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WIENER'S TEST FOR RANDOM WALKS WITH MEAN ZERO AND FINITE VARIANCE

BY KÔHEI UCHIYAMA

Tokyo Institute of Technology

It is shown that an infinite subset of $\mathbf{Z}^N$ is either recurrent for each aperiodic N-dimensional random walk with mean zero and finite variance, or transient for each of such random walks. This is an exact extension of the result by Spitzer in three dimensions to that in the dimensions $N \geq 4$.

1. Introduction. A random walk on the N-dimensional integer lattice $\mathbf{Z}^N$ is a stochastic process Y_n of the form $Y_n = \xi_1 + \cdots + \xi_n$, where ξ_i, $i = 1, 2, \dots,$ is a sequence of independent and identically distributed $\mathbf{Z}^N$-valued random variables defined on a probability space (Ω, F, P). Let $\{p(x),\ x \in \mathbf{Z}^N\}$ be the common distribution of ξ_i: $p(x) = P\{\xi_1 = x\}$. We will assume that $p(x)$ is aperiodic (i.e., the smallest additive subgroup containing $\{x \in \mathbf{Z}^N\colon p(x) > 0\}$ agrees with $\mathbf{Z}^N$) and has zero mean and a finite variance:

$$\sum p(x)x = 0 \quad \text{and} \quad \sum p(x)|x|^2 < \infty. \tag{1.1}$$

A subset A of $\mathbf{Z}^N$ is called *transient* if A is visited by Y_n only finitely many times with probability 1. If A is not transient, then by the Hewitt–Savage 0–1 law it is visited by Y_n infinitely many times with probability 1 and is called *recurrent*. If $N = 1$ or 2, every nonempty subset is recurrent, and if $N \geq 3$, every finite set is transient (cf. [5]). In this paper we prove the following theorem.

THEOREM 1. *Let $N \geq 3$. Then an infinite subset of $\mathbf{Z}^N$ is either recurrent for each aperiodic random walk Y_n on $\mathbf{Z}^N$ with mean zero and a finite variance, or transient for each of such random walks.*

For $N = 3$ this is Theorem 26.2 of [5]. Theorem 1 may be also regarded as an extension of Wiener's test for the N-dimensional standard simple random walk ($N \geq 3$) obtained by Itô and McKean [3] (see Theorem 2 in the present paper), which is a natural analogue of the classical Wiener's test that characterizes a regular boundary point to the Dirichlet problem for the Laplace operator in a Euclidean domain.

According to [3] an infinite subset A of $\mathbf{Z}^N$ is transient for the standard simple random walk X_n if and only if

$$\sum_{k=1}^{\infty} 2^{-k}\,\mathrm{Cap}(A_k) < \infty. \tag{1.2}$$

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Here $A_k = \{x \in A\colon 2^k < |x|^{N-2} \le 2^{k+1}\}$ and $\mathrm{Cap}(F)$ denotes the capacity of a finite set $F \subset \mathbf{Z}^N$ and may be given as follows. Let $G(x, y)$ be the Green's function of X_n: $G(x, y) = \sum_{n=0}^{\infty} P\{X_n = y | X_0 = x\}$. Then

$$\mathrm{Cap}(F) = \sup\left\{ \sum_{x \in F} \mu(y)\colon \sum_{y \in F} G(x, y)\mu(y) \le 1 \text{ for } x \in F \right\}. \tag{1.3}$$

The proof in [3] makes use, in a significant manner, of the fact that $G(x, y)$ decays in the same order as the Newtonian potential, namely,

$$\frac{a}{|y - x|^{N-2} + 1} \le G(x, y) \le \frac{b}{|y - x|^{N-2} + 1},$$

where a and b are positive constants and $|\cdot|$ denotes the Euclidean length. In three dimensions this is true for every aperiodic random walk Y_n having zero mean and a finite variance, and the same proof works to verify Wiener's test for those random walks, which in turn, with the help of (1.3), proves Theorem 1 when $N = 3$ (cf. [5]).

In higher dimensions the Green function $G^*(x, y)$ of the random walk Y_n does not generally behave like the Newtonian potential: in fact $|y - x|^{N-2} G^*(x, y)$ is unbounded for a large class of p satisfying (1.1). One can, however, show that there are a nonnegative function $\pi(x)$ that is summable on $\mathbf{Z}^N$ and a positive constant m such that

$$\frac{m}{|y - x|^{N-2} + 1} \le G^*(x, y) \le \sum_z \frac{\pi(z - x)}{|y - z|^{N-2} + 1}. \tag{1.4}$$

This bound of G^*, although admitting the possibility that $G^*(0, y)$ may decay to zero with arbitrarily slow rate along a suitable sequence of y, turns out to be enough to guarantee that the same condition (1.2) serves as a necessary and sufficient condition for A to be transient with respect to the random walk Y_n; hence Theorem 1 follows. We will provide this part of the proof of Theorem 1 in a general setting as in [4] so that the result can be applied to a large class of transient Markov processes, while the bound (1.4) is virtually a corollary of results of [6].

