)-F(t,\mathrm{q})\big|\\ &\le\Omega_{31}+\Omega_{32}+\Omega_{33},\end{aligned}\tag{5.26}$$

where Ω_{3i} is obtained by replacing the term with absolute values with $|R_i|$. Since $|R_1|\le(\|H\|_\infty/L)$, we have

$$\Omega_{31}\le\frac{c}{L}A_L(\mathrm{q}),\tag{5.27}$$

where $A_L(\mathrm{q})$ is defined by (4.2). We now turn to Ω_{32}:

$$\begin{aligned}
\Omega_{32} \le\ & \frac{c}{L}\sum_{i,\,j} V\big(L(x_i - x_j)\big)\mathbb{1}(v_{\alpha_i} \ne v_{\alpha_j}) \\
& \times \Bigg| \sum_{k\ne i,\,j} \big[W_L(x_i - x_k + z_3)H(x_i + z_1, t) \\
& \qquad - W_L(x_i - x_k)H(x_i, t)\big]\Bigg| \\
\le\ & \frac{c}{L}\sum_{i,\,j,\,k} V\big(L(x_i - x_j)\big)\mathbb{1}(v_{\alpha_i} \ne v_{\alpha_j})\big|W_L(x_i - x_k + z_3)\big| \\
& \times \big|H(x_i + z_2, t) - H(x_i, t)\big| \\
& + \frac{c}{L}\sum_{i,\,j,\,k} V\big(L(x_i - x_j)\big)\mathbb{1}(v_{\alpha_i} \ne v_{\alpha_j}) \\
& \times \big|W_L(x_i - x_k + z_3) - W_L(x_i - x_k)\big|\,\big|H(x_i, t)\big| \\
=:\ & \Omega_{321} + \Omega_{322}.
\end{aligned} \tag{5.28}$$

By smoothness of H,

$$\begin{aligned}
\Omega_{321} &\le \frac{c_1\varepsilon}{L}\sum_{i,\,j,\,k} V(L(x_i - x_j))\mathbb{1}(v_{\alpha_i} \ne v_{\alpha_j}) \\
&= \frac{c_1\varepsilon N}{L}\sum_{i,\,j} V\big(L(x_i - x_j)\big)\mathbb{1}(v_{\alpha_i} \ne v_{\alpha_j}) \\
&\le c_2\varepsilon A_L(\mathrm{q}),
\end{aligned} \tag{5.29}$$

for some constants c_1 and c_2.

Step 7. To treat Ω_{322}, we argue as in Steps 2 and 3. We first write

$$\Omega_{322} = \Omega_{3221} + \Omega_{3222} + \Omega_{3223}, \tag{5.30}$$

where Ω_{3221} is obtained from restricting the summation to $i,\ j,\ k$ with $L(x_i - x_k) \in [-r_0, r_0]$ or $L(x_i - x_k + z_3) \in [-r_0, r_0]$, Ω_{3222} is obtained by restricting to the case of either $L(x_i - x_k)$, $L(x_i - x_k + z_3) < -r_0$ or $L(x_i - x_k)$, $L(x_i - x_k + z_3) > r_0$, and Ω_{3223} is obtained by restricting to the case $L(x_i - x_k)$, $L(x_i - x_k + z_3) \notin [-r_0, r_0]$ but one is less than $-r_0$ and the other greater than r_0. As in (5.15) we can show

$$|\Omega_{3222}| \le c\varepsilon \frac{N}{L} A_L(\mathrm{q}). \tag{5.31}$$

By taking the summation over k first, we can estimate Ω_{3221} as

$$\begin{aligned}\Omega_{3221} &\le \frac{2c}{L}\|W\|_\infty \sum_{i,j} V\big(L(x_i - x_j)\big)\,\mathbb{1}(v_{\alpha_i} \ne v_{\alpha_j})\\ &\qquad\times \mathscr{N}\left(x_j - \frac{r_0}{L} - \varepsilon, x_j + \frac{r_0}{L} + \varepsilon, \mathrm{q}\right)\\ &\le \frac{c_1}{L} A_L(\mathrm{q}) \sup_a \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}\right)\end{aligned} \tag{5.32}$$

for some constant c_1. We can repeat the derivation of (5.16) for Ω_{3223} to obtain

$$\Omega_{3223} \le \frac{c}{L} A_L(\mathrm{q}) \sup_a \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}\right). \tag{5.33}$$

Combining (5.30), (5.31), (5.32) and (5.33) yields

$$\Omega_{322} \le \frac{c_1}{L} A_L(\mathrm{q}) \sup_a \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}\right) + c_1 \varepsilon A_L(\mathrm{q})$$

for some constant c_1. This, (5.29) and (5.28) imply

$$\Omega_{32} \le \frac{c}{L} A_L(\mathrm{q}) \sup_a \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}\right) + c\varepsilon A_L(\mathrm{q}). \tag{5.34}$$

The term Ω_{33} is treated likewise. This, (5.34), (5.26) and (5.27) yield

$$|\Omega_3| \le c A_L(\mathrm{q}) \left(\sup_a \frac{1}{L} \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}\right) + \varepsilon + L^{-1}\right) \tag{5.35}$$

for some constant c. This and (5.24) imply

$$\begin{aligned}&\left|\int_0^{T_L} e^{-F} \mathscr{A}_c e^F(t, \mathrm{q}_t)\, dt\right|\\ &\quad\le \left(\int_0^{T_L} A_L(\mathrm{q})\, dt\right)\\ &\qquad\times \left(\sup_{0\le t\le T} \sup_a \frac{1}{L} \mathscr{N}\left(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\right) + \varepsilon + L^{-1}\right).\end{aligned} \tag{5.36}$$

Final step. From (5.10), (5.17), (5.19), (5.21), (5.22), (5.23) and (5.36), we conclude $E_L e^{X+Y} = 1$ with X equal to $v_\beta - v_\alpha$ times

$$\begin{aligned}&\int_0^{T_L} \sum_{i,j} \big[V(L(x_i - x_j + z_1))H(x_i + z_2, t) - V(L(x_i - x_j))H(x_i, t)\big]\\ &\qquad\times \mathbb{1}(\alpha_i(t) = \alpha, \alpha_j(t) = \beta)\, dt,\end{aligned} \tag{5.37}$$

$$
\begin{aligned}
|Y| \le R := {} & c_1\Big(L + \int_0^{T_L} A_L(\mathrm{q})\,dt\Big) \\
& \times \Big(\sup_{0\le t\le T}\sup_a \frac{1}{L}\mathscr{N}\Big(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\Big) + \varepsilon + L^{-1}\Big) \\
& + c_1\varepsilon L
\end{aligned}
\tag{5.38}
$$

for some constant c_1. We use the Schwarz inequality in the form $E_L e^{X/2} \le (E_L e^{X+Y})^{1/2}(Ee^{-Y})^{1/2}$ to deduce

$$
E_L e^{X/2} \le (E_L e^R)^{1/2}. \tag{5.39}
$$

We then replace H with $-H$ to deduce

$$
E_L e^{-X/2} \le (E_L e^R)^{1/2}.
$$

From this, (5.39) and the elementary inequality $e^{|x|} \le e^x + e^{-x}$ we deduce

$$
E_L e^{|X/2|} \le 2(E_L e^R)^{1/2}. \tag{5.40}
$$

Recall that by (3.7) and (3.4),

$$
\begin{aligned}
& E_L \exp\Big(c_1 \sup_a \sup_{0\le t\le T} \mathscr{N}\Big(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\Big)\Big) \\
& \quad \le E_L \exp\Big[\hat{C}_0 L h\Big(\frac{2r_0}{L} + 2\varepsilon\Big)\Big(1 + \sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Big)\Big] \\
& \quad \le \exp\Big(\hat{C}_0 L h\Big(\frac{2r_0}{L} + 2\varepsilon\Big)\Big)\exp\Big(\frac{\hat{C}_0}{\theta_0} L h\Big(\frac{2r_0}{L} + 2\varepsilon\Big) C_0(T)\Big).
\end{aligned}
$$

Because of this, (5.40), (5.38) and (5.37), the (5.5) follows if we can show

$$
\begin{aligned}
& E_L \exp\Big[c_1 \sup\nolimits_a \sup\nolimits_{0\le t\le T} \tfrac{1}{L}\mathscr{N}\Big(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\Big) \\
& \qquad\qquad \times \int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big] \\
& \quad \le \exp\big[cL(h(L^{-1} + \varepsilon))^{1/2}\big],
\end{aligned}
\tag{5.41}
$$

for some constant c. For every λ, the left-hand side of (5.41) is bounded above by

$$
\begin{aligned}
& E_L \exp\Big(c_1\lambda \int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big) \\
& \quad + E_L \exp\Big(c_1 \frac{N}{L}\int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big) \\
& \qquad \times \mathbb{1}\Big(\frac{1}{L}\sup_a \sup_{0\le t\le T} \mathscr{N}\Big(a, a + \frac{2r_0}{L} + 2\varepsilon, \mathrm{q}(t)\Big) \ge \lambda\Big)
\end{aligned}
\tag{5.42}
$$

because $\mathscr{N}$ is always bounded by N. The first term in (5.42) is bounded above by $\exp(C_1(c_1\lambda, T)L)$ by Theorem 4.1. By Lemma 3.3, the second term in (5.42) is bounded by

$$\begin{aligned}
&E_L \exp\biggl(c_1 Z \int_0^{T_L} A_L(\mathrm{q}(t)\,dt)\biggr)\\
&\qquad\times \mathbb{1}\biggl(\frac{\hat{C}_0}{L} h\biggl(\frac{2r_0}{L}+2\varepsilon\biggr)\Bigl(1+\sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Bigr)\ge\lambda\biggr)\\
&\quad\le\biggl(E_L\exp\biggl(2c_1 Z\int_0^{T_L} A_L(\mathrm{q}(t))\,dt\biggr)\biggr)^{1/2}\\
&\qquad\quad\times P_L\biggl(\frac{\hat{C}_0}{L} h\biggl(\frac{2r_0}{L}+2\varepsilon\biggr)\Bigl(1+\sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Bigr)\ge\lambda\biggr)^{1/2}\\
&\qquad\le\exp\bigl(\tfrac{1}{2}C_1(2c_1 Z, T)L\bigr)\\
&\qquad\quad\times\exp(-\lambda M L)E_L\exp\biggl[M\hat{C}_0 h\biggl(\frac{2r_0}{L}+2\varepsilon\biggr)\biggl(1+\sup_{0\le t\le T}\Phi(\mathrm{q}(t))\biggr)\biggr]
\end{aligned}$$

for every positive constant M, where for the first inequality we used the Schwartz inequality, and for the second inequality we applied (4.4) and then the Chebyshev inequality. We now choose $\lambda = (h(2r_0/L+2\varepsilon))^{1/2}$ and $M = \frac{1}{2}C_1(2c_1 Z, T)(h(2r_0/L+2\varepsilon))^{-1/2}$ and apply (3.4). As a result we have (5.40) and this completes the proof of the theorem. □

6. Uniform integrability, part I.

In this section we establish an exponential bound for the collision term that is slightly stronger than (4.4).

We start with some definitions. Fix a number $b\in(0,1)$:

$$\Gamma_3(x) := x(w_3(x))^b =: \begin{cases} x(\log\log\log x)^b, & x\ge e^e,\\ 0, & x<e^e,\end{cases} \tag{6.1}$$

$$\begin{aligned} X_t(x) := \int_0^t \sum_{i\ne j} & V\bigl(L(x_i(s)-x_j(s))\bigr)\\ &\times\mathbb{1}(\alpha_i(s)=\alpha,\alpha_j(s)=\beta)V(L(x_i(s)-x-vs))\,ds,\end{aligned} \tag{6.2}$$

$$\tau(x)=\tau_l(x) := \inf\{t\colon X_t(x)\ge l\},\qquad \sigma(x)=\sigma_l(x):=\tau(x)\wedge T_L, \tag{6.3}$$

where $v\in\mathbb{R}$, and $l>0$ are given. Throughout this section we assume v, v_α, v_β are distinct.

The following is the main result of this section.

THEOREM 6.1. *There exists a positive constant $\eta_2=\eta_2(b)$ such that for every positive T,*

$$C_6(T) := \sup_L \frac{1}{L}\log E_L\exp\biggl[\eta_2 L\int\Gamma_3(X_{T_L}(x))\,dx\biggr]<\infty. \tag{6.4}$$

The following lemma is the main ingredient for the proof of (6.4).

LEMMA 6.2. *There exist positive constants η_3 and $C_6'(T)$ such that for every positive l and L with $l \le \exp((\log L)^2)$,*

$$\frac{1}{L}\log E_L \exp\Big[2\eta_3 L w_3(l)\int (X_{T_L}(x) - X_{\sigma_l(x)}(x))\,dx\Big] \le C_6'(T). \tag{6.5}$$

We first demonstrate how (6.5) implies (6.4).

PROOF OF THEOREM 6.1. First observe that (6.5) means

$$\frac{1}{L}\log E_L \exp\Big[2\eta_3 L w_3(l)\int (X_{T_L}(x) - l)\,\mathbb{1}(\tau_l(x) \le T_L)\,dx\Big] \le C_6'(T). \tag{6.6}$$

In particular,

$$\frac{1}{L}\log E_L \exp\Big[2\eta_3 L w_3(l)\int (X_{T_L}(x) - l)\,\mathbb{1}(X_{T_L}(x) \ge 2l)\,dx\Big] \le C_6'(T),$$

which in turn implies

$$\frac{1}{L}\log E_L \exp\Big[2\eta_3 L l w_3(l)\int \mathbb{1}\big(X_{T_L}(x) \ge 2l\big)\,dx\Big] \le C_6'(T). \tag{6.7}$$

Moreover,

$$\begin{aligned}
&\frac{1}{L}\log E_L \exp\Big[\eta_3 L w_3(l)\int X_{T_L}(x)\,\mathbb{1}1(X_{T_L}(x) \ge l)\,dx\Big]\\
&\quad= \frac{1}{L}\log E_L \exp\Big[\eta_3 L w_3(l)\int (X_{T_L}(x) - l)\,\mathbb{1}(X_{T_L}(x) \ge l)\,dx\\
&\qquad\qquad + \eta_3 L l w_3(l)\int \mathbb{1}(X_{T_L}(x) \ge l)\,dx\Big]\\
&\quad\le \frac{1}{L}\log\Big\{\frac{1}{2}E_L \exp\Big[2\eta_3 L w_3(l)\int (X_{T_L}(x) - l)\,\mathbb{1}(X_{T_L}(x) \ge l)\,dx\Big]\\
&\qquad\qquad + \frac{1}{2}E_L \exp\Big[2\eta_3 L l w_3(l)\int \mathbb{1}(X_{T_L}(x) \ge l)\,dx\Big]\Big\}\\
&\quad\le C_6'(T),
\end{aligned} \tag{6.8}$$

where for the first inequality we used $e^{a+b} \le \frac{1}{2}e^{2a} + \frac{1}{2}e^{2b}$, and for the second inequality we used (6.7) and (6.6).

Put $\hat{w}_3 = w_3^b$. First observe that since X_{T_L} is bounded above by a constant multiple of L^2, there exists L_0 such that $X_{T_L} \le \exp((\log L)^2)$ for $L \ge L_0$. For $X \ge e^e$,

$$\Gamma_3(X) = \big(\hat{w}_3(X) - \hat{w}_3(e^e)\big)X = \int_{e^e}^{\infty} \hat{w}_3'(l) X\,\mathbb{1}(X \ge l)\,dl.$$

From this we deduce that if $L \geq L_0$,

$$\begin{aligned}
&\frac{1}{L}\log E_L \exp\Big[\eta_2 L \int \Gamma_3(X_{T_L}(x))\,dx\Big] \\
&\qquad = \frac{1}{L}\log E_L \exp\Big[\eta_2 L \int_{e^e}^{\exp((\log L)^2)} \int w_3(l) X_{T_L}(x) \\
&\qquad\qquad \times \mathbb{1}(X_{T_L}(x) \geq l) y(l)\,dx\,dl\Big],
\end{aligned} \tag{6.9}$$

where $y(l) = bw_3(l)^{b-2}(\log\log l)^{-1}(\log l)^{-1}l^{-1}$. Choose $\eta_2 = \eta_3/c_0$ where $c_0 = \int_{e^e}^{\infty} y(l)\,dl$. From Jensen's inequality and (6.8) we conclude that the right-hand side of (6.9) is bounded above by

$$\begin{aligned}
&\frac{1}{L}\log E_L \int_{e^e}^{\exp((\log L)^2)} \exp\Big[\eta_2 c_1 L w_3(l) \int X_{T_L}(x)\mathbb{1}(X_{T_L}(x) \geq l)\,dx\Big] \\
&\qquad \times \frac{y(l)}{c_1}\,dl \leq C_6'(T),
\end{aligned} \tag{6.10}$$

where $c_1 = \int_{e^e}^{\exp((\log L)^2)} y(l)\,dl$. This evidently completes the proof of (6.4). □

In the sequel, $\mathscr{F}_\lambda$ denotes the σ-field generated by $(\mathrm{q}(s)\colon 0 \leq s \leq \lambda)$. Define

$$g(x,t,\mathrm{q}) = \sum_{i,j} V(L(x_i - x_j))\mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta)V(L(x_i - x - vt)). \tag{6.11}$$

Note that we can write

$$\int (X_{T_L}(x) - X_{\sigma(x)}(x))\,dx = \int\int_{\sigma(x)}^{T_L} g(x,t,\mathrm{q}(t))\,dt\,dx. \tag{6.12}$$

PROOF OF LEMMA 6.2.

Step 1. The expression (2.14) suggests looking at the exponential martingale corresponding to the process

$$\sum_{i,j} W_L(x_i - x_j) H\bigg(\frac{v_\beta - v}{v_\beta - v_\alpha}x_i + \frac{v - v_\alpha}{v_\beta - v_\alpha}x_j - vt\bigg)\mathbb{1}(\alpha_i = \alpha,\ \alpha_j = \beta),$$

where H is the indicator function of the set $\{\tau_l > T_L\}$. The randomness of the set and the nonsmoothness of the indicator function will not allow us to choose such H. Because of this, we introduce several approximation procedures. Let ε be a positive constant such that $k_0 = \varepsilon^{-1}$ is an integer and put $\lambda_0 = 0$. We divide the interval $[0, T)$ into smaller subintervals $[\lambda_k, \lambda_{k+1})$, each of which is of length εT. Put

$$\begin{aligned}
H(x,k,\varepsilon) &= \mathbb{1}\big(\tau_l(x) \in [\lambda_k, \lambda_{k+1})\big), \\
H_L(x,k,\varepsilon) &= \int \zeta_L(x - z) H(z,k,\varepsilon)\,dz,
\end{aligned} \tag{6.13}$$

where $z_L(z) = L\zeta(Lz)$ and $\zeta =: \mathbb{R} \to [0, \infty)$ is a smooth function of compact support and will be chosen later. We fix ε and for each integer $k \in [0, k_0)$ we define $F_k(\mathrm{q}, t) = F_k(\mathrm{q}, t, \varepsilon)$ to be

$$\frac{\bar{c}}{L(v_\beta - v_\alpha)} \sum_{i,j} \hat{W}_L(x_i - x_j) H_L \times \left(\frac{v_\beta - v}{v_\beta - v_\alpha} x_i + \frac{v - v_\alpha}{v_\beta - v_\alpha} x_j - vt, k, \varepsilon \right) \mathbb{1}(\alpha_i = \alpha, \ \alpha_j = \beta),$$

where $\hat{W}_L = W_L - 1$ if $v_\beta - v_\alpha > 0$, $\hat{W}_L = W_L + 1$ if $v_\beta - v_\alpha < 0$ and $\bar{c}$ is a positive constant. We also define

$$G(\mathrm{q}, t) = G(\mathrm{q}, t, \varepsilon) = \sum_{k=0}^{k_0-2} F_k(\mathrm{q}, t) \mathbb{1}(t \geq \lambda_{k+1}). \tag{6.14}$$

Note that our choice of $\hat{W}_L$ implies

$$-\frac{2\bar{c}}{|v_\alpha - v_\beta|} \leq \frac{\bar{c}}{v_\beta - v_\alpha} \hat{W}_L \leq 0 \quad \text{and} \quad F_k(\mathrm{q}, t) \leq 0. \tag{6.15}$$

Consider the process

$$\begin{aligned} M(t) = \exp\Bigg[& G(\mathrm{q}(t), t) - \sum_{k=0}^{k_0-2} F_k(\mathrm{q}(\lambda_{k+1}), \lambda_{k+1}) \mathbb{1}(t \geq \lambda_{k+1}) \\ & - \sum_{k=0}^{k_0-2} \mathbb{1}(t \geq \lambda_{k+1}) \int_{\lambda_{k+1}}^t \left(\frac{\partial F_k}{\partial t} + \mathscr{A}_0 F_k \right)(\mathrm{q}(s), s)\, ds \\ & - \int_0^t e^{-G} \mathscr{A}_c e^G(\mathrm{q}(s), s)\, ds \Bigg]. \end{aligned}$$

We claim

$$E_L M(T_L) = 1. \tag{6.16}$$

To see this, we first define the following process for $t \in [\lambda_{k+1}, T]$:

$$\begin{aligned} M_k(t) = \exp\Bigg[& G(\mathrm{q}(t), t) - G(\mathrm{q}(\lambda_{k+1}), \lambda_{k+1}) \\ & - \int_{\lambda_{k+1}}^t \left(\frac{\partial G}{\partial t} + \mathscr{A}_0 G \right)(\mathrm{q}(s), s)\, ds - \int_{\lambda_{k+1}}^t e^{-G} \mathscr{A}_c e^G(\mathrm{q}(s), s)\, ds \Bigg]. \end{aligned}$$

Since $H_L(x,k,\varepsilon)$ is measurable with respect to the σ-field $\mathscr{F}_{\lambda_{k+1}}$, the process $M_k(t)$ is an exponential martingale. As a result,

$$\begin{aligned}
E_L M(T_L) &= E_L E_L\big[M(T_L)|\mathscr{F}_{\lambda_{k_0-1}}\big]\\
&= E_L \exp\bigg(G\big(\mathrm{q}(\lambda_{k_0-1},\lambda_{k_0-1})\big) - \sum_{k=0}^{k_0-3} F_k\big(\mathrm{q}(\lambda_{k+1}),\lambda_{k+1}\big)\\
&\qquad + \sum_{k=0}^{k_0-3}\int_{\lambda_{k+1}}^{\lambda_{k_0-1}}\bigg(\frac{\partial F_k}{\partial t}+\mathscr{A}_0 F_k\bigg)(\mathrm{q}(s),s)\,ds\\
&\qquad - \int_0^{\lambda_{k_0-1}} e^{-G}\mathscr{A}_c e^{G}(\mathrm{q}(s),s)\,ds\bigg)\\
&\qquad \times \mathbb{1}(T_L>\lambda_{k_0-1})E_L\big[M_{k_0-2}(T)|\mathscr{F}_{\lambda_{k_0-1}}\big]\\
&\quad + E_L M(T_L)\mathbb{1}(T_L\le\lambda_{k_0-1})\\
&= E_L M(T_L\wedge\lambda_{k_0-1}),
\end{aligned}$$

where the last equality follows from the fact that we replace $E_L[M_{k_0-2}(T_L)|\mathscr{F}_{\lambda_{k_0-1}}]$ with 1 because M_k is an exponential martingale with $M_k(\lambda_{k+1})=1$. Inductively we can show

$$E_L M(T_L) = E_L M(T_L\wedge\lambda_k).$$

From this we can deduce (6.16) because $M(0)=1$.