2. Wiener's test. In this section we briefly review a result of [4] in which Wiener's test is formulated and proved for a class of Markov chains.

Let X_n, $n = 0, 1, 2, \ldots,$ be a transient Markov chain on a discrete state space S. Let $P^n(x, y)$, $x, y \in S$, be the n-step transition probability of the chain and let $G(x, y)$ be its Green function, $G(x, y) = \sum_{n=0}^{\infty} P^n(x, y)$. Then the hitting probability

$$h(x, F) := P\{\exists\ n \ge 0,\ X_n \in F \mid X_0 = x\}$$

of a finite set $F \subset S$ of the chain starting at x is represented as

$$h(x, F) = Ge(x) := \sum_{y \in S} G(x, y) e(y), \tag{2.1}$$

where $e(y) = 0$ for $y \notin F$ and

$$e(y) = P\{\forall\ n \geq 1,\ X_n \notin F \mid X_0 = y\} \quad \text{for } y \in F.$$

We call $e(\cdot)$ the equilibrium charge of F according to the nomenclature of potential theory. The total charge $e\{F\} := \sum e(x)$ is denoted by $\mathrm{Cap}(F)$, called the capacity of F. [For random walks it has the variational representation (1.3).] For each positive integer k let us define a "ball" $B_k(x)$ of center at x by

$$B_k(x) = \{y \in S\colon G(x, y) \geq 2^{-k}\}.$$

Let us fix a point $x_0 \in S$ and assume that $B_k(x_0)$ is a finite set for each k and that $G(x_0, y) > 0$ for all y. As in [4], we introduce the following condition to be satisfied by G:

(i) there exist a positive integer p and a positive number λ such that, for all sufficiently large k,

$$G(x, y) \leq \lambda G(x_0, y) \quad \text{if } x \notin S'_k \text{ and } y \in S_k,$$

where

$$S_k := B_{k+1}(x_0) \setminus B_k(x_0) \quad \text{and} \quad S'_k := B_{k+p}(x_0) \setminus B_{k-p+1}(x_0).$$

(The condition is exactly the same as that in [4], although given somewhat differently.)

LAMPERTI'S THEOREM [4]. *Suppose that $X_0 = x_0$ with probability 1 and condition (i) is satisfied. Then, for any infinite subset A of S,*

$$P(X_n \in A \text{ infinitely often}) = 0 \quad \text{if and only if } \sum \mathrm{Cap}(A \cap S_k)2^{-k} < \infty.$$

One observes that, in view of (2.1),

$$2^{-k-1}\,\mathrm{Cap}(A_k) \leq h(x_0, A_k) \leq 2^{-k}\,\mathrm{Cap}(A_k), \qquad A_k := A \cap S_k, \tag{2.2}$$

and then realizes that the "if" part of the equivalence of the theorem above is immediate from the trivial half of the Borel–Cantelli lemma. We will apply the following version of the other half (cf. [5], P26.3).

LEMMA 1. *Let E_k be a sequence of events. If $\sum_{k=1}^{\infty} P\{E_k\} = \infty$ and*

$$\limsup_{n\to\infty} \sum_{m=1}^{n} \sum_{k=1}^{n} P\{E_k \cap E_m\} \Bigg/ \left(\sum_{k=1}^{n} P\{E_k\}\right)^2 < \infty, \tag{2.3}$$

then $P\{\limsup E_k\} > 0$.

3. Equivalence between two Markov chains. Let X_n, $G(x, y)$, $B_k(x)$ and so on be as in the preceding section. Let X_n^* be another Markov chain on the same state space S. Denote by $G^*(x, y)$ and $h^*(x, F)$, respectively, the

Green's function and the hitting probability of F relative to X^*. In addition to the condition (i) imposed on G in the preceding section we need the following conditions, where M denotes some positive constant:

(ii)(a) $G(x, y) \le MG^*(x, y)$, for all x, y;
(ii)(b) there exists a function $\pi_x(y) \ge 0$ such that, for all x, y,

$$G^*(x, y) \le \sum_{z \in \mathbf{Z}^N} \pi_x(z) G(z, y)$$

and

$$M_1 := \sup_x \pi_x\{S\} < \infty.$$

Here $\pi_x\{A\} = \sum_{y \in A} \pi_x(y)$.

THEOREM 2. *If $X_0 = X_0^* = x_0$ with probability* 1 *and conditions* (i) *and* (ii) *are satisfied, then, for any infinite subset A of S,*

$$P(X_n^* \in A \textit{ infinitely often}) = 0$$

if and only if

$$P(X_n \in A \textit{ infinitely often}) = 0;$$

and these are the case if and only if $\sum h^*(x_0, A \cap S_k) < \infty$.

LEMMA 2. *If* (ii)(b) *is satisfied, then, for every finite subset F of S,*

$$G^* e_F \le M_1 h^*(\cdot, F),$$

where e_F denotes the equilibrium charge of F relative to X.