Step 2. Put

$$\Omega_1(\varepsilon) = G(\mathrm{q}(T_L),T_L) - \sum_0^{k_0-2} F_k\big(\mathrm{q}(\lambda_{k+1}),\lambda_{k+1}\big)\mathbb{1}(T_L\ge\lambda_{k+1}),$$

$$\Omega_2(\mathrm{q},t,k,\varepsilon) = \bigg(\frac{\partial F_k}{\partial t}+\mathscr{A}_0 F_k\bigg)(\mathrm{q},t), \tag{6.17}$$

$$\Omega_3(\mathrm{q},t,\varepsilon) = e^{-G}\mathscr{A}_c e^{G}(\mathrm{q},t).$$

Using the decomposition (4.1), we have

$$\Omega_2(\mathrm{q},t,k,\varepsilon) = \Omega_{21}(\mathrm{q},t,k,\varepsilon)+\Omega_{22}(\mathrm{q},t,k,\varepsilon),$$

$$\begin{aligned}
\Omega_{21}(\mathrm{q},t,k,\varepsilon) &:= -\bar{c}\sum_{i,j} V(L(x_i-x_j))\\
&\qquad \times H_L\bigg(\frac{v_\beta-v}{v_\beta-v_\alpha}x_i+\frac{v-v_\alpha}{v_\beta-v_\alpha}x_j - vt,k,\varepsilon\bigg)\\
&\qquad \times \mathbb{1}(\alpha_i=\alpha,\alpha_j=\beta),
\end{aligned}\tag{6.18}$$

$$\Omega_{22}(\mathrm{q},t,k,\varepsilon) = -\frac{\bar c}{L}\sum_{i,j} R_L(x_i - x_j) \\ \times H_L\left(\frac{v_\beta - v}{v_\beta - v_\alpha}\; x_i + \frac{v - v_\alpha}{v_\beta - v_\alpha} x_j - vt, k, \varepsilon\right) \\ \times \mathbb{1}(\alpha_i = \alpha, \alpha_j = \beta).$$

Note that the argument of H_L is chosen in such a way that when we apply $\partial/\partial t + \mathscr{A}_0$ on the H_L term, we obtain zero. Use (6.16)–(6.18) and apply the Schwarz inequality in the form $E_L B^{1/2} \le (E_L AB)^{1/2}(E_L A^{-1})^{1/2}$ with $A = \exp\Omega_1(\varepsilon)$. As a result,

$$\begin{aligned} &E_L \exp\left(-\tfrac{1}{2}\sum_{k=0}^{k_0-2} \mathbb{1}(T_L \ge \lambda_{k+1}) \int_{\lambda_{k+1}}^{T_L} \Omega_{21}(\mathrm{q}(t),t,k,\varepsilon)\,dt\right) \\ &\quad\le \Bigg[E_L \exp\Bigg(-\Omega_1(\varepsilon) + \sum_{k=0}^{k_0-1} \mathbb{1}(T_L \ge \lambda_{k+1}) \\ &\qquad\times \int_{\lambda_{k+1}}^{T_L} \Omega_{22}(\mathrm{q}(t),t,k,\varepsilon)\,dt + \int_0^{T_L} \Omega_3(\mathrm{q}(t),t)\,dt\Bigg)\Bigg]^{1/2}. \end{aligned} \tag{6.19}$$

Step 3. We start with bounding Ω_1. We clearly have

$$\begin{aligned} K_L(x) &:= \sum_{k=0}^{k_0-1} H_L(x,k,\varepsilon)\mathbb{1}(T_L \ge \lambda_{k+1}) \\ &\le \int L\zeta(L(x-z))\mathbb{1}(z \in B_l)\,dz, \end{aligned} \tag{6.20}$$

where $B_l = \{z\colon \tau_l(z) < T_L\}$. Put $\omega = (v - v_\alpha)/(v_\beta - v_\alpha)$. We use (6.15) and (6.20) to obtain

$$\begin{aligned} -\Omega_1(\varepsilon) &\le -\sum_{k=0}^{k_0-1} F_k(\mathrm{q}(T_L),T_L)\mathbb{1}(T_L \ge \lambda_{k+1}) \\ &\le c_1 \bar c \sup_z \sum_j K_L(\omega x_j(T_L) + z) \end{aligned} \tag{6.21}$$

for some constant c_1. The term Ω_{22} can be treated likewise; using $|R_L| \le 2$ we obtain

$$\begin{aligned} &\left|\sum_{k=0}^{k_0-1} \int_{\lambda_{k+1}}^{T_L} \Omega_{22}(\mathrm{q}(t),t,k,\varepsilon)\,dt\right| \\ &\qquad\le c_2 \bar c T \sup_{0\le t\le T} \sup_z \sum_j K_L(\omega x_j(t) + z) \end{aligned} \tag{6.22}$$

for some constant c_2.

Step 4. We now turn to Ω_3.

$$\begin{aligned} \Omega_3(\mathrm{q},t) = \tfrac{1}{2}\sum_{i,\,j}\sum_{\gamma\delta} K(\alpha_i\alpha_j,\gamma\delta)V(L(x_i - x_j)) \\ \times \left[\exp\big(G(S_{ij}^{\gamma\delta}\mathrm{q},t) - G(\mathrm{q},t)\big) - 1\right] \end{aligned} \tag{6.23}$$

Moreover $G(S_{ij}^{\gamma\delta}\mathsf{q}, t) - G(\mathsf{q}, t)$ equals

$$
\begin{aligned}
&\sum_{k=0}^{k_0-1} \frac{\bar{c}}{L(v_\beta - v_\alpha)} \hat{W}_L(x_i - x_j) \\
&\qquad \times H_L\left(\frac{v_\beta - v}{v_\beta - v_\alpha}\ x_i + \frac{v - v_\alpha}{v_\beta - v_\alpha}\ x_j - vt, k, \varepsilon \right) \\
&\qquad \times \big(\mathbb{1}(\gamma = \alpha, \delta = \beta) - \mathbb{1}(\alpha_i = \alpha,\ \alpha_j = \beta) \big) \mathbb{1}(t \ge \lambda_{k+1}) \\
&+ \sum_{k=0}^{k_0-1} \frac{\bar{c}}{L(v_\beta - v_\alpha)} \\
&\qquad \times \sum_{m \ne i,\, j} \hat{W}_L(x_m - x_j) H_L\left(\frac{v_\beta - v}{v_\beta - v_\alpha} x_m + \frac{v - v_\alpha}{v_\beta - v_\alpha} x_j - vt, k, \varepsilon \right) \\
&\qquad\qquad \times \big(\mathbb{1}(\alpha_m = \alpha, \delta = \beta) - \mathbb{1}(\alpha_m = \alpha, \alpha_j = \beta) \big) \mathbb{1}(t \ge \lambda_{k+1}) \\
&+ \sum_{k=0}^{k_0-1} \frac{\bar{c}}{L(v_\beta - v_\alpha)} \\
&\qquad \times \sum_{m \ne i,\, j} \hat{W}_L(x_i - x_m) H_L\left(\frac{v_\beta - v}{v_\beta - v_\alpha} x_i + \frac{v - v_\alpha}{v_\beta - v_\alpha} x_m - vt, k, \varepsilon \right) \\
&\qquad\qquad \times \big(\mathbb{1}(\gamma = \alpha, \alpha_m = \beta) - \mathbb{1}(\alpha_i = \alpha, \alpha_m = \beta) \big) \mathbb{1}(t \ge \lambda_{k+1}) \\
&\quad =: R_1 + R_2 + R_3.
\end{aligned} \tag{6.24}
$$

Recall that an inequality $|A| \le B$ implies $|e^A - 1| \le e^B |A|$. On the other hand, since

$$
\sum_{0}^{k_0-1} H_L(x, k, \varepsilon) \le a = \int \zeta \, dx,
$$

we have

$$
\left| G(S_{ij}^{\gamma\delta}\mathsf{q}, t) - G(\mathsf{q}, t) \right| \le \frac{6\bar{c}aN}{|v_\alpha - v_\beta| L} = c_3 \bar{c}.
$$

Hence

$$
\begin{aligned}
\left| \Omega_3(\mathsf{q}, t, \varepsilon) \right| &\le \tfrac{1}{2} \sum_{i \ne j} \sum_{\gamma,\, \delta} K(\alpha_i \alpha_j; \gamma\delta) V(L(x_i - x_j)) e^{c_3 \bar{c}} \\
&\qquad \times \left| G(S_{ij}^{\gamma\delta}\mathsf{q}, t, \varepsilon) - G(\mathsf{q}, t, \varepsilon) \right| \\
&\le \Omega_{31}(\mathsf{q}, t, \varepsilon) + \Omega_{32}(\mathsf{q}, t, \varepsilon) + \Omega_{33}(\mathsf{q}, t, \varepsilon),
\end{aligned} \tag{6.25}
$$

where $\Omega_{3i}(\mathsf{q}, t, k, \varepsilon)$ is obtained by replacing the term with absolute values on the first line with $|R_i|$. Set

$$
\Omega_{3i}(\varepsilon) = \int_0^{T_L} \Omega_{3i}(\mathsf{q}(t), t, \varepsilon) \, dt.
$$

Note that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{k=0}^{k_0-1}\mathbb{1}\big(\tau_l\in[\lambda_k,\lambda_{k+1}),T_L\ge\lambda_{k+1},t\in[\lambda_{k+1},T_L]\big)\\
&\qquad=\mathbb{1}\big(\tau_l<T_L,t\in(\tau_l,T_L]\big),\\
&\sum_{k=0}^{k_0-1}\mathbb{1}\big(\tau_l\in[\lambda_k,\lambda_{k+1}),T_L\ge\lambda_{k+1},t\in[\lambda_{k+1},T_L]\big)\\
&\qquad\le\mathbb{1}\big(\tau_l<T,t\in(\tau_l,T_L]\big).
\end{aligned}\tag{6.26}
$$

Using this and the fact that $L|R_1|$ is uniformly bounded, we deduce

$$
\big|\Omega_{31}(\varepsilon)\big|\le\frac{c_4\bar{c}\exp(c_3\bar{c})}{L}\int_0^{T_L}A_L(\mathrm{q}(t))\,dt\tag{6.27}
$$

for some constant c_4. Next we concentrate on Ω_{32}. For some constant c_5,

$$
\begin{aligned}
\big|\Omega_{32}(\varepsilon)\big|&\le\frac{1}{L}c_5\bar{c}\exp(c_3\bar{c})\sum_{k=0}^{k_0-1}\mathbb{1}(T_L\ge\lambda_{k+1})\\
&\quad\times\int_{\lambda_{k+1}}^{T_L}\sum_{i\ne j\ne m}V\big(L(x_i(t)-x_j(t))\big)\mathbb{1}(v_{\alpha_i(t)}\ne v_{\alpha_j(t)})\\
&\qquad\qquad\times\big|\hat{W}_L(x_i(t)-x_m(t))\big|\\
&\qquad\qquad\times H_L\bigg(\frac{v_\beta-v}{v_\beta-v_\alpha}x_i(t)+\frac{v-v_\alpha}{v_\beta-v_\alpha}x_m(t)-vt,k,\varepsilon\bigg)\,dt\\
&\le\frac{1}{L}2c_5\bar{c}\exp(c_3\bar{c})\int_0^{T_L}\sum_{i\ne j\ne m}V\big(L(x_i(t)-x_j(t))\big)\\
&\qquad\qquad\times\mathbb{1}\big(v_{\alpha_i(t)}\ne v_{\alpha_j(t)}\big)\\
&\qquad\qquad\times K_L\bigg(\frac{v_\beta-v}{v_\beta-v_\alpha}x_i(t)+\frac{v-v_\alpha}{v_\beta-v_\alpha}x_m(t)-vt\bigg)\,dt\\
&\le 2c_5\bar{c}\exp(c_3\bar{c})Z\bigg(\int_0^{T_L}A_L(\mathrm{q})\,dt\bigg)\\
&\quad\times\sup_z\sup_{0\le t\le T}\frac{1}{L}\sum_m K_L\bigg(\frac{v-v_\alpha}{v_\beta-v_\alpha}x_m(t)+z\bigg),
\end{aligned}\tag{6.28}
$$

where for the first inequality we have used (6.26). The term $\Omega_{33}(\varepsilon)$ is treated likewise. This, (6.28), (6.27), (6.26) and (6.25) imply

$$
\begin{aligned}
\int_0^{T_L}\Omega_3(t,\mathrm{q}(t),\varepsilon)\,dt&\le c_6\bar{c}\exp(c_3\bar{c})\int_0^{T_L}A_L(\mathrm{q}(t))\,dt\\
&\qquad\times\bigg[L^{-1}+\sup_z\sup_{0\le t\le T}\frac{1}{L}\sum_k K_L(\omega x_k(t)+z)\bigg]
\end{aligned}\tag{6.29}
$$

for some constant c_6.

Step 5. From (6.29), (6.22), (6.21) and (6.19) we learn

$$
\begin{aligned}
&E_L \exp\biggl(-\frac{1}{2}\sum_{k=0}^{k_0-1} \mathbb{1}(T_L \ge \lambda_{k+1}) \int_{\lambda_{k+1}}^{T_L} \Omega_{21}(\mathrm{q}(t), t, k, \varepsilon)\,dt\biggr) \\
&\quad \le \biggl\{E_L \exp\biggl[c_7\bar{c}\exp(c_3\bar{c})L^{-1}\int_0^{T_L} A_L(\mathrm{q}(t))\,dt \\
&\qquad\qquad + c_7\bar{c}\exp(c_3\bar{c})\sup_z \sup_{0\le t\le T} \frac{1}{L}\sum_m K_L(\omega x_m(t)+z) \\
&\qquad\qquad \times \int_0^{T_L} A_L(\mathrm{q}(t))\,dt + c_7\bar{c}\sup_z \frac{1}{L}\sum_m K_L(\omega x_m(T)+z)\biggr]\biggr\}^{1/2} \\
&\quad =: \bigl\{E_L \exp[X_1+X_2+X_3]\bigr\}^{1/2} \\
&\quad \le \bigl(E_L\exp(3X_1)\bigr)^{1/6}\bigl(E_L\exp(3X_2)\bigr)^{1/6}\bigl(E_L\exp(3X_3)\bigr)^{1/6}.
\end{aligned}
\tag{6.30}
$$

We let ε go to zero. By (6.26) the exponent on the left-hand side of (6.30) converges to

$$
\begin{aligned}
\Omega_{21} := \frac{\bar{c}}{2}\int_{B_l}\int_{\tau_l(z)}^{T_L}\sum_{i,j} V(L(x_i(t)-x_j(t))) \\
\times \zeta_L\biggl(\frac{v_\beta - v}{v_\beta - v_\alpha}x_i(t) + \frac{v-v_\alpha}{v_\beta - v_\alpha}x_j(t) - vt - z\biggr) \\
\times \mathbb{1}\bigl(\alpha_i(t)=\alpha, \alpha_j(t)=\beta\bigr)\,dt\,dz
\end{aligned}
\tag{6.31}
$$

As a result,

$$
E_L\exp\Omega_{21} \le \bigl(E_L\exp(3X_1)\bigr)^{1/6}\bigl(E_L\exp(3X_2)\bigr)^{1/6}\bigl(E_L\exp(3X_3)\bigr)^{1/6}. \tag{6.32}
$$

We now choose $\bar{c} = \eta w_3(l)$ where w_3 is defined by (6.1) and $\eta \in [0,1]$ will be chosen later. Then for $l \in (\exp(e), \exp((\log L)^2))$,

$$
\bar{c}\exp(c_3\bar{c}) = \eta \log\log\log l(\log\log l)^{c_3\eta}. \tag{6.33}
$$

If $l \le \exp((\log L)^2)$, then we can use (6.33) to show that for some constant c_8, $\bar{c}\exp(c_3\bar{c}) \le c_8 L$. Using Theorem 4.1,

$$
\begin{aligned}
E_L\exp(3X_1) &\le E_L\exp\biggl(c_8\int_0^{T_L} A_L(\mathrm{q}(t))\,dt\biggr) \\
&\le \exp\bigl[LC_1(c_8, T)\bigr].
\end{aligned}
\tag{6.34}
$$

Step 6. We now turn to X_2. Recall

$$
E_L\exp(3X_2) = E_L\biggl[3c_7\bar{c}\exp(c_3\bar{c})Y_L\int_0^{T_L} A_L(\mathrm{q}(t))\,dt\biggr], \tag{6.35}
$$

where

$$
Y_L = \sup_z \sup_{0\le t\le T} \frac{1}{N}\sum_k K_L(\omega x_k(t)+z).
$$

By Lemma 3.2,

$$Y_L(\mathrm{q}) \le c_8\Big(1 + \sup_{0\le t\le T} \Phi(\mathrm{q}(t))h(|B_l|)\Big). \tag{6.36}$$

Moreover Y_L is bounded by aZ, where $a = \int \zeta(z)\,dz$. From this, (6.36) and (6.35), we deduce that $E_L \exp(3X_3)$ is bounded above by

$$\begin{aligned} &E_L \exp\Big[3c_7\bar{c}\exp(c_3\bar{c})\lambda \int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big] \\ &\quad + E_L \exp\Big[3c_7\bar{c}\exp(c_3\bar{c})aZ \int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big] \\ &\quad \times \mathbb{1}\Big(c_8\Big(1 + \sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Big)h(|B_l|) > \lambda\Big) \\ &\le E_L \exp\Big[3c_7\bar{c}\exp(c_3\bar{c})\lambda \int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big] \\ &\quad + \Big(E_L \exp\Big[6c_7\bar{c}\exp(c_3\bar{c})aZ \int_0^{T_L} A_L(\mathrm{q}(t))\,dt\Big]\Big)^{1/2} \\ &\quad \times \Big(P_L\Big(c_8\Big(1 + \sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Big)h(|B_l|) > \lambda\Big)\Big)^{1/2}, \end{aligned} \tag{6.37}$$

where for the second inequality we applied the Schwarz inequality. First we choose $\lambda = \bar{c}^{-1}\exp(-c_3\bar{c})$ so that by Theorem 4.1 the first term in (6.37) is bounded above by $\exp(C_1(3c_7, T)L)$. For the second term we would like to apply (4.5) and for this we need

$$6c_7\bar{c}\exp(c_3\bar{c})aZ \le \eta_1 \log\log L. \tag{6.38}$$

First we assume η is small enough so that $c_3\eta \le \frac{1}{2}$. For such a choice we have

$$\bar{c}\exp(c_3\bar{c}) \le c_9\eta \log\log l \tag{6.39}$$

for some constant c_9. This in turn implies that (6.38) is satisfied if

$$6c_7c_9aZ\eta \log\log l \le \eta_1 \log\log L. \tag{6.40}$$

It is not hard to see that if η is sufficiently small then (6.40) is satisfied for all l with $l \le \exp((\log L)^2)$. Using (6.37), (4.4), (4.5) and the Chebyshev inequality,

$$\begin{aligned} E_L \exp(3X_2) &\le \exp\big[C_1(3c_7, T)L\big] \\ &\quad + \exp\Big[\frac{3}{2}C_2(T)c_7c_9aZL\eta \log\log l\Big] \\ &\quad \times \Big(E_L \exp\Big[\frac{1}{2}\theta_0 L\Big(1 + \sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Big) \\ &\qquad - \frac{\theta_0\lambda L}{2c_8h(|B_l|)}\Big]\Big)^{1/2}. \end{aligned} \tag{6.41}$$

On the other hand, by the Chebyshev inequality and the elementary inequality $\log x \le x$,

$$\begin{aligned} -\big(h(|B_l|)\big)^{-1} &= 1 + \log|B_l| \le 1 + \log\frac{1}{l}\int X_{T_L}(x)\,dx \\ &\le 1 - \log l + \log\frac{c_{10}}{L}\int_0^{T_L} A_L(\mathrm{q}(t))\,dt \\ &\le -\log l + \textstyle\int_0^{T_L}\frac{1}{L}A_L(\mathrm{q}(t))\,dt + 1 + c_{10} \end{aligned}$$

for some constant c_{10}. This, (6.41) and the Schwartz inequality imply

$$\begin{aligned} E_L\exp(3X_2) &\le \exp[C_1(3c_7, T)L] \\ &\quad + \exp\bigg[\frac{3}{2}C_2(T)c_7c_9aZ\eta L\log\log l\bigg] \\ &\quad \times\bigg(E_L\exp\bigg[\theta_0L\Big(1+\sup_{0\le t\le T}\Phi(\mathrm{q}(t))\Big)\bigg]\bigg)^{1/4} \\ &\quad \times\bigg(E_L\exp\bigg[\frac{\theta_0\lambda}{c_8}\int_0^{T_L}A_L(\mathrm{q}(t))\,dt\bigg]\bigg)^{1/4} \\ &\quad \times\exp\bigg(-\frac{\theta_0\lambda(\log l - 1 - c_{10})L}{4c_8}\bigg). \end{aligned} \tag{6.42}$$

Recall $\lambda = \exp(-c_3\bar{c})/\bar{c}$. If η is sufficiently small, we can guarantee

$$\lambda\log l = (\log\log l)^{-1}(\log l)^{1-c_3\eta} > c_{11}(\log l)^{1/2}.$$

From this, Theorem 4.1 and Lemma 3.1, we conclude that (6.42) implies

$$E_L\exp(3X_2) \le \exp(c_{12}L) \tag{6.43}$$

for some constant c_{12}. The term X_3 is treated likewise. From this, (6.43), (6.34) and (6.32), we conclude

$$E_L\exp\Omega_{21} \le \exp(c_{13}L) \tag{6.44}$$

for some c_{13}.

Final step. Recall the definition of Ω_{21} given by (6.31). Suppose that $\zeta \ge 1$ on the interval $[-r_1, r_1]$. Then

$$\begin{aligned} \Omega_{21} \ge \frac{\bar{c}}{2}\int_{B_l}\int_{\tau_l(z)}^{T_L} L\sum_{i,j}V\big(L(x_i(t)-x_j(t))\big) \\ \times\,\mathbb{1}\bigg(\frac{v_\beta - v}{v_\beta - v_\alpha}x_i(t) + \frac{v - v_\alpha}{v_\beta - v_\alpha}x_j(t) \\ - vt - z \in \bigg[-\frac{r_1}{L}, \frac{r_1}{L}\bigg]\bigg)\,dt\,dz. \end{aligned} \tag{6.45}$$

If $V(L(x_i - x_j)) \neq 0$, then $x_j \in [x_i - r_0/L, x_i + r_0/L]$. As a result,

$$\left|\frac{v_\beta - v}{v_\beta - v_\alpha} x_i + \frac{v - v_\alpha}{v_\beta - v_\alpha} x_j - x_i\right| = \left|\frac{v - v_\alpha}{v_\beta - v_\alpha}(x_j - x_i)\right| \leq \left|\frac{v - v_\alpha}{v_\beta - v_\alpha}\right| \frac{r_0}{L}.$$

We choose r_1 large enough so that

$$\frac{r_1}{L} - \left|\frac{v - v_\alpha}{v_\beta - v_\alpha}\right| \frac{r_0}{L} \geq \frac{r_0}{L}\ .$$

For such r_1, (6.45) implies

$$\Omega_{21} \geq \int \int_{\sigma_l(z)}^T \frac{\bar{c}}{2} L \sum_{i,j} V\big(L(x_i(t) - x_j(t))\big)\, \mathbb{1}\big(\alpha_i(t) = \alpha, \alpha_j(t) = \beta\big)$$
$$\times\ \mathbb{1}\left(x_i(t) - vt - z \in \left[-\frac{r_0}{L}, \frac{r_0}{L}\right]\right) dt\, dz.$$

This, (6.44), (6.12) and (6.11) imply (6.5). □

7. Uniform integrability, part II. In the previous section, we established the uniform integrability of $X_t(x)$, which is the time average of the collision term. What we really need in the succeeding sections is the uniform integrability of the collision term without a time average. To achieve this, we are forced to replace Γ_3 with a slightly *slower* function Γ_4. Fix a number $b \in (0,1)$:

$$\Gamma_4(x) := x(w_4(x))^b = \begin{cases} x(\log\log\log\log x)^b, & x \geq e^{e^e}, \\ 0, & x < e^{e^e}. \end{cases} \tag{7.1}$$

The main result of this section is the following theorem.

THEOREM 7.1. *For every ρ and T, there exists a constant $C_7(\rho, T)$ such that if*

$$\limsup_{L\to\infty} \frac{1}{L} \log \mathscr{P}_L(\mathscr{G}) \geq -\rho \tag{7.2}$$

for every open neighborhood $\mathscr{G}$ of $\bar{m}$, then $\bar{m}(t, dx) = \bar{g}(x,t)\,dx$ and for any pair (α, β) with $v_\alpha \neq v_\beta$,

$$\sup_{z\in\mathbb{T}} \int_0^T \int \Gamma_4\big(\bar{g}_\alpha(x+z,t)\bar{g}_\beta(x,t)\big)\, dx\, dt \leq C_7(\rho, T). \tag{7.3}$$

To prepare for the proof of Theorem 7.1, we start with some definitions. For any pair (α, β) with $v_\alpha \neq v_\beta$, define

$$\begin{aligned} Y_t(x, \alpha, \beta) &= \sum_{i\neq j} V\big(L(x_i(t) - x_j(t))\big) \\ &\qquad \times\ \mathbb{1}\big(\alpha_i(t) = \alpha, \alpha_j(t) = \beta\big) V\big(L(x_i(t) - x)\big), \\ X_t(x, \alpha, \beta, \gamma) &= \int_0^{T_L \wedge t} Y_s(x + v_\gamma s, s)\, ds. \end{aligned} \tag{7.4}$$

Clearly Y and X are random functions that depend on the trajectory $(\mathrm{q}(t)\colon 0 \le t \le T)$. We define the vector valued random measure

$$\mu(dx, dt) = \big(\mu_1(dx, dt), \dots, \mu_n(dx, dt)\big),$$
$$\mu_\alpha(dx, dt) = \sum_{v_\gamma,\, v_\delta \ne v_\alpha} Y_t(x + v_\alpha t, \gamma, \delta)\, dx\, dt. \tag{7.5}$$

Recall m_α defined by (1.11). Put $M_\alpha(dx, dt) = m_\alpha(t, dx)\, dt$ and $M = (M_1, \dots, M_n)$. The transformation $\mathrm{q} \mapsto (M, \mu)$, with q distributed according to $\mathscr{P}_L$, induces a probability measure $\tilde{\mathscr{P}}_L$ on $\mathscr{D} \times \mathscr{M}$ where $\mathscr{M}$ denotes the space of measures on $\mathbb{T} \times [0, T]$. The space $\mathscr{M}$ with weak convergence is *not* a good topological space. We therefore consider the spaces $\mathscr{M}^k$ of measures on $\mathbb{T} \times [0, T]$ of total mass at most k:

$$\mathscr{M}^k = \big\{\mu \in \mathscr{M}\colon \mu(\mathbb{T} \times [0, T]) \le k\big\}.$$

Each $\mathscr{M}^k$ is a compact metric space with respect to weak convergence. In this connection we have the following:

$$\lim_{k\to\infty} \limsup_{L\to\infty} \frac{1}{L} \log \tilde{\mathscr{P}}_L(\mathscr{D} \times \mathscr{M} - \mathscr{D} \times \mathscr{M}^k) = -\infty. \tag{7.6}$$

Roughly speaking, (7.6) says that the large deviation principles take place on $\mathscr{M}^k$ with k large but fixed. We omit the proof of (7.6) because it follows from Lemma 7.3 below.

The measure $\tilde{\mathscr{P}}_L$ is concentrated on (M, μ) with μ absolutely continuous with respect to the Lebesgue measure. Unfortunately, the space of absolutely continuous measures is not closed with respect to weak convergence. For our purposes in this section we would rather restrict $\tilde{\mathscr{P}}_L$ to the following closed subset of $\mathscr{M}$:

$$\mathscr{M}_k = \bigg\{\mu \in \mathscr{M}\colon \mu(dx, dt) = \nu(x, dt)\, dx,$$
$$\sup_\alpha \int \Gamma_3(\nu_\alpha(x, [0, T])\, dx \le k\bigg\}. \tag{7.7}$$

In other words, μ is absolutely continuous in the x-variable and we have a bound on its total variation in the t-variable.

LEMMA 7.2. *The set $\mathscr{M}_k$ is closed.*

PROOF. Suppose $\mu^i(dx, dt) = \nu^i(x, dt)\, dx$, and

$$\sup_i \sup_\alpha \int \Gamma_3\big(\nu^i_\alpha(x, [0, T])\big)\, dx \le k.$$

Since Γ_3 grows faster than the linear function at infinity, we can choose a subsequence $\nu^{i'}$, such that $\nu^{i'}(x, [0, t]) \to f(x, t)$ weakly for each rational number

$t \in [0, T]$, where f is a measureable function with

$$\sup_{\alpha} \int \Gamma_3(f_\alpha(x,t))\,dx \le k.$$

It is not hard to see that f is nondecreasing in t. Therefore it can be extended to all of $[0, T]$ in such a way that $f(x,t)$ is right continuous in the t-variable. Hence there exists a measure $\nu(x, dt)$ with $\nu(x, [0,t]) = f(x,t)$. It is not hard to see that in fact $\mu^i \Rightarrow \mu$ where $\mu(dx, dt) = \nu(x, dt)\,dx$. □

LEMMA 7.3.

$$\lim_{k\to\infty} \limsup_{L\to\infty} \frac{1}{L} \log \tilde{\mathscr{P}}_L(\mathscr{D} \times \mathscr{M} - \mathscr{D} \times \mathscr{M}_k) = -\infty. \tag{7.8}$$

PROOF. By the Chebyshev inequality and Theorem 6.1,

$$\begin{aligned}\tilde{\mathscr{P}}_L(\mathscr{D} \times \mathscr{M} - \mathscr{D} \times \mathscr{M}_k) &\le \exp(k\eta_3 L) \int \exp\biggl(\eta_3 L \int \Gamma_3\bigl(\nu(x, [0,T])\bigr)\,dx\biggr)\,d\tilde{\mathscr{P}}_L \\ &\le \exp(-k\eta_3 L + c_1 L)\end{aligned}$$

for some constant c_1. This evidently implies (7.8). □

On several occasions in this section and the next section we will use the exponential martingale

$$M_t = \exp\biggl(F(t, \mathsf{q}(t)) - F'(\mathsf{q}(0)) - \int_0^t e^{-F}\biggl(\frac{\partial}{\partial s} + \mathscr{A}\biggr)e^F(s, \mathsf{q}(s))\,ds\biggr)$$

with

$$F(t, \mathsf{q}) = \sum_\alpha \sum_{i=1}^N p_\alpha(x_i, t)\,\mathbb{1}(\alpha_i = \alpha),$$

$$F'(\mathsf{q}) = \sum_\alpha \sum_{i=1}^N G_\alpha(x_i)\,\mathbb{1}(\alpha_i = \alpha),$$

where p is a smooth function. A straightforward calculation yields

$$\begin{aligned}M_t = \exp\biggl\{ &\sum_\alpha \sum_{i=1}^N p_\alpha(x_i(T), T)\,\mathbb{1}(\alpha_i(T) = \alpha) \\ &- G(x_i(0))\,\mathbb{1}(\alpha_i(0) = \alpha) \\ &- \int_0^T \sum_\alpha \sum_{i=1}^N D_\alpha p(x_i(t), t)\,\mathbb{1}(\alpha_i(t) = \alpha)\,dt \\ &- \int_0^T \tfrac{1}{2} \sum_{i\ne j} V\bigl(L(x_i(t) - x_j(t))\bigr)\end{aligned} \tag{7.9}$$

$$\times \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta)\big[\exp\big(p_\gamma(x_i(t),t)+p_\delta(x_j(t),t) - p_\alpha(x_i(t),t)-p_\beta(x_j(t),t)\big)-1\big] \times \mathbb{1}\big(\alpha_i(t)=\alpha,\alpha_j(t)=\beta\big)\,dt\bigg\}.$$

If $F(t,\mathrm{q})=\sum_{i=1}^N H(x_i,t)\mathbb{1}(\alpha_i=\alpha)$ and $G=0$, where α is given and H is a smooth function with support in $\mathbb{T}\times(0,T)$, then M_t simplifies to

$$\begin{aligned} M_t = \exp\bigg[&\sum_{i=1}^N H(x_i(t),t)\mathbb{1}(\alpha_i(t)=\alpha) \\ &-\int_0^t \sum_{i=1}^N D_\alpha H(x_i(t),t)\mathbb{1}(\alpha_i(t)=\alpha)\,dt \\ &-\int_0^t \sum_{i\neq j} V\big(L(x_i(t)-x_j(t))\big) \\ &\times \sum_{\beta\gamma\delta}\Big[K(\alpha\beta,\gamma\delta)(\exp(-H(x_i(t),t))-1) \\ &\times \mathbb{1}(\alpha_i(t)=\alpha,\alpha_j(t)=\beta) \\ &+K(\gamma\delta,\alpha\beta)(\exp(H(x_i(t),t))-1) \\ &\times \mathbb{1}(\alpha_i(t)=\gamma,\alpha_j(t)=\delta)\Big]\,dt\bigg]. \end{aligned} \tag{7.10}$$

Recall the definition of α-differentiability as in (1.12). As a step toward Theorem 7.1, we state and prove a lemma.

LEMMA 7.4. *Suppose that there exists a positive number ρ for which (7.2) holds for every open neighborhood of $\bar m$. Then $\bar m(t,dx)=\bar g(x,t)\,dx$ for some $\bar g$, and $\bar g_\alpha$ is α-differentiable for each α. Moreover, there exists a constant $C_8(T)$ such that*

$$\sup_{0\le t\le T}\int \phi\bigg(\frac{1}{2}\sum_\alpha \bar g_\alpha(x,t)\bigg)\,dx \le \frac{C_0(T)+\rho}{\theta_0}, \tag{7.11}$$

where θ_0 and $C_0(T)$ are defined in Lemma 3.1, and

$$\int_0^T\int |D_\alpha g_\alpha|\,dx\,dt \le C_8(T)+\frac{\rho}{\log 2}. \tag{7.12}$$

PROOF.

Step 1. Since for every $x\in\mathbb{T}$,

$$\begin{aligned} m_\alpha\bigg(t,\bigg(x,x+\frac{1}{L}\bigg)\bigg) \le{}& m_\alpha\bigg(t,\bigg(-\frac{1}{2}+\frac{i}{L},-\frac{1}{2}+\frac{i+1}{L}\bigg)\bigg) \\ &+m_\alpha\bigg(t,\bigg(-\frac{1}{2}+\frac{i+1}{L},-\frac{1}{2}+\frac{i+2}{L}\bigg)\bigg) \end{aligned}$$

for some integer i, and since ϕ is convex, Lemma 3.1 implies

$$\sup_{0\le t\le T}\int \exp\biggl[L\theta_0\int\phi\biggl(\frac{L}{2}\sum_\alpha m_\alpha\biggl(t,\biggl(x,x+\frac{1}{L}\biggr)\biggr)\biggr)\,dx\biggr]\mathscr{P}_L(dm) \le \exp(C_0(T)L). \tag{7.13}$$

Recall that ζ is a smooth function with $\int\zeta\,dx=1$, $\zeta_\varepsilon(z)=\varepsilon^{-1}\zeta(\varepsilon^{-1}z)$ and $m_\varepsilon=m*\zeta_\varepsilon$ [see (5.1)]. Set $m^{(L)}(x,t)=Lm(t,(x,x+1/L))$. A straightforward calculation yields

$$\biggl|\sum_\alpha m_{\alpha,\varepsilon}(x,t)-m_\alpha^{(L)}*\zeta_\varepsilon(x,t)\biggr|\le\frac{c_1}{L\varepsilon} \tag{7.14}$$

for some constant c_1. Integrate both sides of (7.13) with respect to ζ_ε and apply the Jensen inequality. As a result,

$$\sup_{0\le t\le T}\int\exp\biggl[L\theta_0\int\phi\biggl(\tfrac{1}{2}\sum_\alpha m_\alpha^{(L)}*\zeta_\varepsilon(x,t)\biggr)\,dx\biggr]\mathscr{P}_L(dm)\le\exp(C_0(T)L).$$

From this and (7.14) we learn that for every $t\in[0,T]$,

$$\limsup_{L\to\infty}\frac{1}{L}\log\int\exp\biggl[L\theta_0\int\phi\biggl(\sum_\alpha m_{\alpha,\varepsilon}(x,t)\biggr)\,dx\biggr]\mathscr{P}_L(dm)\le C_0(T). \tag{7.15}$$

Let $\mathscr{G}$ be an open neighborhood of $\bar m$. By the Chebyshev inequality,

$$\frac{1}{L}\log\mathscr{P}_L(\mathscr{G})\le\frac{1}{L}\log\int\exp\biggl[L\theta_0\int\phi\biggl(\frac{1}{2}\sum_\alpha m_{\alpha,\varepsilon}(x,t)\biggr)\,dx\biggr]\mathscr{P}_L(dm) - \inf_{m\in\mathscr{G}}\theta_0\int\phi\biggl(\frac{1}{2}\sum_\alpha m_{\alpha,\varepsilon}(x,t)\biggr)\,dx.$$

Hence, by (7.2) and (7.15),

$$\sup_{\varepsilon>0}\sup_{\mathscr{G}}\inf_{m\in\mathscr{G}}\int\phi\biggl(\frac{1}{2}\sum_\alpha m_{\alpha,\varepsilon}(x,t)\biggr)\,dx\le\frac{C_0(T)+\rho}{\theta_0}, \tag{7.16}$$

where the second supremum is over all open neighborhood of $\bar m(x,t)$. Choose a sequence of open sets $\mathscr{G}_i=\{m\colon d(m,\bar m)\le 1/i\}$ where d is a metric for the weak convergence. Since ϕ is convex, the functional

$$m\mapsto\int\phi\biggl(\tfrac{1}{2}\sum_\alpha m_{\alpha,\varepsilon}\biggr)\,dx$$

is lower semicontinuous. Because of this,

$$\lim_{i\to\infty}\inf_{m\in\mathscr{G}_i}\int\phi\biggl(\tfrac{1}{2}\sum_\alpha m_{\alpha,\varepsilon}(x,t)\biggr)\,dx\ge\int\phi\biggl(\tfrac{1}{2}\sum_\alpha \bar m_{\alpha,\varepsilon}(x,t)\biggr)\,dx.$$

From this and (7.14) we learn that for every positive ε and every $t\in[0,T]$,

$$\sup_{\varepsilon>0}\int\phi\biggl(\frac{1}{2}\sum_\alpha m_{\alpha,\varepsilon}(x,t)\biggr)\,dx\le\frac{C_0(T)+\rho}{\theta_0}. \tag{7.17}$$

Since ϕ grows faster than the linear function at infinity, a bound of the form (7.1) implies that $\bar{m}(t, dx) = \bar{g}(x, t)\, dx$ for some $\bar{g}$ for which (7.11) holds.

Step 2. Consider the martingale M_t given by (7.10). From $E_L M_{T_L} = 1$ and the inequality $E_L B^{1/2} \le (E_L AB)^{1/2}(E_L A^{-1})^{1/2}$ we learn

$$\begin{aligned} E_L \exp\biggl[\tfrac{1}{2} \sum_{i=1}^{N} H(x_i(T_L), T_L)\, \mathbb{1}(\alpha_i(T_L) = \alpha) \\ - \tfrac{1}{2} \int_0^{T_L} \sum_{i=1}^{N} D_\alpha H(x_i(t), t)\, \mathbb{1}(\alpha_i(t) = \alpha)\, dt \biggr] \\ \le E_L \exp\biggl[c_1 \exp(\|H\|_\infty) \int_0^{T_L} A_L(\mathsf{q}(t))\, dt \biggr], \end{aligned}$$

for some constant c_1. If we replace T_L with T on the left-hand side, an error will appear that is bounded above by exp(const.$(T - T_L)L$). Hence,

$$\begin{aligned} E_L \exp\biggl[-\tfrac{1}{2} \int_0^{T} \sum_{i=1}^{N} D_\alpha H(x_i(t), t)\, \mathbb{1}(\alpha_i(t) = \alpha)\, dt \biggr] \\ \le E_L \exp(T \|D_\alpha H\|_\infty N)\, \mathbb{1}(T_L \ne T) \\ + E_L \exp\biggl[c_1 \exp(\|H\|_\infty) \int_0^{T_L} A_L(\mathsf{q}(t))\, dt \biggr]. \end{aligned}$$

From this, (4.4) and (4.5), we deduce

$$\begin{aligned} \limsup_{L\to\infty} \frac{1}{L} \log \int \exp\biggl[-\frac{1}{2} \int_0^T \int D_\alpha H(x, t) m_\alpha(t, dx) \biggr] \mathscr{P}_L(dm). \\ \le C_2 c_1 \exp(\|H\|_\infty). \end{aligned} \tag{7.18}$$

Let $\mathscr{G}$ be an open neighborhood of $\bar{m}$. By Chebyshev's inequality,

$$\begin{aligned} \frac{1}{L} \log \mathscr{P}_L(\mathscr{G}) \le \frac{1}{L} \log \int \exp\biggl[-\frac{1}{2} \int_0^T \int D_\alpha H m_\alpha(t, dx) \biggr] \mathscr{P}_L(dm) \\ - \inf_{m \in G} \biggl[-\frac{1}{2} \int_0^T \int D_\alpha H m_\alpha(t, dx) \biggr]. \end{aligned}$$

By (7.2) and (7.18),

$$\sup_{\mathscr{G}} \sup_{m \in \mathscr{G}} \biggl[-C_2 c_1 \exp(\|H\|_\infty) - \tfrac{1}{2} \int_0^T \int D_\alpha H m_\alpha(t, dx) \biggr] \le \rho, \tag{7.19}$$

where the first supremum is over open neighborhoods of $\bar{m}$. Fix a positive ε and choose

$$\mathscr{G} = \biggl\{ m\colon \tfrac{1}{2} \biggl| \int_0^T \int D_\alpha H m_\alpha(t, dx) - \int_0^T \int D_\alpha H \bar{m}_\alpha(t, dx) \biggr| < \varepsilon \biggr\}.$$

Then (7.19) implies

$$\left[-C_2 c_1 \exp(\|H\|_\infty) - \tfrac{1}{2}\int_0^T \int D_\alpha H \bar{m}_\alpha(t, dx)\right] \le \rho + \varepsilon.$$

This inequality is also true if we replace H with $-H$. We let ε go to zero. Recall that $\bar{m}_\alpha(t, dx) = g_\alpha(x, t)\, dx$. Hence

$$\left|\tfrac{1}{2}\int_0^T \int g_\alpha D_\alpha H \, dx\, dt\right| - C_2 c_1 \exp(\|H\|_\infty) \le \rho \tag{7.20}$$

for every smooth H with support in $\mathbb{T} \times (0, T)$. We write $H = \lambda r$ where λ is a positive scalar and r is a smooth function with $\|r\|_\infty \le 1$. Since

$$\psi(a, b) := \sup_{\lambda \ge 0}[a\lambda - be^\lambda] = \begin{cases} -b, & \text{if } a \le b, \\ a \log \dfrac{a}{b} - a, & \text{if } a > b, \end{cases}$$

we learn from (7.20) that

$$\psi\left(\left|\tfrac{1}{2}\int_0^T \int g_\alpha D_\alpha r \, dx\, dt\right|, C_3(T) c_1\right) \le \rho,$$

for every smooth r with $\|r\|_\infty \le 1$. Note that if $a \ge 2eb$ then $\psi(a, b) \ge a \log 2$. Hence

$$\left|\frac{1}{2}\int_0^T \int g_\alpha D_\alpha r \, dx\, dt\right| \le \max\left(2eC_3(T)c_1, \frac{\rho}{\log 2}\right).$$

From this we can conclude that g_α is α-differentiable, and that (7.12) holds. □

In the previous lemma we established the α-differentiability of g_α. We will derive (7.3) by establishing a bound for the $D_\alpha g_\alpha$, using the results of the previous section. The measure $\tilde{\mathscr{P}}_L$ will allow us to relate $D_\alpha g_\alpha$ to $\mu_\alpha(dx, dt) = \nu_\alpha(x, dt)\, dx$ for which an estimate of the form (7.7) is available. The next lemma ensures that a condition of the form (7.2) for $\mathscr{P}_L$ would imply a similar condition for $\tilde{\mathscr{P}}_L$. Recall the definition of $\mathscr{M}_k$ given by (7.7).

LEMMA 7.5. *There exists a constant $C_9(\rho, T)$ such that if (7.2) holds for every open neighborhood $\mathscr{G}$ of $\bar{m}$, then there exists a measure $\bar{\mu}$ in $\mathscr{M}_{C_9(\rho)}$ for which*

$$\limsup_{L\to\infty} \frac{1}{L} \log \tilde{\mathscr{P}}_L(\tilde{G}) \ge -2\rho \tag{7.21}$$

for every open neighborhood $\tilde{\mathscr{G}}$ of $(\bar{m}, \tilde{\mu})$.

PROOF. For each k,

$$\begin{aligned}\frac{1}{L}\log\tilde{\mathscr{P}}_L(\mathscr{G}) &\le \frac{1}{L}\log\big[\tilde{\mathscr{P}}_L(\mathscr{G}\times\mathscr{M}_k)+\tilde{\mathscr{P}}_L(\mathscr{D}\times\mathscr{M}_k^c)\big]\\ &\le \frac{1}{L}\log 2\\ &\quad+\max\left[\frac{1}{L}\log\tilde{\mathscr{P}}_L(G\times\mathscr{M}_k),\frac{1}{L}\log\tilde{\mathscr{P}}_L(\mathscr{D}\times\mathscr{M}_k^c)\right],\end{aligned}\tag{7.22}$$

using the elementary inequality $\log(a+b)\le\log 2+\max(\log a,\log b)$. By Lemma 7.2, there exists $k=C_9(\rho)$ so that

$$\frac{1}{L}\log\tilde{\mathscr{P}}_L(\mathscr{D}\times\mathscr{M}_k^c)\le -2\rho\ .$$

This and (7.22) imply

$$\limsup_{L\to\infty}\frac{1}{L}\log\tilde{\mathscr{P}}_L(\mathscr{G}\times\mathscr{M}_k)\ge-\rho\tag{7.23}$$

for every open neighborhood $\mathscr{G}$ of $\bar{m}$. We would like to establish (7.21) by contradiction. Suppose to the contrary, for every $\mu\in\mathscr{M}_k$, there exists an open neighborhood $A(\mu)\times B(\mu)\subseteq\mathscr{D}\times\mathscr{M}$ such that

$$\limsup_{L\to\infty}\frac{1}{L}\log\tilde{\mathscr{P}}_L(A(\mu)\times B(\mu))\le-2\rho,\tag{7.24}$$

where $\bar{m}\in A(\mu)$, $\mu\in B(\mu)$. Since by Lemma 7.2 the set $\mathscr{M}_k$ is compact, there exists a finite collection $\{\mu_1,\dots,\mu_r\}$ such that

$$\mathscr{M}_k\subseteq B(\mu_1)\cup\cdots\cup B(\mu_r)\ .$$

We set $\mathscr{G}=A(\mu_1)\cap\cdots\cap A(\mu_r)$. Then using the inequality $\log(a_1+\cdots+a_r)\le\log r+\max_{1\le i\le r}\log a_i$ and (7.24),

$$\begin{aligned}\limsup_{L\to\infty}\frac{1}{L}\log\tilde{\mathscr{P}}_L(\mathscr{G}\times\mathscr{M}_k)&\le\limsup_{L\to\infty}\frac{1}{L}\log\tilde{\mathscr{P}}_L\left(\bigcup_{i=1}^r A(\mu_i)\times B(\mu_i)\right)\\ &\le\max_{1\le i\le r}\limsup_{L\to\infty}\frac{1}{L}\log\tilde{\mathscr{P}}_L\big(A(\mu_i)\times B(\mu_i)\big)\le-2\rho,\end{aligned}$$

which contradicts (7.23). □

PROOF OF THEOREM 7.1.

Step 1. Recall that $E_L M_{T_L}=1$ where M_t is given by (7.10). Assume $H\ge 0$. For such H, $e^{-H}-1\le 0$. Moreover, by (1.5), if $K(\gamma\delta,\alpha\beta)\ne 0$ then $v_\gamma\ne v_\alpha$,

$v_\delta \neq v_\alpha$. Hence, for some constant c_1,

$$
\begin{aligned}
E_L \exp\Bigg\{ & - \int_0^{T_L} \sum_{i=1}^{N} D_\alpha H(x_i(t), t)\, \mathbb{1}(\alpha_i(t) = \alpha)\, dt \\
& - \int_0^{T_L} c_1 \sum_{i \neq j} V\big(L(x_i(t) - x_j(t))\big) \\
& \qquad \times \sum_{v_\delta,\, v_\gamma \neq v_\alpha} (\exp(H(x_i(t), t)) - 1) \\
& \qquad\qquad \times \mathbb{1}(\alpha_i(t) = \gamma, \alpha_j(t) = \delta)\, dt \Bigg\} \le 1.
\end{aligned}
\tag{7.25}
$$

Let $c_2(H)$ denote the Lipschitz constant of e^H:

$$
\left| e^{H(x,t)} - e^{H(y,t)} \right| \le c_2(H)|x - y|
$$

for all x, y and $t \in [0, T]$. Then, from (7.4),

$$
\begin{aligned}
& \sum_{i \neq j} V\big(L(x_i - x_j)\big)(\exp(H(x_i, t)) - 1)\, \mathbb{1}(\alpha_i = \gamma, \alpha_j = \delta) \\
& \qquad = \int L Y_t(x, \gamma, \delta)(\exp(H(x, t)) - 1)\, dx + R_L(\mathrm{q}, t),
\end{aligned}
$$

where

$$
|R_L(\mathrm{q}, t)| \le c_3 c_2(H) L^{-1} A_L(\mathrm{q})
$$

for some constant c_3. From this and (7.25) we learn

$$
\begin{aligned}
E_L \exp\Bigg\{ & -L \int_0^{T_L} \int D_\alpha H(x, t) m_\alpha(t, dx)\, dt \\
& - L \int_0^{T} \int \sum_{v_\gamma,\, v_\delta \neq v_\alpha} c_1 (\exp(H(x, t)) - 1) Y_t(x, \gamma, \delta)\, dx\, dt \\
& \qquad - c_4 c_2(H) L^{-1} \int_0^{T_L} A_L(\mathrm{q}(t))\, dt \Bigg\} \le 1
\end{aligned}
$$

for some constant c_4. If we replace T_L with T in the first integral, the error will be small. In fact, we may use (7.6) and the inequality $E_L B^{1/2} \le (E_L AB)^{1/2} (E_L A^{-1})^{1/2}$ to deduce

$$
\begin{aligned}
\int \exp\Bigg\{ & -\frac{L}{2} \int_0^{T_L} \int D_\alpha H(x, t) m_\alpha(t, dx)\, dt \\
& \times \frac{c_1 L}{2} \int_0^{T_L} \int (\exp(H(x + v_\alpha t, t)) - 1) \mu_\alpha(dx, dt) \Bigg\} \tilde{\mathscr{P}}_L(dm, d\mu) \\
\le {} & E_L \exp(T \|D_\alpha H\|_\infty N)\, \mathbb{1}(T_L \neq T) \\
& + \left\{ E_L \exp\left(c_4 c_2(H) L^{-1} \int_0^{T_L} A_L(\mathrm{q}(t)\, dt) \right) \right\}^{1/2}.
\end{aligned}
$$

We now use (4.4) and (4.5) of Theorem 4.1 to conclude

$$\begin{aligned}\limsup_{L\to\infty}\frac{1}{L}\log\int\exp\biggl\{-\frac{L}{2}\int_0^T\int D_\alpha H m_\alpha(t,dx)\,dt\\
-\frac{c_1 L}{2}\int_0^T\int(\exp(H(x-v_\alpha t,t))-1)\,d\mu_\alpha\biggr\}\\
\times\mathscr{P}_L(dm,d\mu)\le 0.\end{aligned}\tag{7.26}$$

Step 2. Choose $\bar{\mu}$ as in Lemma 7.4 and let $\tilde{\mathscr{G}}$ be a neighborhood of $(\bar{m},\bar{\mu})$. By Chebyshev's inequality

$$\begin{aligned}\tilde{\mathscr{P}}_L(\tilde{G})\le\exp\biggl\{L\sup_{(m,\mu)\in\tilde{\mathscr{G}}}\biggl[\frac{1}{2}\int_0^T\int D_\alpha H m_\alpha(t,dx)\,dt\\
+\frac{c_1}{2}\int_0^T\int(\exp(H(x+v_\alpha t,t))-1)\,d\mu_\alpha\biggr]\biggr\}\\
\times\int\exp\biggl\{-\frac{L}{2}\int_0^T\int D_\alpha H m_\alpha(t,dx)\,dt\\
-\frac{c_1 L}{2}\int_0^T\int\bigl(\exp(H(x-v_\alpha t,t))-1\bigr)\,d\mu_\alpha\biggr\}\tilde{\mathscr{P}}_L(dx\,dt).\end{aligned}$$

From this and (7.26) we obtain

$$\begin{aligned}\limsup_{L\to\infty}\frac{1}{L}\tilde{\mathscr{P}}_L(\tilde{G})\le\sup_{(m,\mu)\in\tilde{\mathscr{G}}}\biggl[\frac{1}{2}\int_0^T\int D_\alpha H m_\alpha(t,dx)\,dt\\
+\frac{c_1}{2}\int_0^T\int(\exp(H(x+v_\alpha t,t))-1)\,d\mu_\alpha\biggr].\end{aligned}\tag{7.27}$$

We choose

$$\begin{aligned}\tilde{G}=\biggl\{(m,\mu)\colon\biggl|\frac{1}{2}\int_0^T\int D_\alpha H m_\alpha(t,dx)\,dt-\frac{1}{2}\int_0^T\int D_\alpha H\bar{m}_\alpha(t,dx)\,dt\biggr|<\varepsilon,\\
\biggl|\frac{c_1}{2}\int_0^T\int(\exp(H(x-v_\alpha t,t))-1)\,d\mu_\alpha\\
-\frac{1}{2}\int_0^T\int(\exp(H(x-v_\alpha t,t))-1)\,d\bar{\mu}_\alpha\biggr|<\varepsilon\biggr\},\end{aligned}$$

we use (7.21) and then we let $\varepsilon\to 0$ in (7.27). As a result we get

$$\begin{aligned}\sup_H\biggl[-\frac{1}{2}\int_0^T\int D_\alpha H\bar{m}_\alpha(t,dx)\,dt\\
-\frac{c_1}{2}\int_0^T\int(\exp(H(x+v_\alpha t,t))-1)\,d\bar{\mu}_\alpha\biggr]\le 2\rho.\end{aligned}\tag{7.28}$$

Step 3. By Lemmas 7.3–7.5 we know

$$\bar{m}_\alpha(t, dx) = \bar{g}_\alpha(x,t)\,dx, \qquad \bar{\mu}_\alpha(dx, dt) = \bar{\nu}_\alpha(x, dt)\,dx,$$

$$\sup_t \int \phi\left(\frac{1}{2}\sum_\alpha \bar{g}_\alpha(x,t)\right) dx \le \frac{C_0(T)+\rho}{\theta}, \tag{7.29}$$

$$\sup_\alpha \int \Gamma_3\big(\bar{\nu}_\alpha(x,[0,T])\big)\,dx \le C_9(\rho, T).$$

After an integration by parts, (7.28) implies

$$\begin{aligned} &\frac{1}{2}\int_0^T \int H D_\alpha \bar{g}_\alpha\,dx\,dt \\ &- \frac{c_1}{2}\int_0^T \int (\exp(H(x+v_\alpha t, t)) - 1)\bar{\nu}_\alpha(x, dt)\,dx \le 2\rho \end{aligned} \tag{7.30}$$

for nonnegative smooth H with support in $\mathbb{T} \times (0,T)$. Let H be a nonnegative bounded measurable function on $\mathbb{T}\times[0,T]$. Choose a sequence of smooth functions H_n with support in $(0,T)$ such that

$$H_n \ge 0, \qquad \sup_n \|H_n\|_\infty < \infty \qquad \text{and } H_n \to H \text{ a.e.}$$

Since (7.30) holds for H_n, we can use the dominated convergence to deduce (7.30) holds for H, which is an arbitrary nonnegative bounded measurable function. Set $\hat{g}_\alpha(x,t) = \bar{g}_\alpha(x + v_\alpha t, t)$. Since

$$\int_0^T \int H D_\alpha \bar{g}_\alpha\,dx\,dt = \int_0^T \int H(x+v_\alpha t, t)\frac{\partial \hat{g}_\alpha}{\partial t}\ (x,t)\,dx\,dt,$$

(7.30) implies

$$\frac{1}{2}\int_0^T \int H\frac{\partial \hat{g}_\alpha}{\partial t}\,dx\,dt - \frac{c_1}{2}\int_0^T \int (\exp(H(x,t)) - 1)\bar{\mu}_\alpha(dx, dt) \le 2\rho \tag{7.31}$$

for every nonnegative $H \in L^\infty$. Set $\tau(dx,dt) = (\partial \hat{g}_\alpha/\partial t)\,dx\,dt$, $\tau^+(dx,dt) = (\partial \hat{g}_\alpha/\partial t)^+ dx\,dt$. We claim that $\tau^+ \ll \bar{\mu}_\alpha$. To see this, take a set A with $\bar{\mu}_\alpha(A) = 0$, and then choose $H = c\mathbb{1}_{A\cap B}$ where $B = \{(x,t)\text{: } (\partial \hat{g}_\alpha/\partial t)(x,t) > 0\}$, and c is a positive number. We deduce $\tau^+(A) = 0$ from (7.31) after letting $c \to +\infty$. Next, we choose

$$H(x,t) = \begin{cases} \log h(x,t), & \text{if } h(x,t) \ge 1, \\ 0, & \text{if } h(x,t) < 1, \end{cases}$$

where $h = d\tau^+/d\bar{\mu}_\alpha$ is the Radon–Nykodym derivative of τ^+ with respect to $\bar{\mu}_\alpha$. Of course, H may not be a bounded function but it can be approximated by a sequence of bounded functions H_n that increases to H. By the monotone convergence theorem we still have (7.31) for such H. As a result

$$\tfrac{1}{2}\int_0^T \int \big[h\log h - c_1(h-1)\big]\mathbb{1}(h\ge 1)\,d\bar{\mu}_\alpha \le 2\rho.$$

Note that

$$\int_0^T h\,d\bar{\mu}_\alpha = \int_0^T \int \left(\frac{\partial \hat{g}_\alpha}{\partial t}\right)^+ dx\,dt \le \int_0^T |D_\alpha \bar{g}_\alpha|\,dx\,dt.$$

From this and (7.12) we learn

$$\int_0^T \int h \log^+ h\,d\bar{\mu}_\alpha \le \frac{c_1}{2}\left(C_8(T) + \frac{\rho}{\log 2}\right) + 2\rho. \tag{7.32}$$

Step 4. By the entropy inequality,

$$\int_0^T \int h \log h\,d\bar{\mu}_\alpha = Z_1 \log \frac{Z_1}{Z_2} + \sup_{F\ge 0}\left[\int F\,d\tau^+ - Z_1 \log \int \frac{e^F}{Z_2}\,d\bar{\mu}_\alpha\right], \tag{7.33}$$
$$Z_1 = \tau^+(\mathbb{T}\times[0,T]), \qquad Z_2 = \bar{\mu}_\alpha(\mathbb{T}\times[0,T]).$$

If we restrict the supremum to functions F that depend on x only, we get

$$Z_1 \log \frac{Z_1}{Z_2} + \sup_{F\ge 0}\left[\int F(x)\sigma_\alpha(dx) - Z_1 \log \int \frac{e^{F(x)}}{Z_2}\hat{\nu}_\alpha(x)\,dx\right]$$
$$\le \int_0^T \int h \log h\,d\bar{\mu}_\alpha \le \int_0^T \int h \log^+ h\,d\bar{\mu}_\alpha,$$

where $\sigma_\alpha(dx) = \hat{\sigma}_\alpha(x)\,dx$ and

$$\hat{\sigma}_\alpha(x) := \int_0^T \left(\frac{\partial \hat{g}_\alpha}{\partial t}\right)^+ (x,t)\,dt, \tag{7.34}$$
$$\hat{\nu}_\alpha(x) := \bar{\nu}_\alpha(x,[0,T]).$$

From this and (7.32) we learn that

$$\int \hat{\sigma}_\alpha(x) \log \frac{\hat{\sigma}_\alpha(x)}{\hat{\nu}_\alpha(x)}\,dx \le c_2 + c_3\rho$$

for some constants c_2 and c_3. Hence

$$\int \hat{\sigma}_\alpha \log \frac{Z_2\hat{\sigma}_\alpha}{Z_1\hat{\nu}_\alpha}\,dx \le c_2 + c_3\rho + Z_1 \log \frac{Z_2}{Z_1}. \tag{7.35}$$

By (7.12) and Lemma 7.5, there exists a function $C_{10}(\rho)$ such that

$$Z_1 \log \frac{Z_2}{Z_1} \le Z_1 \log Z_2 + 1 \le C_{10}(\rho),$$

because $Z_2 = C_9(\rho)$. Hence (7.35) implies

$$\int \hat{\sigma}_\alpha \log \frac{Z_2\hat{\sigma}_\alpha}{Z_1\hat{\nu}_\alpha}\,dx \le c_2 + c_3\rho + C_{10}(\rho). \tag{7.36}$$

By Lemma 7.6 below, (7.36) implies

$$\int \Gamma_4(\hat{\sigma}_\alpha)\,dx \le C_{11}(\rho,T) \tag{7.37}$$

for some constant $C_{11}(\rho, T)$.

Final step. Recall the definition of $\hat{\sigma}_\alpha$ given by (7.34). We certainly have

$$\begin{aligned}\bar{g}_\alpha(x+v_\alpha t, t) &= \hat{g}_\alpha(x,t) \\ &= \hat{g}_\alpha(x,0) + \int_0^t \frac{\partial \hat{g}_\alpha}{\partial s}(x,s)\,ds \le \bar{g}_\alpha(x,0) + \hat{\sigma}_\alpha(x).\end{aligned} \tag{7.38}$$

Put $\tau_\alpha(x) = \bar{g}_\alpha(x,0) + \hat{\sigma}_\alpha(x)$. Then (7.38) says

$$\bar{g}_\alpha(x,t) \le \tau_\alpha(x - v_\alpha t). \tag{7.39}$$

On the other hand, we can use (7.37) and the second equation of (7.29) to claim

$$\int \Gamma_4(\tau_\alpha(x))\,dx \le C_{12}(\rho, T) \tag{7.40}$$

for some constant C_{12}, because Γ_4 is bounded above by a multiple of ϕ. Let x, y be two positive numbers. Using the elementary inequality $\log(A+B) \le \log 2 + \log A + \log B$, it is not hard to show that

$$w_4(xy) \le c_4 + w_4(x) + w_4(y) \le 3\max\bigl(c_4, w_4(x), w_4(y)\bigr)$$

for some constant c_4. Therefore,

$$\Gamma_4(xy) = xy(w_4(xy))^b \le xy3^b\bigl[c_4^b + (w_4(x))^b + (w_4(y))^b\bigr]. \tag{7.41}$$

Using this we have that whenever $v_\alpha \ne v_\beta$,

$$\begin{aligned}&\int_0^T \int \Gamma_4(\bar{g}_\alpha(x+z,t)\bar{g}_\beta(x,t))\,dx\,dt \\ &\quad \le 3^b c_4^b \int_0^T \int \bar{g}_\alpha(x+z,t)\bar{g}_\beta(x,t))\,dx\,dt \\ &\qquad + \int_0^T \int 3^b \bar{g}_\alpha(x+z,t)\Gamma_4(\bar{g}_\beta(x,t))\,dx\,dt \\ &\qquad + \int_0^T \int 3^b \bar{g}_\beta(x,t)\Gamma_4(\bar{g}_\alpha(x+z,t))\,dx\,dt.\end{aligned}$$

We can bound each term on the right-hand side. For example,

$$\begin{aligned}&\int_0^T \int \bar{g}_\alpha(x+z,t)\Gamma_4(\bar{g}_\beta(x,t))\,dx\,dt \\ &\quad \le \int_0^T \int \tau_\alpha(x - v_\alpha t + z)\Gamma_4(\tau_\beta(x - v_\beta t))\,dx\,dt \\ &\quad = \frac{1}{|v_\alpha - v_\beta|}\int\int \tau_\alpha(z_1)\Gamma_4(z_2)\,dz_1\,dz_2 \\ &\quad = \frac{1}{|v_\alpha - v_\beta|}\int \tau_\alpha(z_1)\,dz_1 \int \Gamma_4(z_2)\,dz_2,\end{aligned} \tag{7.42}$$

where for the first inequality we used (7.39) and for the second equality we made a change of variables $(x - v_\alpha t + z, x - v_\beta t) \mapsto (z_1, z_2)$. By (7.40), the left-hand side of (7.42) is bounded by a constant that depends on T and ρ. This evidently completes the proof. □

LEMMA 7.6. *There exists a constant $C(\rho_1,\rho_2,Z_1,Z_2)$ such that if for two nonnegative measurable functions f and g,*

$$\int f\log\frac{fZ_2}{gZ_1}\,dx\le\rho_1,\qquad \int\Gamma_3(g)\,dx\le\rho_2,\qquad \int f\,dx=Z_1,\qquad \int g\,dy=Z_2, \tag{7.43}$$

then $\int\Gamma_4(f)\le C(\rho_1,\rho_2,Z_1,Z_2)$.

PROOF. First observe that

$$\rho_1=\int\biggl[f\log\frac{fZ_2}{gZ_1}-f+\frac{gZ_1}{Z_2}\biggr]dx=\int\psi\biggl(\frac{fZ_2}{gZ_1}\biggr)\frac{gZ_1}{Z_2}\,dx,$$

where $\psi(z)=z\log z-z+1$. Note that $\psi(z)\ge 0$ for $z\ge 0$. Hence for any k and l,

$$\begin{aligned}\int f\,\mathbb{1}(f\ge l)\,dx&=\int\frac{fZ_2}{gZ_1}\mathbb{1}\biggl(f\ge l,\frac{fZ_2}{gZ_1}\ge k\biggr)\frac{gZ_1}{Z_2}\,dx\\&\quad+\int\frac{fZ_2}{gZ_1}\mathbb{1}\biggl(f\ge l,\frac{fZ_2}{gZ_1}\le k\biggr)\frac{gZ_1}{Z_2}\,dx\\&\le\frac{1}{\log k-1}\int\psi\biggl(\frac{fZ_2}{gZ_1}\biggr)\frac{gZ_1}{Z_2}\,dx\\&\quad+\int\frac{fZ_2}{gZ_1}\mathbb{1}\biggl(\frac{gZ_1}{Z_2}\ge\frac{l}{k},\frac{fZ_2}{gZ_1}\le k\biggr)\frac{gZ_1}{Z_2}\,dx\\&\le\frac{\rho_1}{\log k-1}+k\int\frac{gZ_1}{Z_2}\mathbb{1}\biggl(\frac{gZ_1}{Z_2}\ge\frac{l}{k}\biggr)dx\\&\le\frac{\rho_1}{\log k-1}+\frac{k}{(w_3(\frac{l}{k}))^b}\int\Gamma_3\biggl(\frac{gZ_1}{Z_2}\biggr)dx\end{aligned}\tag{7.44}$$

provided $l/k\ge e^e$. It is not hard to see that for some constant c_1,

$$\Gamma_3\biggl(\frac{gZ_1}{Z_2}\biggr)\le c_1\biggl[\frac{gZ_1}{Z_2}+\frac{Z_1}{Z_2}\Gamma_3(g)+g\Gamma_3\biggl(\frac{Z_1}{Z_2}\biggr)\biggr].$$

[See, e.g., (7.41).] Hence

$$\int\Gamma_3\biggl(\frac{gZ_1}{Z_2}\biggr)dx\le c_1\biggl[Z_1+\frac{Z_1}{Z_2}\rho_2+Z_2\Gamma_3\biggl(\frac{Z_1}{Z_2}\biggr)\biggr]=:C'(Z_1,Z_2,\rho_2).$$

From this and (7.44) we deduce

$$\int f\,\mathbb{1}(f\ge l)\,dx\le\frac{\rho_1}{\log k-1}+\frac{k}{(w_3(\frac{l}{k}))^b}C'(Z_1,Z_2,\rho_2).$$

We choose $k=(w_3(l))^a$ for some $a\in(0,b)$. Then

$$\int f\,\mathbb{1}(f>l)\le\frac{C''(Z_1,Z_2,\rho_2,\rho_1)}{w_4(l)}$$

for some constant C''. Set $\hat{w}_4(l) = (w_4(l))^b$. Then

$$\begin{aligned}\int \Gamma_4(f)\,dx &= \int\int_{e^{e^e}}^{\infty} \hat{w}_4'(l) f\,\mathbb{1}(f > l)\,dl\,dx\\ &\le \int_{e^{e^e}}^{\infty} \hat{w}_4'(l)\frac{C''}{w_4(l)}\,dl\\ &= \int_{e^{e^e}}^{\infty} C'' b w_4(l)^{b-2} w_3(l)^{-1} w_2(l)^{-1} (\log l)^{-1} l^{-1}\,dl\\ &= c_2 C''(Z_1, Z_2, \rho_1, \rho_2)\end{aligned}$$

for some constant c_2 and this completes the proof. □

8. The upper bound This section is devoted to the proof of (1.18). Take smooth functions p: $\mathbb{T}\times[0,T]\to\mathbb{R}^n$ and G: $\mathbb{T}\to\mathbb{R}^n$. Define

$$\begin{aligned}B(p, G, \varepsilon; m)&\\ &= \int\sum_\alpha p_\alpha(x,T) m_\alpha(T,dx) - \int\sum_\alpha G_\alpha(x) m_\alpha(0,dx)\\ &\quad - \int_0^T\int\sum_\alpha D_\alpha p_\alpha(x,t) m_\alpha(t,dx)\,dt \qquad (8.1)\\ &\quad - \int_0^T\int \tfrac12 \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) m_{\alpha,\varepsilon}(x,t) m_{\beta,\varepsilon}(x,t)\\ &\qquad\qquad \times\big[\exp(p_\gamma(x,t)+p_\delta(x,t)-p_\alpha(x,t)-p_\beta(x,t))-1\big]\,dx\,dt\end{aligned}$$

where $m\in\mathscr{D}$ and, as in (5.1),

$$m_{\alpha,\varepsilon}(x,t) = \int \zeta_\varepsilon(x-y) m_\alpha(t,dy)\ .$$

LEMMA 8.1. *For every $r\in(0,1)$, define*

$$\hat{B}(p,G,\varepsilon,r) = \limsup_{L\to\infty}\frac{1}{L}\int \exp\big[LrB(p,G,\varepsilon;m)\big]\mathscr{P}_L(dm). \tag{8.2}$$

Then

$$\limsup_{\varepsilon\to 0}\hat{B}(p,G,\varepsilon,r) \le \frac{Z}{r}\log\sum_\alpha\int\exp(p_\alpha(x,0)-G_\alpha(x)) f_\alpha^0(x)\,dx. \tag{8.3}$$

PROOF. Recall the exponential martingale M_t given by (7.9). We certainly have

$$\begin{aligned}E_L M_T = E_L M_0 &= E_L \exp\Big[\sum_i (p_{\alpha_i}(x_i,0) - G_{\alpha_i}(x_i)\Big]\\ &= \Big[\sum_\alpha\int\exp(p_\alpha(x,0)-G_\alpha(x)) f_\alpha^0(x)\,dx\Big]^N\end{aligned}$$

As a result, for (8.3), it suffices to show

$$\limsup_{\varepsilon\to 0}\hat{B}(p,G,\varepsilon)\le\frac{1}{r}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{1}{L}\log E_L M_T. \tag{8.4}$$

By the definition of $\mathscr{P}_L$, the expectation $\int\exp[LrB(p,G,\varepsilon;m)]\mathscr{P}_L(dm)$ equals

$$\begin{aligned}\mathscr{X}=E_L\exp\Big[r\sum_i p_{\alpha_i(T)}\big(x_i(T),T\big)-G_{\alpha_i(0)}(x_i(0))\\ -r\int_0^T\sum_i D_{\alpha_i(t)}p_{\alpha_i(t)}(x_i(t),t)\,dt\\ -r\int_0^T\int\tfrac{1}{2}\sum_{\alpha\beta\gamma\delta}K(\alpha\beta,\gamma\delta)m_{\alpha,\varepsilon}(x,t)m_{\beta,\varepsilon}(x,t)\\ \times\big(\exp(p_\gamma(x,t)+p_\delta(x,t)\\ -p_\alpha(x,t)-p_\beta(x,t))-1\big)\,dx\,dt\Big],\end{aligned}\tag{8.5}$$

where $m_{\alpha,\varepsilon}$ is defined by (5.1). We replace T in (8.4) with T_L. The resulting expression will be denoted by $\mathscr{Y}$. Since $m_{\alpha,\varepsilon}$ is bounded by Z/ε, we have

$$\mathscr{X}\le\mathscr{Y}+\exp\left(\frac{cL}{\varepsilon^2}\right)P_L(T_L\ne T), \tag{8.6}$$

for constant c. Set

$$\begin{aligned}X_L=Lr\int_0^T\int\tfrac{1}{2}\sum_{\alpha\beta\gamma\delta}K(\alpha\beta,\gamma\delta)m_{\alpha,\varepsilon}(x,t)m_{\beta,\varepsilon}(x,t)\\ \times\big[\exp(p_\gamma(x,t)+p_\delta(x,t)-p_\alpha(x,t)-p_\beta(x,t))-1\big]\,dx\,dt\\ -r\int_0^T\tfrac{1}{2}\sum_{i,j}V(L(x_i-x_j))\sum_{\gamma\delta}K(\alpha_i(t)\alpha_j(t),\gamma\delta)\\ \times\big[\exp(p_\gamma(x_i(t),t)+p_\delta(x_j(t),t)-p_{\alpha_i(t)}(x_i(t),t)\\ -p_{\alpha_j(t)}(x_j(t),t))-1\big]\,dt\end{aligned}$$

and choose q such that $r+1/q=1$. From (8.5), (8.6), (4.4), (7.10) and the Hölder inequality, we deduce that for (8.4) it suffices to show

$$\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{1}{L}\log E_L\exp(qX_L)\le 0. \tag{8.7}$$

In X_L, first we replace $p(x_j(t),t)$ with $p(x_i(t),t)$. Since p is smooth and $V(L(x_i(t)-x_j(t)))\neq 0$, the error would be bounded above by

$$\frac{1}{L}\log E_L \exp\left(\frac{c}{L}\int_0^{T_L} A_L(\mathrm{q}(t))\,dt\right)$$

for some constant c. Hence the error goes to zero as L goes to infinity. Finally, we apply (5.2) to establish (8.7). □

PROOF OF (1.18).

Step 1. Let $\mathscr{F}$ be a compact subset of $\mathscr{D}$. Let $\mathscr{F}_0$ denote the set of m that has a neighborhood $\mathscr{G}_m$ with $\limsup_{L\to\infty}(1/L\log)\mathscr{P}_L(\mathscr{G}_m)=-\infty$. Clearly $\mathscr{F}_0$ is open and the set $\mathscr{F}\cap\mathscr{F}_0$ can be covered by finitely many such $\mathscr{G}_m$. As a result, $\mathscr{F}-\mathscr{F}_0$ is compact and

$$\limsup_{L\to\infty}\frac{1}{L}\log\mathscr{P}_L(\mathscr{F}\cap\mathscr{F}_0)=-\infty.$$

This implies that for (1.18) without loss of generality we can take a compact set $\mathscr{F}$ with $\mathscr{F}\cap\mathscr{F}_0=\varnothing$. In view of Theorem 7.1, we have that for such $\mathscr{F}$, if $m\in\mathscr{F}$ then $m(t,dx)=g(x,t)\,dx$, the function g_α is α-differentiable and for any pair (α,β) with $v_\alpha\neq v_\beta$,

$$\sup_{z\in\mathbb{T}}\int_0^T\int\Gamma_4(g_\alpha(x+z,t)g_\beta(x,t))\,dx\,dt<\infty. \tag{8.8}$$

Step 2. Pick a point $\overline{m}\in\mathscr{D}$ and let $\mathscr{U}$ be a neighborhood of $\overline{m}$. From (8.3) and Chebyshev's inequality, it is not hard to show

$$\limsup_{L\to\infty}\frac{1}{L}\log\mathscr{P}_L(\mathscr{U})\le\hat{B}(p,G,\varepsilon,r)-\inf_{m\in\mathscr{U}}B(p,G,\varepsilon;m).$$

Since this holds for all permissible p,G,ε and r, we have

$$\limsup_{L\to\infty}\frac{1}{L}\log\mathscr{P}_L(\mathscr{U})\le\inf_{p,G,\varepsilon,r}\sup_{m\in\mathscr{U}}\big(\hat{B}(p,G,\varepsilon,r)-B(p,G,\varepsilon;m)\big).$$

Note that the functional $B(p,G,\varepsilon;m)$ is continuous in m. This allows us to use the usual large deviation arguments [see, for example, Section 4 of [17]) to show that for any compact set $\mathscr{F}$,

$$\limsup_{L\to\infty}\frac{1}{L}\log\mathscr{P}_L(\mathscr{F})\le\sup_{m\in\mathscr{F}}\inf_{p,G,\varepsilon,r}(\hat{B}-B).$$

To complete the proof, it suffices to show that for any m for which (8.8) holds,

$$\inf_{p,G,\varepsilon,r}(\hat{B}-B)\le -J(m). \tag{8.9}$$

Final step. Clearly $m_{\alpha,\varepsilon}$ converges to g_α almost everywhere. Since (8.8) holds, we deduce

$$\lim_{\varepsilon\to 0}\int_0^T\int r m_{\alpha,\varepsilon} m_{\beta,\varepsilon}\,dx\,dt = \int_0^T\int r g_\alpha g_\beta\,dx\,dt.$$

By letting $\varepsilon \to 0$ and $r \to 1$,

$$\begin{aligned}\inf_{\varepsilon,r}(\hat{B}-B) \le Z\log\sum_\alpha\int \exp\bigl(p_\alpha(x,0)-G_\alpha(x)\bigr) f_\alpha^0(x)\,dx\\ + \int\sum_\alpha\bigl(p_\alpha(x,T)g_\alpha(x,T)-G_\alpha(x)g_\alpha(x,0)\bigr)\,dx\\ - \int_0^T\int\sum_\alpha g_\alpha D_\alpha p_\alpha\,dx\,dt\\ - \int_0^T\int \tfrac{1}{2}\sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta)g_\alpha g_\beta\\ \times\bigl[\exp(p_\gamma+p_\delta-p_\alpha-p_\beta)-1\bigr]\,dx\,dt.\end{aligned}$$

We then let $G_\alpha(x) = p_\alpha(x,0) - K_\alpha(x)$ and take the infimum over p and K separately. As a result we get that the left-hand side of (8.9) is less than $-J_0(m) - J_d(m;p)$ for any smooth p. By the dominated convergence theorem we can show that

$$\lim_{k\to\infty} J_d(m,p_k) = J_d(m,p)$$

if the sequence p_k is uniformly bounded and p_k converges pointwise to p. This clearly completes the proof of (8.9). □

9. The lower bound.

First we construct a class of processes for which the corresponding macroscopic densities satisfy a perturbed equation of the form (1.16).

Let $\hat{p}\colon \mathbb{T}\times[0.T]\to\mathbb{R}^n$ be a smooth function, and define the new jump rate

$$\begin{aligned}K(\alpha\beta,\gamma\delta;x,y,t)\\ = K(\alpha\beta,\gamma\delta)\exp\bigl(\hat{p}_\gamma(x,t)+\hat{p}_\delta(y,t)-\hat{p}_\alpha(x,t)-\hat{p}_\beta(y,t)\bigr).\end{aligned}\tag{9.1}$$

A new process is characterized as an inhomogeneous Markov process with the infinitesimal generator $\hat{\mathscr{A}}^{(L)} = \mathscr{A}_0 + \hat{\mathscr{A}}_c$ where

$$\begin{aligned}\hat{\mathscr{A}}_c F(\mathrm{q}) = \tfrac{1}{2}\sum_{i,j} V(L(x_i-x_j))\\ \times\sum_{\gamma,\delta} K(\alpha_i\alpha_j,\gamma\delta;x_i,x_j,t)(F(S_{i,j}^{\gamma,\delta}\mathrm{q})-F(\mathrm{q})).\end{aligned}\tag{9.2}$$

The relationship between $\mathscr{A}^{(L)}$ and $\hat{\mathscr{A}}^{(L)}$ is

$$\hat{\mathscr{A}}^{(L)}F = \frac{\mathscr{A}^{(L)}(Fu)}{u} - \frac{\mathscr{A}^{(L)}u}{u}F,$$

where $u = e^w$ and

$$w(\mathrm{q}, t) = \sum_i \hat{p}_{\alpha_i}(x_i, t).$$

The new process $\mathrm{q}(t)$ induces a probability measure Q_L on the space $D([0, T], E)$. The initial distribution for the new process is $\hat{\mu}_L$ given as in (1.7) with $g_\alpha(x, 0)$ instead of $f^0(x)$. By Girsanov's theorem, Q_L is absolutely continuous with respect to P_L and

$$\begin{aligned} &\frac{dQ_L}{dP_L} = X_0 M_T, \\ &M_t = \exp\biggl(w(t, \mathrm{q}(t)) - w(0, \mathrm{q}(0)) - \int_0^t e^{-w}(\partial_t + \mathscr{A}^{(L)}) e^w(s, \mathrm{q}(s))\, ds\biggr), \\ &X_0 = \exp\biggl(\sum_i \log \frac{g_{\alpha_i(0)}(x_i(0), 0)}{f^0_{\alpha_i(0)}(x_i(0))}\biggr). \end{aligned} \tag{9.3}$$

Using Theorem 4.1, it is not hard to see that

$$E_L M_{T_L}^2 \le \exp(c_1 L) \tag{9.4}$$

for some constant c_1. We use this and Lemma 3.1 to deduce

$$\begin{aligned} &\int \exp\biggl[L \frac{\theta_0}{2} \sup_{0 \le t \le T_L} \Phi(\mathrm{q}(t))\biggr] Q_L(d\mathrm{q}) \\ &\quad \le (E_L M_{T_L}^2)^{1/2} \Bigl\{E_L \exp\Bigl[L\theta_0 \sup_{0 \le t \le T} \Phi(\mathrm{q}(t))\Bigr]\Bigr\}^{1/2} \\ &\quad \le \exp\biggl(\frac{1}{2} C_0(T) L + \frac{1}{2} c_1 L\biggr). \end{aligned} \tag{9.5}$$

By standard arguments, one can show that (1.16) has a unique solution with the initial density $g(\cdot, 0)$ (the uniqueness proof of [4] for the case $p = 0$ can be readily applied for the case bounded p). Once (9.5), (4.4) are available for our new process, we can apply the arguments of [13] to establish the kinetic limit. Let $\mathscr{Q}_L$ denote the distribution of m with respect to Q_L.

THEOREM 9.1. *The sequence $\mathscr{Q}_L$ converges to $\mathscr{Q}$ that is concentrated at a single g, the unique solution to (1.16).*

PROOF OF (1.19). Let $\mathscr{G}$ be an open set. Suppose $g \in \mathscr{G}$ and that g satisfies (1.16) for a continuously differentiable $\hat{p}$. Construct the corresponding process $\mathscr{Q}_L$. Since $g \in \mathscr{G}$,

$$\lim_{L \to \infty} \mathscr{Q}_L(\mathscr{G}) = 1. \tag{9.6}$$

On the other hand, it is well known that

$$
\begin{aligned}
&\big(1-\mathscr{Q}_L(\mathscr{G})\big)\log\frac{1-\mathscr{Q}_L(\mathscr{G})}{1-\mathscr{P}_L(\mathscr{G})}+\mathscr{Q}_L(\mathscr{G})\log\frac{\mathscr{Q}_L(\mathscr{G})}{\mathscr{P}_L(\mathscr{G})}\\
&\qquad\le H(\mathscr{Q}_L,\mathscr{P}_L)=\int\log\frac{d\mathscr{Q}_L}{d\mathscr{P}_L}\,d\mathscr{Q}_L\le H(Q_L,P_L).
\end{aligned}\tag{9.7}
$$

We then let $L\to\infty$. From (9.6) and (9.7) we deduce

$$
\liminf_{L\to\infty}\frac{1}{L}\log\mathscr{P}_L(\mathscr{G})\ge-\liminf_{L\to\infty}\frac{1}{L}H(Q_L,P_L).
$$

Now, using (9.3), we have

$$
\begin{aligned}
H(Q_L,P_L)&=J_0(g)+\int\bigg[w(t,\mathsf{q}(T))-w(0,\mathsf{q}(0))\\
&\qquad\qquad-\int_0^T e^{-w}(\partial_t+\mathscr{A}^{(L)})e^{w}(s,\mathsf{q}(s))\,ds\bigg]\,dQ_L\\
&=J_0(g)+\int\bigg[w(t,\mathsf{q}(T))-u(0,\mathsf{q}(0))\\
&\qquad\qquad-\int_0^T\big(\partial_t+\hat{\mathscr{A}}^{(L)}\big)w(s,\mathsf{q}(s))\,ds\bigg]\,Q_L(d\mathsf{q})\\
&\quad+\int\int_0^T(\hat{\mathscr{A}}^{(L)}w-e^{-w}\mathscr{A}^{(L)}e^{w})(s,\mathsf{q}(s))\,ds\;Q_L(d\mathsf{q})\\
&=J_0(g)+\int\int(\hat{\mathscr{A}_c}w-e^{-w}\mathscr{A}_c e^{w})(s,\mathsf{q}(s))\,ds\;Q_L(d\mathsf{q})\\
&=J_0(g)+\tfrac{1}{2}\int\int\sum_{i,j}V\big(L(x_i(s)-x_j(s))\big)\\
&\qquad\times\sum_{\gamma\delta}\big(K(\alpha_i(s)\alpha_j(s),\gamma\delta)\psi\big(\hat p(x_i(s),\gamma)\\
&\qquad\qquad+\hat p(x_j(s),\delta)\big)\\
&\qquad-\hat p(x_i(s),\alpha_i(s))-\hat p(x_j(s),\alpha_j(s))\big)\,ds\,Q_L(d\mathsf{q}),
\end{aligned}
$$

where for the second equality we used the fact that the expression in the brackets is a Q-martingale and $\psi(z)=(z-1)e^z+1$. Once more we can apply Theorems 4.1 and 1.1 of [13] to our new process to conclude that

$$
\begin{aligned}
&\lim_{L\to\infty}\frac{1}{L}H(Q_L,P_L)\\
&\quad=J_0(g)+\int_0^T\int\frac{1}{2}\sum_{\alpha\beta\gamma\delta}K(\alpha\beta,\gamma\delta)g_\alpha g_\beta\big(\exp(\hat p_\gamma+\hat p_\delta-\hat p_\alpha-\hat p_\beta)-1\big)\,dx\,dt\\
&\quad=J(m).
\end{aligned}
$$

In Section 11 we will establish (1.20) and this will complete the proof of (1.19).

10. The rate function, part I. In this section we establish some of the properties of the rate functions. A bound of the form $J(g) \le k$ will yield several useful bounds on g. These bounds are the macroscopic counterparts of the estimates we obtained in Sections 3–7. We first state the main results of this section.

THEOREM 10.1. *For every positive number k, the set $\{g\colon J(g) \le k\}$ is compact.*

THEOREM 10.2. (i) *Let $e = (e_1, \dots, e_n) \in \mathbb{R}^n$ such that*

$$e_\alpha + e_\beta = e_\gamma + e_\delta \quad \text{if } K(\alpha\beta, \gamma\delta) \neq 0.$$

Then, whenever $J_d(g) < \infty$, we have

$$\frac{\partial}{\partial t}\Bigl(\sum_\alpha e_\alpha g_\alpha\Bigr) + \frac{\partial}{\partial x}\Bigl(\sum_\alpha v_\alpha e_\alpha g_\alpha\Bigr) = 0. \tag{10.1}$$

in distribution. In particular,

$$\int \sum_\alpha e_\alpha g_\alpha(x,t)\,dx = \int \sum_\alpha e_\alpha g_\alpha(x,0)\,dx. \tag{10.2}$$

(ii) *Suppose $J_d(g) < \infty$ and $\int \sum_\alpha g_\alpha\,dx < \infty$. Then*

$$\begin{aligned} &\int_0^T \int \sum_{\alpha,\beta} (v_\alpha - v_\beta)^2 g_\alpha g_\beta\,dx\,dt \\ &\quad \le \int_0^T \int\int \sum_{\alpha,\beta} (v_\alpha - v_\beta)^2 g_\alpha(x,t) g_\beta(y,t)\,dx\,dy\,dt \\ &\qquad + J_d(g) + \int\int \sum_{\alpha,\beta} (v_\alpha - v_\beta)\bigl(g_\alpha(x,0) g_\beta(y,0) \\ &\qquad\qquad - g_\alpha(x,T) g_\beta(y,T)\bigr)\xi(x-y)\,dx\,dy. \end{aligned} \tag{10.3}$$

THEOREM 10.3. *Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be as in* (1.4)(vi). *Then*

$$\begin{aligned} &\sup_{0 \le t \le T} \int \sum_\alpha g_\alpha(x,t) \log \frac{g_\alpha(x,t)}{\lambda_\alpha}\,dx \\ &\quad \le J_d(g) + \int \sum_\alpha g_\alpha(x,0) \log \frac{g_\alpha(x,0)}{\lambda_\alpha}\,dx. \end{aligned} \tag{10.4}$$

THEOREM 10.4. *There exists a constant $B_1(k,T)$ such that for every pair (α,β) with $v_\alpha \neq v_\beta$,*

$$\sup_{|z|\le\varepsilon}\left|\int_0^T\int H(x,t)g_\alpha(x,t)(g_\beta(x+z,t)-g_\beta(x,t))\,dx\,dt\right| \tag{10.5}$$
$$\le J_d(g)+B_1(k,T)|\log\varepsilon|^{-1}$$
$$\times \sup_{0\le t\le T}\left(1+\int\sum_\alpha g_\alpha(x,t)\log^+ g_\alpha(x,t)\,dx\right)$$

for every smooth H with $\|H\|_\infty+\|\partial H/\partial t\|_\infty+\|\partial H/\partial x\|_\infty\le k$.

THEOREM 10.5. *There exists a constant $B_2(k_1,k_2;T)$ such that if*

$$J_d(g)\le k_1,\qquad \int\sum_\alpha g_\alpha(x,0)\log^+ g_\alpha(x,0)\,dx\le k_2, \tag{10.6}$$

then for every triplet $(v_\alpha,v_\beta,v_\gamma)$ with $v_\alpha\neq v_\beta\neq v_\gamma$,

$$\int\Gamma_1\left(\int_0^T g_\alpha g_\beta(x+v_\gamma t,t)\,dt\right)dx\le B_2(k_1,k_2;T). \tag{10.7}$$

THEOREM 10.6. *There exists a constant $B_3(k_1,k_2;T)$ such that if (10.6) holds, then*

$$\sup_z\int_0^T\int\sum_{\check\alpha\neq\check\beta}\Gamma_2(g_\alpha(x+z,t)g_\beta(x,t))\,dx\,dt\le B_3(k_1,k_2;T). \tag{10.8}$$

PROOF OF THEOREM 10.2. (i) Let $r\colon \mathbb{T}\times[0,T]\to\mathbb{R}$ be a smooth function and choose $p_\alpha(x,t)=cr(x,t)e_\alpha$ where c is a constant. Since $p_\alpha+p_\beta=p_\gamma+p_\delta$ whenever $K(\alpha\beta,\gamma\delta)\neq 0$, we have

$$J_d(g)\ge c\int_0^T\int r\sum_\alpha e_\alpha D_\alpha g_\alpha\,dx\,dt.$$

We let $c\to\pm\infty$. Since $J_d(g)<\infty$, we obtain

$$\int_0^T\int r\sum_\alpha e_\alpha D_\alpha g_\alpha\,dx\,dt=0.$$

This is precisely (10.1). The statement (10.2) is an immediate consequence of (10.1).

(ii) Define $m(x,t)=\sum_\alpha g_\alpha(x,t)$, $u(x,t)=\sum_\alpha v_\alpha g_\alpha(x,t)$ and

$$p_\alpha(x,t)=v_\alpha\int m(y,t)\xi(x-y)\,dy-\int u(y,t)\xi(x-y)\,dy,$$

where ξ is defined right after (2.1). Clearly, for some constant c_1,

$$p_\alpha(x,t)\le c_1\int m(y,t)\,dy=c_1\int m(y,0)\,dy, \tag{10.9}$$

where for the last equality we have used (10.2). By our assumption $\int m(y,0)\,dy<\infty$, hence p_α is bounded. Moreover, by conservation of momentum,

$$p_\alpha+p_\beta=p_\gamma+p_\delta,$$

whenever $K(\alpha\beta,\gamma\delta)\neq 0$. We set $p_\alpha(x,t)=p_\alpha(x,0)$ for $t<0$ and $p_\alpha(x,t)=p_\alpha(x,T)$ for $t>T$. In this way p is defined for all t. We pick a smooth mollifier $\phi^{(k)}(x,t)$ and define $p_\alpha^{(k)}=p_\alpha*\phi^{(k)}$. As a result $p_\alpha^{(k)}$ is smooth and we still have $p_\alpha^{(k)}+p_\beta^{(k)}=p_\gamma^{(k)}+p_\delta^{(k)}$ whenever $K(\alpha\beta,\gamma\delta)\neq 0$. Using this and the definition of J_d,

$$\begin{aligned}&-\int_0^T\int\sum_\alpha g_\alpha D_\alpha p_\alpha^{(k)}\,dx\,dt\\ &\qquad\le J_d(g)+\int\sum_\alpha\big(p_\alpha^{(k)}(x,0)g_\alpha(x,0)-p_\alpha^{(k)}(x,T)g_\alpha(x,T)\big)\,dx.\end{aligned}\tag{10.10}$$

We let k go to infinity. The right-hand side of (10.10) converges to the last line of (10.3) by the dominated convergence theorem. We now concentrate on the left-hand side of (10.10). The conservation of mass and momentum can be used for (10.1) to deduce

$$\frac{\partial m}{\partial t}+\frac{\partial u}{\partial x}=0,\qquad \frac{\partial u}{\partial t}+\frac{\partial E}{\partial x}=0,\tag{10.11}$$

where $E(x,t)=\sum_\alpha v_\alpha^2 g_\alpha(x,t)$. We would like to use (10.11) in order to calculate $D_\alpha p_\alpha$. From the definition of ξ, it is not hard to see that

$$\frac{\partial p_\alpha}{\partial x}=-v_\alpha m+u+v_\alpha\int m(y,t)\,dy-\int u(y,t)\,dy\tag{10.12}$$

in the distributional sense. Moreover, by (10.11),

$$\begin{aligned}\frac{\partial p_\alpha}{\partial t}&=v_\alpha\int\frac{\partial m}{\partial t}(y,t)\xi(x-y)\,dy-\int\frac{\partial u}{\partial t}(y,t)\xi(x-y)\,dy\\ &=-v_\alpha\int\frac{\partial u}{\partial y}(y,t)\xi(x-y)\,dy+\int\frac{\partial E}{\partial y}(y,t)\xi(x-y)\,dy\\ &=v_\alpha u-E-v_\alpha\int u(y,t)\,dy+\int E(y,t)\,dy\end{aligned}\tag{10.13}$$

in the distributional sense. The formal calculation (10.13) can be readily made rigorous by first multiplying both sides by a smooth function and integrating by parts. From (10.12) and (10.13) we learn that

$$\begin{aligned}-D_\alpha p_\alpha&=v_\alpha^2 m-2v_\alpha u+E-v_\alpha^2\int m(y,t)\,dy\\ &\qquad+2v_\alpha\int u(y,t)-\int E(y,t)\,dy.\end{aligned}\tag{10.14}$$

We convolve both sides of (10.14) with $\phi^{(k)}$ to obtain an expression for $D_\alpha p_\alpha^{(k)}$. Note that

$$v_\alpha^2 m - 2v_\alpha u + E = \sum_\beta (v_\alpha - v_\beta)^2 g_\beta.$$

Since this is nonnegative, we can use Fatou's lemma to pass to the limit as $k \to \infty$ on the left-hand side of (10.10) to yield the first term of (10.3). For the second term we use the dominated convergence theorem. This clearly implies (10.3). □

The following elementary lemma will be used for the proof of Theorem 10.3.

LEMMA 10.7. *Let a_1, a_2, a_3, a_4 be four positive numbers and let $a_i' = \min(a_i, k)$. Then*

$$a_1 a_2 \frac{a_3' a_4'}{a_1' a_2'} + a_3 a_4 \frac{a_1' a_2'}{a_3' a_4'} - a_3 a_4 - a_1 a_2 \le (a_1 + a_2)(a_3 + a_4). \tag{10.15}$$

PROOF. Let x denote the left-hand side of (10.15). If all a_i's are less than k, then $x = 0$ and (10.15) holds. If $a_1, a_2, a_3 \le k \le a_4$, then

$$x = a_3(k - a_4) + \left(\frac{a_4}{k} - 1\right) a_1 a_2 \le a_1 a_4.$$

If $a_1, a_3, a_4 \ge k \ge a_2$, then

$$x = a_1 k + a_3 a_4 \frac{a_2}{k} - a_3 a_4 - a_1 a_2 \le a_1 a_3.$$

If either a_1, $a_2 \le k \le a_3$, a_4 or a_3, $a_4 \le k \le a_1$, a_2, then $x \le 0$. If a_1, $a_3 \le k \le a_2$, a_4, then

$$x = a_2 a_3 + a_1 a_4 - a_3 a_4 - a_1 a_2 \le a_2 a_3 + a_1 a_4.$$

The remaining cases can be treated likewise. □

PROOF OF THEOREM 10.3. We first explain the idea of the proof. If we can allow $p_\alpha = \log(g_\alpha/\lambda_\alpha)$, then

$$\sum_{\alpha\beta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \big(\exp(p_\gamma + p_\delta - p_\alpha - p_\beta) - 1\big)$$
$$= \sum_{\alpha\beta} K(\alpha\beta, \gamma\delta) \lambda_\alpha \lambda_\beta \left(\frac{g_\gamma}{\lambda_\gamma}\frac{g_\delta}{\lambda_\delta} - \frac{g_\alpha}{\lambda_\alpha}\frac{g_\beta}{\lambda_\beta}\right) = 0$$

because $K(\alpha\beta, \gamma\delta)\lambda_\alpha\lambda_\beta = K(\gamma\delta, \alpha\beta)\lambda_\gamma\lambda_\delta$. If g_α is differentiable,

$$\int_0^T \int p_\alpha D_\alpha g_\alpha \, dx \, dt = \int \bigg[g_\alpha(x, T) \log \frac{g_\alpha(x, T)}{\lambda_\alpha} - g_\alpha(x, T)$$
$$- g_\alpha(x, 0) \log \frac{g_\alpha(x, 0)}{\lambda_\alpha} + g_\alpha(x, 0) \bigg] dx.$$

To make the above computation rigorous, we introduce some cutoff functions:

$$\phi(\varepsilon, k; g) = \begin{cases} \log g, & \text{if } \varepsilon < g < k, \\ \log \varepsilon, & \text{if } 0 \le g \le \varepsilon, \\ \log k, & \text{if } g \ge k, \end{cases}$$

$$\psi(k; g) = \begin{cases} \log g, & \text{if } 0 \le g < k, \\ \log k, & \text{if } g \ge k. \end{cases}$$

We then choose $p_\alpha = \phi(\varepsilon, k; \hat{g}_\alpha)$ where $\hat{g}_\alpha = (g_\alpha/\lambda_\alpha)$. Since p_α is bounded,

$$\begin{aligned} J_d(g) \ge& \int_0^T \int \sum_\alpha \phi(\varepsilon, k; \hat{g}_\alpha) D_\alpha g_\alpha \, dx \, dt \\ &- \int_0^T \int \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \\ &\quad \times \big[\exp(\phi(\varepsilon, k; \hat{g}_\gamma) + \phi(\varepsilon, k; \hat{g}_\delta) \\ &\quad - \phi(\varepsilon, k; \hat{g}_\alpha) - \phi(\varepsilon, k; \hat{g}_\beta)) - 1\big] \, dx \, dt \\ =:& \ \Omega_1(\varepsilon, k) - \Omega_2(\varepsilon, k). \end{aligned} \tag{10.16}$$

Note that if $v_\alpha \ne v_\beta$,

$$\begin{aligned} &\max_{\varepsilon \le \varepsilon_1} g_\alpha g_\beta \exp\big(\phi(\varepsilon, k; \hat{g}_\gamma) + \phi(\varepsilon, k; \hat{g}_\delta) - \phi(\varepsilon, k; \hat{g}_\alpha) - \phi(\varepsilon, k; \hat{g}_\beta)\big) \\ &\quad \le g_\alpha g_\beta \exp\big(\phi(\varepsilon_1, k; \hat{g}_\gamma) + \phi(\varepsilon_1, k; \hat{g}_\delta) - \psi(k; \hat{g}_\alpha) - \psi(k; \hat{g}_\beta)\big) \end{aligned}$$

is integrable. Hence we can apply the dominated convergence theorem:

$$\begin{aligned} \lim_{\varepsilon \to 0} \Omega_2(\varepsilon, k) &= \Omega_2(k) \\ &:= \int_0^T \int \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta (\Delta(\gamma\delta, \alpha\beta) - 1) \, dx \, dt, \end{aligned} \tag{10.17}$$

where $\Delta(\gamma\delta, \alpha\beta) = \exp(\psi(k, \hat{g}_\gamma) + \psi(k, \hat{g}_\delta) - \psi(k, \hat{g}_\alpha) - \psi(k, \hat{g}_\beta))$. Furthermore, by (1.4)(vi),

$$\begin{aligned} \Omega_2(k) = \int_0^T \int \frac{1}{4} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) \lambda_\alpha \lambda_\beta \\ \times \left[\hat{g}_\alpha \hat{g}_\beta (\Delta(\gamma\delta, \alpha\beta) - 1) + \hat{g}_\gamma \hat{g}_\delta \left(\frac{1}{\Delta(\gamma\delta, \alpha\beta)} - 1 \right) \right] dx \, dt. \end{aligned} \tag{10.18}$$

By Lemma 10.7, we know that

$$\begin{aligned} &\hat{g}_\alpha \hat{g}_\beta (\Delta(\gamma\delta, \alpha\beta) - 1) + \hat{g}_\gamma \hat{g}_\delta \left(\frac{1}{\Delta(\gamma\delta, \alpha\beta)} - 1 \right) \\ &\quad \le (\hat{g}_\alpha + \hat{g}_\beta)(\hat{g}_\gamma + \hat{g}_\delta). \end{aligned} \tag{10.19}$$

Since $K(\alpha\beta,\gamma\delta)\neq 0$, by (1.5) we know that $v_\alpha\neq v_\gamma, v_\delta$ and $v_\beta\neq v_\gamma, v_\delta$. Therefore the right-hand side of (10.19) is integrable. As a result, we can apply Fatou's lemma to deduce

$$\limsup_{k\to\infty}\Omega_2(k)\le 0$$

because the left-hand side of (10.19) converges pointwise to zero. From this, (10.17) and (10.16), we deduce

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\int_0^T\int\sum_\alpha \phi(\varepsilon,k;\hat g_\alpha)D_\alpha g_\alpha\,dx\,dt\le J_d(g). \tag{10.20}$$

Define

$$\Phi_\alpha(\varepsilon,k;z)=\begin{cases} z\log\dfrac{z}{\lambda_\alpha}-z+\lambda_\alpha, & \text{if } \varepsilon\le\dfrac{z}{\lambda_\alpha}\le k,\\[2mm] z\log\dfrac{k}{\lambda_\alpha}-k+\lambda_\alpha, & \text{if } \dfrac{z}{\lambda_\alpha}\ge k,\\[2mm] z\log\dfrac{\varepsilon}{\lambda_\alpha}-\varepsilon+\lambda_\alpha, & \text{if } 0\le\dfrac{z}{\lambda_\alpha}\le\varepsilon.\end{cases}$$

Note that ϕ_α is nonnegative and $(d\Phi_\alpha/dz)(\varepsilon,k;z)=\phi(\varepsilon,k,z/\lambda_\alpha)$. It is not hard to show that in fact $\Phi_\alpha(\varepsilon,k;g_\alpha)$ is α-differentiable with

$$D_\alpha\Phi_\alpha(\varepsilon,k;g_\alpha)=\phi\left(\varepsilon,k;\frac{g_\alpha}{\lambda_\alpha}\right)D_\alpha g_\alpha.$$

From this and (10.20) we learn that

$$\limsup_{k\to\infty}\ \limsup_{\varepsilon\to 0}\int\sum_\alpha \Phi_\alpha(\varepsilon,k;g_\alpha(x,T))-\Phi_\alpha(\varepsilon,k;g_\alpha(x,0))\,dx\le J_d(g).$$

Since Φ_α is increasing in ε and k, we use the monotone convergence theorem to deduce

$$\begin{aligned}&\int\sum_\alpha\left[g_\alpha(x,T)\log\frac{g_\alpha(x,T)}{\lambda_\alpha}-g_\alpha(x,T)+\lambda_\alpha\right]dx\\ &\qquad-\int\sum_\alpha\left[g_\alpha(x,0)\log\frac{g_\alpha(x,0)}{\lambda_\alpha}-g_\alpha(x,0)+\lambda_\alpha\right]dx\le J_d(g).\end{aligned}$$

Finally, we use the conservation of mass

$$\int\sum_\alpha g_\alpha(x,T)dx=\int\sum_\alpha g_\alpha(x,0)\,dx$$

to conclude (10.4). □

To prepare for Theorem 10.4, we state and prove a lemma.

LEMMA 10.8. *Let f be a positive measurable funciton. Then for any measurable set B,*

$$\int_B f\,dx \le 3h(|B|)\Big[1+\int f\log^+ f\,dx\Big].$$

Moreover, there exists a constant c_0 such that for every measurable function g with $\|g\|_\infty \le 1$,

$$\int fg\,dx \le c_0 h(\|g\|_\infty)\Big[1+\int f\log^+ f\,dx\Big].$$

PROOF. For any $l \ge 1$,

$$\int_B f\,dx \le l|B| + \int f\,\mathbb{1}(f\ge l)\,dx \le l|B| + \frac{1}{\log l}\int f\log^+ f\,dx.$$

The lemma follows if we choose

$$l = \frac{1}{|B|\log|B|}\Big[1+\int f\log^+ f\,dx\Big].$$

The second inequality can be established in the same way. □

PROOF OF THEOREM 10.4.
Step 1. Suppose $|z|\le\varepsilon$. Define

$$p_\alpha(x,t) = H(x,t)\int (g_\beta(y+z,t)-g_\beta(y,t))\xi(x-y)\,dy,$$

$$p_\beta(x,t) = -\int H(y,t)(g_\alpha(y-z,t)-g_\alpha(y,t))\xi(x-y)\,dy.$$

We set $p_{\alpha'}\equiv 0$ for $\alpha'\ne\alpha,\beta$. Since H is bounded and by the conservation of mass,

$$\|p_\alpha\|_\infty \le 2\|H\|_\infty\bar m,\qquad \|p_\beta\|_\infty \le 2\|H\|_\infty\bar m, \tag{10.21}$$

where $\bar m = \int\sum_{\alpha'} g_{\alpha'}(x,0)\,dx$. From the definition of J_d,

$$\begin{aligned}
J_d(g) \ge{}& \int\big(p_\alpha(x,T)g_\alpha(x,T)+p_\beta(x,T)g_\beta(x,T)\\
&\qquad - p_\alpha(x,0)g_\alpha(x,0)-p_\beta(x,0)g_\beta(x,0)\big)\,dx\\
&- \int_0^T\int(g_\alpha D_\alpha p_\alpha + g_\beta D_\beta p_\beta)\,dx\,dt\\
&- \int_0^T\int \tfrac12\sum_{\alpha'\beta'\gamma'\delta'} K(\alpha'\beta',\gamma'\delta')g_{\alpha'}g_{\beta'}\\
&\qquad\qquad \times[\exp(p_{\gamma'}+p_{\delta'}-p_{\alpha'}-p_{\beta'})-1]\,dx\,dt\\
=:{}& \Omega_1-\Omega_2-\Omega_3.
\end{aligned} \tag{10.22}$$

Clearly we can write

$$p_\alpha(x,t) = H(x,t)\int g_\beta(y,t)\big(\xi(x-y+z)-\xi(x-y)\big)\,dy.$$

Moreover, it is not hard to see that if a is not in $(-\varepsilon,\varepsilon)$ then $\xi(a+z)-\xi(a)=z$. Hence

$$
\begin{aligned}
|p_\alpha(x,t)| &\le \|H\|_\infty\left[\varepsilon\int g_\beta(y,t)\,dy+\int g_\beta(y,t)\mathbb{1}(x-y\in(-\varepsilon,\varepsilon))\,dy\right]\\
&\le \|H\|_\infty\left[\varepsilon\bar m+3h(2\varepsilon)\left(1+\int g_\beta(y,t)\log^+ g_\beta(y,t)\,dy\right)\right],
\end{aligned}
\tag{10.23}
$$

where for the second inequality we used the conservation of mass and Lemma 10.8. From (10.23) we deduce

$$
\begin{aligned}
|\Omega_1| &\le 4\bar m\|H\|_\infty\\
&\quad\times\left[\varepsilon\bar m+3h(2\varepsilon)\left(1+\sup_{0\le t\le T}\int\sum_\gamma g_\gamma(y,t)\log^+ g_\gamma(y,t)\,dy\right)\right]
\end{aligned}
\tag{10.24}
$$

Step 2. A straightforward calculation yields

$$
\begin{aligned}
D_\alpha p_\alpha(x,t) &= D_\alpha H(x,t)\int g_\beta(y,t)\big(\xi(x-y+z)-\xi(x-y)\big)\,dy\\
&\quad+H(x,t)\frac{\partial}{\partial t}\int g_\beta(y,t)\big(\xi(x-y+z)-\xi(x-y)\big)\,dy\\
&\quad+H(x,t)v_\alpha\big(g_\beta(x,t)-g_\beta(x+z,t)\big).
\end{aligned}
\tag{10.25}
$$

Some explanation is needed for the meaning of the second term on the right-hand side of (10.25). Since g_α is α-differentiable, the distributional derivative $D_\alpha g_\alpha$ is meaningful as an integrable function. From this, it is not hard to show that any spatial average of g_α is weakly differentiable in t-variable. Similarly,

$$
\begin{aligned}
D_\beta p_\beta(x,t) &= -\frac{\partial}{\partial t}\int H(y,t)g_\alpha(y,t)\big(\xi(x-y-z)-\xi(x-y)\big)\,dy\\
&\quad-v_\beta\big(H(x,t)g_\alpha(x,t)-H(x-z,t)g_\alpha(x-z,t)\big).
\end{aligned}
\tag{10.26}
$$

Using this and (10.25) we have

$$
\begin{aligned}
\Omega_2 &= \Omega_{21}+\Omega_{22}+\Omega_{23}+\Omega_{24}+\Omega_{25}\\
&=: \int_0^T g_\alpha(x,t)D_\alpha H(x,t)\\
&\qquad\times\int g_\beta(y,t)\big(\xi(x-y+z)-\xi(x-y)\big)\,dy\,dx\,dt\\
&\quad+\int_0^T g_\alpha(x,t)H(x,t)\\
&\qquad\times\frac{\partial}{\partial t}\int g_\beta(y,t)\big(\xi(x-y+z)-\xi(x-y)\big)\,dy\,dx\,dt\\
&\quad-\int_0^T\int g_\beta(x,t)
\end{aligned}
\tag{10.27}
$$

$$\times \frac{\partial}{\partial t} \int H(y,t) g_\alpha(y,t)\big(\xi(x-y-z) - \xi(x-y)\big)\, dy\, dx\, dt$$
$$+ \int_0^T \int v_\alpha g_\alpha(x,t) H(x,t)\big(g_\beta(x,t) - g_\beta(x+z,t)\big)\, dx\, dt$$
$$- \int_0^T \int v_\beta g_\beta(x,t)\big(H(x,t) g_\alpha(x,t)$$
$$- H(x-z,t) g_\alpha(x-z,t)\big)\, dx\, dt.$$

We can show

$$\begin{aligned}|\Omega_{21}| &\le 8\bar{m}\|D_\alpha H\|_\infty \\ &\quad \times \left[\varepsilon \bar{m} + 3h(2\varepsilon)\left(1 + \sup_{0\le t\le T} \int \sum_\gamma g_\gamma(y,t) \log^+ g_\gamma(y,t)\, dy\right)\right],\end{aligned} \tag{10.28}$$

in just the same way that we established (10.24). Furthermore, since ξ is odd,

$$\begin{aligned}\Omega_{22} + \Omega_{23} &= \int_0^T \int g_\alpha(x,t) H(x,t) \\ &\qquad \times \frac{\partial}{\partial t} \int g_\beta(y,t)\big(\xi(x-y+z) - \xi(x-y)\big)\, dy\, dx\, dt \\ &\quad - \int_0^T \int g_\beta(y,t) \\ &\qquad \times \frac{\partial}{\partial t} \int H(x,t) g_\alpha(x,t)\big(\xi(y-x-z) - \xi(y-x)\big)\, dx\, dy\, dt \\ &= \int_0^T \left[\frac{d}{dt} \int\int g_\alpha(x,t) H(x,t) g_\beta(y,t) \right. \\ &\qquad \left. \times \left(\xi(x-y+z) - \xi(x-y)\right) dy\, dx\right] dt \\ &= \int\int \big(g_\alpha(x,T) H(x,T) g_\beta(y,T) - g_\alpha(x,0) H(x,0) g_\beta(y,0)\big) \\ &\qquad \times \big(\xi(x-y+z) - \xi(x-y)\big)\, dx\, dy.\end{aligned}$$

Once more we can apply the argument of the first step to derive

$$\begin{aligned}&|\Omega_{22} + \Omega_{23}| \\ &\quad \le c_1 \bar{m} \|H\|_\infty \\ &\qquad \times \left[\varepsilon \bar{m} + h(2\varepsilon)\left(1 + \sup_{0\le t\le T} \int \sum_\gamma g_\gamma(y,t) \log^+ g_\gamma(y,t)\, dy\right)\right]\end{aligned} \tag{10.29}$$

for some constant c_1. Moreover,

$$\Omega_{24} + \Omega_{25} = \int_0^T \int (v_\alpha - v_\beta) H(x,t) g_\alpha(x,t)(g_\beta(x,t) - g_\beta(x+z,t))\, dx\, dt.$$

From this, (10.29), (10.28), (10.27) and (10.22) we learn that

$$\begin{aligned}&\int_0^T\int(v_\beta-v_\alpha)H(x,t)g_\alpha(x,t)\big(g_\beta(x,t)-g_\beta(x+z,t)\big)\,dx\,dt\\&\qquad\le J_d(g)+\Omega_3\\&\qquad\quad+c_2\bar m k\left[\varepsilon\bar m+h(2\varepsilon)\left(1+\sup_{0\le t\le T}\int\sum_\gamma g_\gamma(y,t)\log^+ g_\gamma(y,t)\,dy\right)\right]\end{aligned}\tag{10.30}$$

Final step. Recall that an inequality of the form $|A|\le B$ implies $|e^A-1|\le e^B|A|$. This and (10.21) yields

$$|\Omega_3|\le c_3\int_0^T\int\sum_{v_\gamma\ne v_\delta}g_\gamma g_\delta\exp(4\bar m\|H\|_\infty)\big(|p_\gamma|+|p_\delta|\big)\,dx\,dt$$

for some constant c_3. We then use (10.23) to deduce

$$\begin{aligned}|\Omega_3|&\le c_3\exp(4\bar m\|H\|_\infty)\|H\|_\infty\\&\quad\times\left[\varepsilon\bar m+h(2\varepsilon)\left(1+\sup_{0\le t\le T}\int\sum_\gamma g_\gamma(x,t)\log^+ g_\gamma(x,t)\,dx\right)\right]\\&\quad\times\int_0^T\int\sum_{v_\gamma\ne v_\beta}g_\gamma g_\delta\,dx\,dt.\end{aligned}$$

This and (10.30) evidently imply (10.5). □

PROOF OF THEOREM 10.5.

Step 1. Without loss of generality we may assume $v_\alpha-v_\beta\ge0$. Let H be a smooth function and define

$$\begin{aligned}p_\alpha(x,t)&=\int g_\beta(y,t)\xi(x-y)\\&\qquad\times H\left(\frac{v_\beta-v_\gamma}{v_\beta-v_\alpha}x+\frac{v_\gamma-v_\alpha}{v_\beta-v_\alpha}y-v_\gamma t\right)dy,\\p_\beta(x,t)&=\int g_\alpha(y,t)\xi(y-x)\\&\qquad\times H\left(\frac{v_\beta-v_\gamma}{v_\beta-v_\alpha}\ y+\frac{v_\gamma-v_\alpha}{v_\beta-v_\alpha}\ x-v_\gamma t\right)dy.\end{aligned}\tag{10.31}$$

Assume first that g_α and g_β are differentiable functions. Then

$$\begin{aligned}&\frac{d}{dt}\int\int g_\alpha(x,t)g_\beta(y,t)\xi(x-y)\\&\qquad\times H\left(\frac{v_\beta-v_\gamma}{v_\beta-v_\alpha}x+\frac{v_\gamma-v_\alpha}{v_\beta-v_\alpha}y-v_\gamma t\right)dy\,dx\\&\quad=\int\int\frac{\partial g_\alpha}{\partial t}(x,t)g_\beta(y,t)\xi(x-y)\end{aligned}$$

$$\times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}\ y - v_\gamma t\right) dy\, dx$$

$$+ \int\int g_\alpha(x,t)\frac{\partial g_\beta}{\partial t}(y,t)\xi(x-y)$$

$$\times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\, dx$$

$$- \int\int g_\alpha(x,t)g_\beta(y,t)\xi(x-y)$$

$$\times v_\gamma H'\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\, dx$$

$$= \int p_\alpha D_\alpha g_\alpha dx + \int p_\beta D_\beta g_\beta\, dx \tag{10.32}$$

$$- \int\int v_\alpha \frac{\partial g_\alpha}{\partial x}(x,t)g_\beta(y,t)\xi(x-y)$$

$$\times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\, dx$$

$$- \int\int g_\alpha(x,t)v_\beta \frac{\partial g_\beta}{\partial y}(y,t)\xi(x-y)$$

$$\times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\, dx$$

$$- \int\int g_\alpha(x,t)g_\beta(y,t)\xi(x-y)$$

$$\times v_\gamma H'\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\, dx.$$

After an integration by parts, (10.32) equals

$$\begin{aligned} &\int p_\alpha D_\alpha g_\alpha\, dx + \int p_\beta D_\beta g_\beta\, dx \\ &\qquad + (v_\beta - v_\alpha)\int g_\alpha(x,t)g_\beta(x,t)H(x - v_\gamma t)\, dx \\ &\qquad + (v_\alpha - v_\beta)\int\int g_\alpha(x,t)g_\beta(y,t) \\ &\qquad\qquad \times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\, dx. \end{aligned} \tag{10.33}$$

In the case of an arbitrary g with $J(g) < \infty$, we still have that the left-hand side of (10.32) equals (10.33) in the weak sense. To show this, we multiply both sides by a smooth function of t, integrate with respect to t, replace ξ by a smooth approximation and rearrange terms so that only $D_\alpha g_\alpha$ and $D_\beta g_\beta$ appear (avoiding $\partial g_\alpha/\partial t$ and $\partial g_\alpha/\partial x$). Then by a standard argument we pass to the limit and derive (10.33). In summary,

$$
\begin{aligned}
&\int_0^T \int (p_\alpha D_\alpha g_\alpha + p_\beta D_\beta g_\beta)\,dx\,dt \\
&\quad = (v_\alpha - v_\beta)\int_0^T \int g_\alpha(x,t) g_\beta(x,t) H(x - v_\gamma t)\,dx \\
&\qquad + (v_\beta - v_\alpha)\int_0^T \int\int g_\alpha(x,t) g_\beta(y,t) \\
&\qquad\qquad \times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma t\right) dy\,dx\,dt \\
&\qquad + \int\int g_\beta(x,T) g_\beta(y,T)\xi(x-y) \\
&\qquad\qquad \times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y - v_\gamma T\right) dy\,dx \\
&\qquad - \int\int g_\beta(x,0) g_\beta(y,0)\xi(x-y) \\
&\qquad\qquad \times H\left(\frac{v_\beta - v_\gamma}{v_\beta - v_\alpha}x + \frac{v_\gamma - v_\alpha}{v_\beta - v_\alpha}y\right) dy\,dx \\
&\quad = (v_\alpha - v_\beta)\int_0^T \int g_\alpha(x,t) g_\beta(x,t) H(x - v_\gamma t)\,dx\,dt \\
&\qquad + \Omega_1(T) + \Omega_2(T) - \Omega_2(0),
\end{aligned}
\tag{10.34}
$$

for any smooth function H. Since $D_\alpha g_\alpha$, $D_\beta g_\beta$ and $g_\alpha g_\beta$ are integrable functions, it is not hard to establish (10.34) for an H that is bounded and measurable.

Step 2. We set $p_{\alpha'} = 0$ if $\alpha' \neq \alpha, \beta$. We have

$$
\begin{aligned}
J_d(g) \geq &\int_0^T \int \sum_{\alpha'} p_{\alpha'} D_{\alpha'} g_{\alpha'}\,dx\,dt \\
&- \int_0^T \int \tfrac{1}{2} \sum_{\alpha'\beta'\gamma'\delta'} K(\alpha'\beta', \gamma'\delta') g_{\alpha'} g_{\beta'} \\
&\qquad \times \left[\exp(p_{\gamma'} + p_{\delta'} - p_{\alpha'} - p_{\beta'}) - 1\right] dx\,dt.
\end{aligned}
\tag{10.35}
$$

We choose $H(z) = \lambda \mathbb{1}_B(z)$ where λ is a constant and B is a measurable subset of $\mathbb{T}$. From Lemma 10.8 we have that p_α and p_β are bounded above by λ times

$$
c_1 h(c_1|B|)\left[1 + \sum_{\alpha'} \int g_{\alpha'} \log^+ g_{\alpha'}(x,t)\,dx\right]
\tag{10.36}
$$

for some constant c_1. Since $x\log^+ x \le x\log x - x + 1$, we can use (10.4) to deduce that (10.36) is bounded above by λ times

$$c_1 h(c_1|B|)\left[1 + \sum_{\alpha'} \int g_{\alpha'}(x,0)\log^+ g_{\alpha'}(x,0)dx + J_d(g)\right]. \tag{10.37}$$

We now choose λ to be the reciprocal of (10.36). For such a choice of λ, p_α and p_β are bounded by 1. Hence

$$J_d(g) \ge \int_0^T \int (p_\alpha D_\alpha g_\alpha + p_\beta D_\beta g_\beta)\,dx\,dt - c_2\int_0^T \int \sum_{v_{\alpha'}\ne v_{\beta'}} g_{\alpha'}g_{\beta'}\,dx\,dt$$

for some constant c_2. From this, (10.34) and (10.3), we learn that for some constants c_3 and c_4,

$$\begin{aligned}(v_\alpha - v_\beta)&\int_0^T \int g_\alpha(x,t)g_\beta(x,t)H(x-v_\gamma t)\,dx\,dt\\ &\le J_d(g) - \Omega_1(T) - \Omega_2(T) + \Omega_2(0)\\ &\quad + c_3 J_d(g) + c_3\left(\int\sum_\alpha g_\alpha(x,0)\,dx\right)^2\\ &\le c_4 J_d(g) + |\Omega_1(T)| + |\Omega_2(T)| + |\Omega_2(0)| + c_3\left(\int\sum_\alpha g_\alpha(x,0)\,dx\right)^2.\end{aligned} \tag{10.38}$$

Once more we apply Lemma 10.8 to yield that for every t,

$$\begin{aligned}|\Omega_1(T)| &\le c_1\lambda\int_0^T\int g_\alpha(x,t)\,dx\,dt\cdot h(c_1|B|)\\ &\qquad\times\left[1+\sup_{0\le t\le T}\int\sum_{\alpha'} g_{\alpha'}\log^+ g_{\alpha'}(x,t)\,dx\right]\\ &= T\sum_{\alpha'}\int g_{\alpha'}(x,0)\,dx,\\ |\Omega_2(t)| &\le \sum_{\alpha'}\int g_{\alpha'}(x,0)\,dx,\end{aligned}$$

where for the equality we used the conservation of mass and the fact that λ is the reciprocal of (10.36). From this and (10.37) we conclude

$$\begin{aligned}&\int_0^T\int g_\alpha(x,t)g_\beta(x,t)\mathbb{1}_B(x-v_\gamma t)\,dx\,dt\\ &\qquad\le \lambda^{-1}\left[c_5 J_d(g) + c_5\left(1+\int\sum_{\alpha'} g_{\alpha'}(x,0)\,dx\right)^2\right],\end{aligned} \tag{10.39}$$

where λ^{-1} is given by (10.36).

Final step. Let $G(x) = \int g_\alpha g_\beta(x + v_\gamma t, t)\, dt$. Set $B_l = \{x\colon G(x) \geq l\}$. Then, by (10.39),

$$
\begin{aligned}
\int \Gamma_1(G(x))\, dx &= \int \int_1^\infty b \frac{(\log l)^{b-1}}{l} \mathbb{1}(G(x) \geq l)\, dl\, dx \\
&\leq \left[c_5 J_d(g) + c_5 \left(1 + \int \sum_{\alpha'} g_{\alpha'}(x, 0)\, dx\right)^2\right] \\
&\quad \times c_1 \left[1 + \int \left(\sum_{\alpha'} g_{\alpha'}(x, 0) \log g_{\alpha'}(x, 0) \right.\right. \\
&\qquad \left.\left. - g_\alpha(x, 0) + 1 \right) dx + J_d(g) \right] \\
&\quad \times \int_1^\infty b h(c_1 |B_l|) \frac{(\log l)^{b-1}}{l}\, dl.
\end{aligned}
\tag{10.40}
$$

On the other hand,

$$
|B_l| \leq \frac{1}{l} \int G(x)\, dx.
$$

Hence

$$
\begin{aligned}
&\int_1^\infty h(c_1 |B_l|) \frac{(\log l)^{b-1}}{l}\, dl \\
&\qquad \leq \int_1^a h(c_1) \frac{dl}{l} + \int_a^\infty h\left(\frac{c_1}{l} \int G\, dx\right) \frac{(\log l)^{b-1}}{l}.
\end{aligned}
$$

If we choose $a = \max(1, (c_1 \int G\ dx)^2)$, we deduce

$$
\int_1^\infty h(c_1 |B_l|) \frac{(\log l)^{b-1}}{l}\, dl \leq c_6 \left(\log^+ \int G\, dx + 1\right)
$$

for some constant c_6 because $l > a$ implies

$$
h\left(\frac{c_1}{l} \int G\, dx\right) \leq 2 (\log l)^{-1}. \tag{10.41}
$$

On the other hand, by (10.3), the integral $\int G\, dx$ is bounded by a constant that depends on k_1 and k_2. This, (10.40) and (10.41) imply (10.6). □

We now turn to the macroscopic counterpart of our results in Section 8. Set

$$
Y(g; x) = \int_0^T \sum_{v_\alpha \neq v_\beta \neq v_\gamma} g_\alpha g_\beta(x + v_\gamma t, t)\, dt, \qquad \hat{g}_\alpha(x, t) = g_\alpha(x + v_\alpha t, t),
$$

where the summation is over distinct triplets of velocities $(v_\alpha, v_\beta, v_\gamma)$.

LEMMA 10.9. *Let* $\sigma_\alpha(x) = \int_0^T (\partial \hat{g}_\alpha/\partial t)^+(x,t)\,dx$. *Then*

$$\int_0^T \int |D_\alpha g_\alpha|\,dx\,dt \le n^2 \bar{K} \sum_{v_\alpha \ne v_\beta} \int_0^T \int g_\alpha g_\beta\,dx\,dt + \frac{J_0(g)}{\log 2}. \tag{10.42}$$

$$\int \sigma_\alpha \log \frac{\sigma_\alpha}{Y(g)}\,dx \le n^2 \bar{K} \int_0^T \int |D_\alpha g|\,dx\,dt + 2J_0(g). \tag{10.43}$$

PROOF. For every bounded measurable function $p_\alpha(x,t)$,

$$\int_0^T \int p_\alpha D_\alpha g_\alpha\,dx\,dt \le J_0(g) + n^2 \bar{K} \sum_{v_\alpha \ne v_\beta} \int_0^T \int g_\alpha g_\beta\,dx\,dt.$$

A repetition of the final step of Lemma 7.4 would lead to (10.42).

One can derive (10.41) in just the same way we derived (7.35) from (7.28). □

PROOF OF THEOREM 10.5. We first note that in fact if in Lemma 7.5 we assume $\int \Gamma_1(g)dx \le \rho_2$ instead, then we can replace Γ_4 with Γ_2 in the conclusion of that lemma. Now (10.8) follows from Lemma 10.9 and Lemma 7.5. □

PROOF OF THEOREM 10.1. It follows from Theorem 11.2 of the next section that for functions g with $J(g) \le k$, the family $\{\int g_\alpha(x,t) p_\alpha(x)\,dx\}$ is equicontinuous as a function of t. This clearly implies that the set $\{g\colon J(g) \le k\}$ is precompact. Hence it remains to show the closedness. Let $g^{(l)}$ be a sequence with $J(g^{(l)}) \le k$ such that $g^{(l)} \to g$. We would like to conclude $J(g) \le k$. For this, it suffices to verify

$$J_0(g) + J(g;p) \le k \tag{10.44}$$

for any smooth function p. Let ξ_ε be an approximation to identity and set $g^{(l)}_{\alpha,\varepsilon} = g^{(l)}_\alpha * \xi_\varepsilon$. Since $J(g^{(l)}) \le k$, we can apply Theorem 10.4 to deduce

$$\begin{aligned} &J_0(g^{(l)}) + \int_0^T \int \sum_\alpha p_\alpha D_\alpha g^{(l)}_\alpha\,dx\,dt \\ &\quad - \tfrac{1}{2} \int_0^T \int \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g^{(l)}_{\alpha,\varepsilon} g^{(l)}_{\beta,\varepsilon} \\ &\qquad\qquad \times \left[\exp(p_\gamma + p_\delta - p_\alpha - p_\beta) - 1\right] dx\,dt \le c_1 h(\varepsilon) + k, \end{aligned} \tag{10.45}$$

where c_1 is a constant that depends on T, H and k only. Since the functional $g_\alpha \mapsto g_{\alpha,\varepsilon}$ is continuous, we can pass to the limit in (10.45) to conclude

$$\begin{aligned} &J_0(g^{(l)}) + \int_0^T \int \sum_\alpha p_\alpha D_\alpha g_\alpha\,dx\,dt \\ &\quad - \tfrac{1}{2} \int_0^T \int \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g_{\alpha,\varepsilon} g_{\beta,\varepsilon} \\ &\qquad\qquad \times \left[\exp(p_\gamma + p_\delta - p_\alpha - p_\beta) - 1\right] \le c_1 h(\varepsilon) + k. \end{aligned} \tag{10.46}$$

We then let $\varepsilon \to 0$. Using Theorem 10.6, we know that the sequence $\{g_{\alpha,\varepsilon} g_{\beta,\varepsilon}\}$ is uniformly integrable. Since $g_{\alpha,\varepsilon} g_{\beta,\varepsilon} \to g_\alpha g_\beta$ almost everywhere, (10.46) implies (10.44). □

11. The rate function, part II. Recall that $\mathscr{B}$ (resp. $\mathscr{C}$) denotes the set of profiles g for which there exists a bounded (resp. smooth) $\hat{p}$ such that (1.16) holds. To complete the proof of lower bound, we need the following theorem.

THEOREM 11.1. *If $J(g) < \infty$ and $g \in \mathscr{B}$, then there exists a sequence $g_k \in \mathscr{C}$ such that g_k converges to g and $J(g_k)$ converges to $J(g)$ as k goes to infinity.*

We also establish a temporal regularity of profiles with finite rate function. Recall the function w_3 defined by (6.1).

THEOREM 11.2. *There exists a constant $B_4(k, T)$ such that if $J(g) \le k$, then for any smooth $p_\alpha(x)$,*

$$\begin{aligned}&\left|\int (g_\alpha(x,t_1) - g_\alpha(x,t_2)) p_\alpha(x)\,dx\right| \\ &\quad\le |v_\alpha| \left\|\frac{\partial p_\alpha}{\partial x}\right\|_\infty \left(\int \sum_\alpha g_\alpha(x,0)\,dx\right)|t_1 - t_2| \\ &\qquad + \|p_\alpha\|_\infty B_2(k,T)\left(w_3\left(\frac{1}{t_1 - t_2}\right)\right)^{-1}.\end{aligned} \tag{11.1}$$

As a step toward the above theorems, we derive a nonvariational formula for the rate function. Recall the function $H(g, p)$ defined by (0.5). It is not hard to show that H is convex in the p variable (see also Lemma 11.5 below). Let G denote its convex conjugate:

$$G(g, d) = \sup_p \big(p \cdot d - H(g, p)\big). \tag{11.2}$$

PROPOSITION 11.3.

$$J_d(g) = \int_0^T \int G(g, Dg)\,dx\,dt. \tag{11.3}$$

The proof of this is based on some properties of G that are formulated in the next lemma. We first state some definitions. Recall the definition of conserved vectors as in the previous section. In this section we need a g-dependent definition of conserved vectors.

DEFINITION 11.4. *Let $g \in \mathbb{R}^n$. We write A_g for the set of vectors $e \in \mathbb{R}^n$ such that*

$$e_\alpha + e_\beta = e_\gamma + e_\delta$$

whenever $K(\alpha\beta, \gamma\delta)g_\alpha g_\beta \neq 0$. *We also define*

$$A_g^\perp = \{d \in \mathbb{R}^n\colon d \cdot e = 0 \text{ for every } e \in A_g\}.$$

Finally we set $D_g = \{d\colon G(g,d) < \infty\}$.

LEMMA 11.5. (i) *If* $J(g) < \infty$, *then for almost all* (x,t),

$$(g(x,t), Dg(x,t)) \in \{(g,d)\colon d \in A_g^\perp\}. \tag{11.4}$$

(ii) $D_g \subseteq A_g^\perp$. *Moreover, if* $g_\alpha \neq 0$ *for all* α, *then* $D_g = A_g^\perp$.
(iii) $H(g,p)$ *is strictly convex on the set* $A_g^\perp$.

PROOF. (i) From the definition, $H(g,e) = 0$ if $e \in A_g$. Now take a bounded measurable function $p\colon \mathbb{T} \times [0,T] \to \mathbb{R}^n$ such that for every (x,t), $p(x,t) \in A_{g(x,t)}$. For such p we clearly have

$$\int_0^T \int \lambda p \cdot Dg\, dx\, dt \le J_d(g)$$

for every scalar-valued measurable function $\lambda(x,t)$. If $J_d(g) < \infty$, we deduce that for such p, $p \cdot Dg = 0$ for almost all (x,t). From this, it is not hard to conclude (11.4).

(ii) A version of the previous proof (i) shows that if $G(p,d) < \infty$ then $(p,d) \in A_g^\perp$.

Suppose $g_\alpha \neq 0$ for all α. To show $G(g,d) < \infty$ for all $d \in A_g^\perp$, it suffices to verify

$$\lim_{\substack{p \to \infty \\ p \in A_g^\perp}} \frac{H(g,p)}{|p|} = +\infty, \tag{11.5}$$

because (11.5) would allow us to restrict the supremum in (11.2) to a sufficiently large bounded set of p. Let $S = \{p \in A_g^\perp\colon |p| = 1\}$. If $\bar p \in S$, then for some $(\alpha,\beta,\gamma,\delta)$ we have $\bar p_\alpha + \bar p_\beta - \bar p_\gamma - \bar p_\delta \neq 0$. Since S is compact, we can find a positive ε such that whenever $\bar p \in S$ then for some $(\alpha,\beta,\gamma,\delta)$, $|\bar p_\alpha + \bar p_\beta - \bar p_\gamma - \bar p_\delta| > \varepsilon$. From (1.4)(vi), we know that if $K(\alpha\beta,\gamma\delta) \neq 0$ then $K(\gamma\delta,\alpha\beta) \neq 0$. Hence, for $t > 0$,

$$\begin{aligned} K(\alpha\beta,\gamma\delta)&g_\alpha g_\beta \exp\big(t(\bar p_\alpha + \bar p_\beta - \bar p_\gamma - \bar p_\delta)\big) \\ &+ K(\gamma\delta,\alpha\beta)g_\gamma g_\delta \exp\big(t(\bar p_\gamma + \bar p_\delta - \bar p_\alpha - \bar p_\beta)\big) \ge ce^{t\varepsilon} \end{aligned}$$

for some constant c. If $p \in A_g^\perp$, we can choose $\bar p = p/|p|$ and $t = |p|$ to conclude

$$H(g,p) \ge ce^{\varepsilon|p|} - c_1,$$

for some constant c_1 that is independent of p. This clearly implies (11.5).

(iii) A straightforward calculation yields

$$\begin{aligned}
\sum_{\alpha} \frac{\partial H}{\partial p_\alpha}(g, p) b_\alpha
&= \sum_{\alpha\beta\gamma\delta} K(\gamma\delta, \alpha\beta) g_\gamma g_\delta \exp(p_\alpha + p_\beta - p_\gamma - p_\delta) b_\alpha \\
&\quad - \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \exp(p_\gamma + p_\delta - p_\alpha - p_\beta) b_\alpha \\
&= \frac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \exp(p_\gamma + p_\delta - p_\alpha - p_\beta)(b_\gamma + b_\delta - b_\alpha - b_\beta),
\end{aligned}$$

$$\begin{aligned}
\sum_{\alpha\beta} \frac{\partial^2 H}{\partial p_\beta \partial p_\alpha}(g, p) b_\alpha b_\beta
&= \frac{1}{2} \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \exp(p_\gamma + p_\delta - p_\alpha - p_\beta)(b_\gamma + b_\delta - b_\alpha - b_\beta)^2.
\end{aligned}$$

This clearly implies the strict convexity of $H(g, \cdot)$ on $A_g^\perp$. $\square$

PROOF OF PROPOSITION 11.3. Evidently we have

$$J_d(g) \le \int_0^T \int G(g, Dg)\, dx\, dt. \tag{11.6}$$

Hence, for (11.3), we only need to show that if $J_d(g) < \infty$ then

$$J_d(g) \ge \int_0^T \int G(g, Dg)\, dx\, dt. \tag{11.7}$$

By (11.4), we may assume $Dg \in A_g^\perp$ almost everywhere. Suppose $d \in A_g^\perp$ and $p = p^1 + p^2$ with $p^1 \in A_g$ and $p^2 \in A_g^\perp$. We clearly have

$$d \cdot p - H(g, p) = d \cdot p^2 - H(g, p^2).$$

Therefore

$$G(g, d) = \sup_{p \in A_g^\perp} (d \cdot p - H(g, p)).$$

For every $d \in A_g^\perp$, define

$$G_k(g, d) = \max_{\substack{p:\ |p| \le k \\ p \in A_g^\perp}} (d \cdot p - H(g, p)). \tag{11.8}$$

Recall that $H(g, \cdot)$ is strictly convex on $A_g^\perp$. Hence the maximizer in (11.8) is unique. We denote the maximizer by $P^{(k)}(g, d)$. Clearly $P^{(k)}$ is a continuous function. We then define $p^{(k)}(x, t) = P^{(k)}(g(x, t), Dg(x, t))$. Then

$$J_d(g) \ge J_d(g; p^{(k)}) = \int_0^T \int G_k(g, Dg)\, dx\, dt.$$

Since $G_k \uparrow G$, by the monotone convergence theorem (11.7) holds. $\square$

We continue with an example to illustrate G.

EXAMPLE 11.6. Consider the left–right model of Example 1.7 with $I_1 = \{1, 2\}$ and $I_2 = \{3, 4\}$. We also assume the conservation of momentum. Assume $K(\alpha\beta, \gamma, \delta) = 1$ whenever K is nonzero. We then have $A_g = \{e\colon e_1 + e_2 = e_3 + e_4\}$ if $g_1g_2 + g_3g_4 \neq 0$ and $A_g = \mathbb{R}^4$ otherwise. Hence if $g_1g_2 + g_3g_4 \neq 0$,

$$A_g^\perp = \{d\colon d = (r, r, -r, -r) \text{ for some } r \in \mathbb{R}\}.$$

Moreover

$$D_g = A_g^\perp \quad \text{if } g_1g_2g_3g_4 \neq 0,$$
$$D_g = \{(r, r, -r, -r)\colon r \geq 0\} \quad \text{if } g_1g_2 = 0,\ g_3g_4 \neq 0,$$
$$D_g = \{(r, r, -r, -r)\colon r \leq 0\} \quad \text{if } g_1g_2 \neq 0,\ g_3g_4 = 0.$$

In the same order, the corresponding $G(g, d)$'s are

$$G(g, d) = g_1g_2\psi(\Delta) + g_3g_4\psi(-\Delta), \quad \Delta = \log \frac{-r + \sqrt{r^2 + 4g_1g_2g_3g_4}}{2g_1g_2},$$
$$G(g, d) = |r| \log \frac{|r|}{g_3g_4} - |r| + g_3g_4,$$
$$G(g, d) = |r| \log \frac{|r|}{g_1g_2} - |r| + g_1g_2.$$

The next issue we would like to address is the existence of $\hat{p}$ in (1.16). As the previous example illustrated, it is possible to have $D_g \neq A_g^\perp$. This in fact corresponds to examples for which $\hat{p}$ becomes infinite. Hence some care is needed for the meaning of (1.16).

PROPOSITION 11.7. *Suppose $J(g) < \infty$. Then there exist measurable functions $\Delta(\alpha\beta, \gamma\delta)\colon \mathbb{T} \times [0, T] \to [-\infty, +\infty)$ such that for all α,*

$$D_\alpha g_\alpha = \sum_{\beta,\gamma,\delta} K(\gamma\delta, \alpha\beta) g_\gamma g_\delta \exp(\Delta(\alpha\beta, \gamma\delta)) - \sum_{\beta,\gamma,\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \exp(\Delta(\gamma\delta, \alpha\beta)), \tag{11.19}$$

$$J(g) = \int_0^T \int \sum_{\alpha\beta\gamma\delta} K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \psi(\Delta(\gamma\delta, \alpha\beta))\, dx\, dt. \tag{11.20}$$

As Proposition 11.7 indicates, it is more convenient to deal with the differences $p_\gamma + p_\delta - p_\alpha - p_\beta$ than p_α. Note that by (11.20) we know that $\psi(\Delta(\gamma\delta, \alpha\beta)) < +\infty$ almost everywhere. Hence $\Delta(\gamma\delta, \alpha\beta) < +\infty$ whenever $K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \neq 0$. Because of this, the expression $K(\alpha\beta, \gamma\delta) g_\alpha g_\beta \exp(\Delta(\gamma\delta, \alpha\beta))$ is well defined.

PROOF OF PROPOSITION 11.7. Fix g and consider the convex functions $H(g,\cdot)$ and $G(g,\cdot)$. Let B_g denote the *affine hull* of D_g, that is, the intersection of all affine sets which contain D_g. It is known that the topological interior of D_g as a subset of B_g is contained in the range of the gradient $\partial H/\partial p$ (see, for example, page 224 of [3]). Hence if $G(g,d)<\infty$, then there exists a sequence $p^{(k)}$ such that

$$d^{(k)} := \frac{\partial H}{\partial p}(g, p^{(k)}) \to d. \tag{11.21}$$

It is not hard to show that in fact $d^{(k)} = d^{(k)}(g,d) \in D_g$ can be choosen in a measurable manner. Since H is strictly convex on $A_g^{\perp}$, the function $p^{(k)} = p^{(k)}(g,d)$ is uniquely defined and measurable. Since $G(g,\cdot)$ is lower semicontinuous,

$$\liminf_k G(g, d^{(k)}) \geq G(g,d). \tag{11.22}$$

Clearly, $d^{(k)} \in D_f$. As a result,

$$G(g, d^{(k)}) = d^{(k)} \cdot p^{(k)} - H(g, p^{(k)}) \leq G(g,d).$$

This and (11.22) imply $\lim_k G(g, d^{(k)}) = G(g,d)$. On the other hand,

$$\begin{aligned} G(g, d^{(k)}) &= \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g_\alpha g_\beta \psi\big(\Delta^{(k)}(\alpha\beta,\gamma\delta)\big), \\ d_\alpha^{(k)} &= \sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta) g_\gamma g_\delta \exp\big(\Delta^{(k)}(\alpha\beta,\gamma\delta)\big) \\ &\quad - \sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta) g_\alpha g_\beta \exp\big(\Delta^{(k)}(\gamma\delta,\alpha\beta)\big), \end{aligned} \tag{11.23}$$

where

$$\Delta^{(k)}(\alpha\beta,\gamma\delta) = \Delta^{(k)}(\alpha\beta,\gamma\delta; g, d) = p_\alpha^{(k)} + p_\beta^{(k)} - p_\gamma^{(k)} - p_\delta^{(k)}.$$

From (11.22) and (11.23) we learn that the sequence $\exp(\Delta^{(k)}(\alpha\beta,\gamma\delta))$ is uniformly bounded. We then use a subsequence so that $\exp(\Delta^{(k)}(\alpha\beta,\gamma\delta))$ converges. The limit will be denoted by $\exp(\Delta(\alpha\beta,\gamma\delta; g, d))$. We clearly have

$$\begin{aligned} d_\alpha &= \sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta) g_\gamma g_\delta \exp\big(\Delta(\alpha\beta,\gamma\delta)\big) \\ &\quad - \sum_{\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g_\alpha g_\beta \exp\big(\Delta(\gamma\delta,\alpha\beta)\big), \\ G(g,d) &= \sum_{\alpha\beta\gamma\delta} K(\alpha\beta,\gamma\delta) g_\alpha g_\beta \psi\big(\Delta(\alpha\beta,\gamma\delta)\big). \end{aligned}$$

From the measurability of $p^{(k)}$, we conclude that Δ is a measurable function in (g,d) variable. Finally we choose $\Delta(\alpha\beta,\gamma\delta)(x,t) = \Delta(\alpha\beta,\gamma\delta; g(x,t), Dg(x,t))$. $\square$

PROOF OF THEOREM 11.2. We certainly have

$$\left|\int (g_\alpha(x,t_2)-g_\alpha(x,t_1))p_\alpha(x)\right|$$

$$\leq \left|\int_{t_1}^{t_2}\int p_\alpha D_\alpha g_\alpha\,dx\,dt\right| + \left|\int_{t_1}^{t_2}\int v_\alpha \frac{\partial p_\alpha}{\partial x} g_\alpha\,dx\,dt\right| \tag{11.24}$$

$$\leq \left|\int_{t_1}^{t_2}\int p_\alpha D_\alpha g_\alpha\,dx\,dt\right| + |v_\alpha|\left\|\frac{\partial p_\alpha}{\partial x}\right\|_\infty |t_2-t_1|\int\sum_\alpha g_\alpha(x,0)\,dx,$$

where, for the second inequality, we used the conservation of mass. From (11.19) we learn that

$$\left|\int_{t_1}^{t_2}\int p_\alpha D_\alpha g_\alpha\,dx\,dt\right|$$

$$\leq \int_{t_1}^{t_2}\int |p_\alpha|\sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta)g_\gamma g_\delta \exp(\Delta(\alpha\beta,\gamma\delta))\,dx\,dt \tag{11.25}$$

$$+\int_{t_1}^{t_2}\int |p_\alpha|\sum_{\beta\gamma\delta} K(\alpha\beta,\gamma\delta)g_\alpha g_\beta \exp(\Delta(\gamma\delta,\alpha\beta))\,dx\,dt$$

$$=: \Omega_1+\Omega_2.$$

Furthermore, for every l greater than 2,

$$\Omega_1 = \int_{t_1}^{t_2}\int |p_\alpha|\sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta)g_\gamma g_\delta \exp(\Delta(\alpha\beta,\gamma\delta)) \times \mathbb{1}(\Delta(\alpha\beta,\gamma\delta)\leq l)\,dx\,dt$$

$$+\int_{t_1}^{t_2}\int |p_\alpha|\sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta)g_\gamma g_\delta \exp(\Delta(\alpha\beta,\gamma\delta)) \times \mathbb{1}(\Delta(\alpha\beta,\gamma\delta)\geq l)\,dx\,dt$$

$$\leq c_1\|p_\alpha\|_\infty e^l\int_{t_1}^{t_2}\int\sum_{v_\beta\neq v_\delta} g_\gamma g_\delta\,dx\,dt \tag{11.26}$$

$$+\frac{1}{l}\|p_\alpha\|_\infty\int_{t_1}^{t_2}\int\sum_{\beta\gamma\delta} K(\gamma\delta,\alpha\beta)g_\gamma g_\delta \exp(\Delta(\alpha\beta,\gamma\delta))\Delta(\alpha\beta,\gamma\delta) \times \mathbb{1}(\Delta(\alpha\beta,\gamma\delta)\geq 2)\,dx\,dt,$$

$$\leq c_1\|p_\alpha\|_\infty e^l\int_{t_1}^{t_2}\int\sum_{v_\beta\neq v_\gamma} g_\beta g_\gamma\,dx\,dt + \frac{2}{l}\|p_\alpha\|_\infty J_d(g)$$

$$=: c_1\|p_\alpha\|_\infty e^l\Omega_{11}(t_1,t_2)+\tfrac{2}{l}\|p_\alpha\|_\infty J_d(g),$$

where for the last inequality we used (11.20) and the elementary inequality $2\psi(\Delta)\geq \Delta e^\Delta$ for $\Delta\geq 2$. On the other hand, we can use Theorem 10.5 to assert that for any r greater than e^e,

$$\begin{aligned}\Omega_{11}(t_1,t_2) &= \int_{t_1}^{t_2}\int \sum_{v_\beta\neq v_\gamma} g_\beta g_\gamma \mathbb{1}(g_\beta g_\gamma \le r)\,dx\,dt \\ &\quad + \int_{t_1}^{t_2}\int \sum_{v_\beta\neq v_\gamma} g_\beta g_\gamma \mathbb{1}(g_\beta g_\gamma > r)\,dx\,dt \\ &\le (t_2-t_1)r + \frac{1}{(w_2(r))^b}\int_{t_1}^{t_2}\int \sum_{v_\beta\neq v_\gamma} \Gamma(g_\beta g_\gamma)\,dx\,dt \\ &\le (t_2-t_1)r + \frac{1}{(w_2(r))^b}c_1(k)\end{aligned} \tag{11.27}$$

for a constant $c_1(k)$ that depends on k only. If we choose

$$r = (t_2-t_1)^{-1}\left(w_2\left(\frac{1}{t_2-t_1}\right)\right)^{-b}$$

in (11.27), we obtain

$$\Omega_{11}(t_1,t_2) \le c_2(k)\left(w_2\left(\frac{1}{t_2-t_1}\right)\right)^{-b}$$

for a suitable constant $c_2(k)$. Substituting this in (11.26) yields

$$\Omega_1 \le \|p_\alpha\|_\infty\left[c_2(k)c_1\left(w_2\left(\frac{1}{t_2-t_1}\right)\right)^{-b}e^l + \frac{2k}{l}\right].$$

We now choose

$$l = \log\left(\log\log\frac{1}{\delta}\right)^{b/2}$$

with $\delta = t_2 - t_1$ to conclude

$$\Omega_1 \le \|p_\alpha\|_\infty c_3(k)\left(w_3\left(\frac{1}{t_2-t_1}\right)\right)^{-1} \tag{11.28}$$

for some constant $c_3(k)$. The term Ω_2 can be treated likewise. This, (11.28), (11.25) and (11.24) imply (11.1). □

For Theorem 11.1 we need the following lemma that would guarantee the boundedness of g when $g \in \mathscr{B}$.

LEMMA 11.8. *There exists a constant $B_5(k,T)$ such that if (11.19) holds for a family of measurable functions $\Delta(\alpha\beta,\gamma\delta)$ with*

$$\sum_{\alpha,\beta,\gamma,\delta}\|\Delta(\alpha\beta,\gamma\delta)\|_\infty \le k,$$

$$\sum_\alpha \int g_\alpha(x,0)\,dx \le k,$$

then

$$\big|g_\alpha(x,t)\big| \le B_5(k,T)\big\|g_\alpha(\cdot,0)\big\|_\infty \tag{11.29}$$

for almost all $(x,t) \in \mathbb{T}\times[0,T]$.

The proof of this lemma is omitted because it follows the proof of Theorem 1 in [1].

The proof of the following lemma can be found in [16] in the case of $\Delta = 0$. Toscani's proof can be readily generalized to treat the general case.

LEMMA 11.9. *Let* $g^{(k)}$ *be a sequence of solutions to (11.19) with*

$$\sup_t \sup_k \int \sum_\alpha g_\alpha^{(k)}(x,t)\log^+ g_\alpha^{(k)}(x,t)\,dx < \infty. \tag{11.30}$$

Suppose that the function Δ *is bounded and*

$$\lim_{k\to\infty}\int\big|g^{(k)}(x,0)-g(x,0)\big|\,dx = 0, \tag{11.31}$$

where g *is another solution to (11.19) of finite entropy. Then,*

$$\lim_{k\to\infty}\int_0^T\int\big|g^{(k)}(x,t)-g(x,t)\big|\,dx\,dt = 0. \tag{11.32}$$

PROOF OF THEOREM 11.1.

Step 1. Assume $J(g)<\infty$ and $g\in\mathscr{B}$. First we want to replace the initial data by a smooth function. More precisely, let $g^{(k)}(x,0)$ be a sequence of smooth functions such that (11.31) holds. Let $g^{(k)}(x,t)$ [resp. $g(x,t)$] be the unique solution of (11.19) with the initial data $g^{(k)}(x,0)$ [resp. $g(x,0)$]. We would like to conclude that for a subsequence of $g^{(k)}$, we have

$$\lim_{k\to\infty} J(g^{(k)}) = J(g). \tag{11.33}$$

By Lemma 11.8 we have that $g^{(k)}$ converges to g in the L^1-norm. As a result, for a subsequence $g^{(k)}$ converges to g almost everywhere. Using the conservation of momentum, as in the proof of (10.3) we deduce that

$$\sup_k \sum_{v_\beta\ne v_\gamma}\int_0^T\int g_\alpha^{(k)}g_\beta^{(k)}\,dx\,dt < \infty. \tag{11.34}$$

Since the corresponding Δ is bounded and independent of k, (11.20) implies

$$\sup_k J(g^{(k)}) < \infty. \tag{11.35}$$

This and Theorem 10.3 imply (11.30). By Lemma 11.8 we have that $g^{(k)}$ converges to g in the L^1-norm. As a result, for a subsequence $g^{(k)}$ converges to g almost everywhere. Moreover, by Theorem 10.6 we have

$$\sup_k \int_0^T\int\sum_{v_\beta\ne v_\gamma}\Gamma_2\big(g_\alpha^{(k)}g_\beta^{(k)}\big)\,dx\,dt < \infty. \tag{11.36}$$

This implies the uniform integrability of the sequence $g_\alpha^{(k)} g_\beta^{(k)}$. Since Δ is bounded and $g^{(k)}$ converges to g almost everywhere for a subsequence, we have (11.33) for a subsequence.

Step 2. From the previous step we learn that we may assume $g(\cdot, 0)$ is smooth. By Lemma 11.8 we deduce that $g(x,t)$ is bounded for $(x,t) \in \mathbb{T} \times [0, T]$. We then approximate p by a sequence of smooth functions $p^{(k)}$:

$$\sup_k \|p^{(k)}\|_\infty < \infty,$$
$$\lim_{k\to\infty} \|p - p^{(k)}\|_{L^1} = 0. \tag{11.37}$$

Let $g^{(k)}$ denote the corresponding solution to (1.16) where p is replaced with $p^{(k)}$ and the initial density is $g(\cdot, 0)$. From Lemma 11.7, we have

$$\sup_k \|g^{(k)}\|_{L^\infty(\mathbb{T}\times[0,T])} < \infty.$$

Moreover, by standard arguments we can show that $g^{(k)}$ is smooth (see, e.g., [4]). Using the analog of (1.10) for our equation (1.16), it is not hard to show that for almost all t,

$$\begin{aligned} &\int \sum_\alpha \left| g_\alpha^{(k)}(x,t) - g_\alpha(x,t) \right| dx \\ &\quad \le c_1 \int_0^t \int \sum_{v_\alpha \ne v_\beta} \Big| \exp\big(p_\gamma^{(k)} + p_\delta^{(k)} - p_\alpha^{(k)} - p_\beta^{(k)}\big) g_\alpha^{(k)} g_\beta^{(k)} \\ &\qquad\qquad - \exp(p_\gamma + p_\delta - p_\alpha - p_\beta) g_\alpha g_\beta \Big| \, dx \, dt \end{aligned} \tag{11.38}$$

for some constant c_1. Since $|p^{(k)}|$, $|g^{(k)}|$, $|g|$ are all bounded by a constant, the right-hand side of (11.38) is bounded above by a constant multiple of

$$\int_0^t \int \sum_\alpha |p_\alpha^{(k)} - p_\alpha| + \sum_\alpha |g_\alpha^{(k)} - g_\alpha| \, dx \, dt.$$

From this, (11.37) and Gronwall's inequality we deduce

$$\lim_{k\to\infty} \int_0^T \int \sum_\alpha \left| g_\alpha^{(k)}(x,t) - g_\alpha(x,t) \right| dx \, dt = 0.$$

As a result, a subsequence of $g_\alpha^{(k)}$ converges to g_α almost everywhere. From this, (11.20), and the bounded convergence theorem, we conclude

$$\lim_{k\to\infty} J(g^{(k)}) = J(g). \qquad \square$$

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