PROOF. Let $e = e_F$. Noticing $Ge = h(\cdot, F) \le 1$, we deduce from condition (ii)(b) that

$$G^* e(x) \le \pi_x\{S\} \le M_1. \tag{3.1}$$

Let $H_F^*(x, \cdot)$ denote the hitting distribution of F for X^* starting at x. Then for any function φ that vanishes outside F we have $G^*\varphi = H_F^* G^* \varphi$. This verifies that

$$G^* e(x) \le \left[\sup_y G^* e(y)\right] h^*(x, F).$$

The inequality of the lemma now follows if one substitutes the bound (3.1) for the supremum. □

LEMMA 3. *Suppose that conditions* (i) *and* (ii) *hold. Then there exist constants C_1, C_1' and K such that if $A_k = A \cap S_k$ and $k \ge K$,*

$$h^*(x, A_k) \le C_1 h(x_0, A_k) + M\pi_x\{S_k'\} \quad \textit{for all } x, \tag{3.2}$$

$$h(x_0, A_k) \le C_1' h^*(x_0, A_k). \tag{3.3}$$

In particular,

$$\sum_{k=1}^{\infty} h^*(x_0, A_k) < \infty \quad \textit{if and only if} \quad \sum_{k=1}^{\infty} h(x_0, A_k) < \infty.$$

PROOF. Let e and e^* be the equilibrium charge of A_k relative to X and X^*, respectively. We apply condition (ii)(a) and Lemma 2 in turn, to obtain

$$h(x, A_k) = Ge(x) \le MG^*e(x) \le MM_1 h^*(x, A_k).$$

Thus (3.3) is proved. For the proof of (3.2) we apply (ii)(b) to observe

$$h^*(x, A_k) = G^*e^*(x) \le \sum_{z \in \mathbf{Z}^N} \pi_x(z) Ge^*(z). \tag{3.4}$$

By applying Lemma 2 with the roles of X and X^* interchanged it follows from (ii)(a) that

$$Ge^* \le Mh(x, A_k). \tag{3.5}$$

If $z \notin S'_k$, then, owing to (i), $G(z, y) \le \lambda G(x_0, y)$ for $y \in A_k$, implying $Ge^*(z) \le \lambda Ge^*(x_0)$. This together with (3.5) shows that

$$Ge^*(z) \le \lambda Mh(x_0, A_k) \quad \text{if } z \notin S'_k. \tag{3.6}$$

Decomposing the sum in (3.4) into that over S'_k and the rest and applying (3.5) and (3.6) we obtain (3.2) with $C_1 = \lambda MM_1$. The proof of Lemma 3 is complete. □

PROOF OF THEOREM 2. Set $A_k = A \cap S_k$. According to Lamperti's theorem together with relation (2.2) and Lemma 3 it suffices to show that

$$P(X_n^* \in A \text{ infinitely often}) > 0 \quad \text{if } \sum_k h^*(x_0, A_k) = \infty.$$

For the proof we may assume that $\sum_k h^*(x_0, A_{pk}) = \infty$: otherwise one may replace A_{pk} by one of $A_{pk+1}, A_{pk+1}, \dots,$ or A_{pk+p-1}. We are to apply Lemma 1 to the event $E_k := \{A_{pk}$ is hit by $X^*\}$. We must verify its hypothesis (2.3). It follows from (3.2) and (3.3) that, for all x,

$$h^*(x, A_{pk}) \le Ch^*(x_0, A_{pk}) + M\pi_x\{S'_{pk}\}$$

$(C = C_1 C'_1)$ and from this and condition (ii)(b) with the help of the strong Markov property that, for $k > m > K$,

$$\begin{aligned} &P\{E_k \cap E_m\} \\ &\quad \le P\{E_k \cap E_m;\ E_k \text{ occurs first}\} + P\{E_k \cap E_m;\ E_m \text{ occurs first}\} \\ &\quad \le 2CP\{E_m\}P\{E_k\} + M\sum_{x \in A_{pm}} \nu^{(m)}(x)\pi_x\{S'_{pk}\} + M\sum_{x \in A_{pk}} \nu^{(k)}(x)\pi_x\{S'_{pm}\}, \end{aligned}$$

where $\nu^{(k)}(x)$ is the probability that the chain X_n^* hits A_{pk} and the first hitting site is x. Since $\sum_{x \in A_{pk}} \nu^{(k)}(x) = P\{E_k\}$, summing up both sides of this inequality we get

$$\sum_{m=1}^{n}\sum_{k=1}^{n} P\{E_k \cap E_m\} \leq 2C\left(\sum_{k=1}^{n} P\{E_k\}\right)^2 + \left(4M\sup_x \pi_x\{S\}\right)\sum_{k=1}^{n} P\{E_k\},$$

and hence (2.3) as required. The proof of Theorem 2 is complete. □

In the next theorem, which is not used in the proof of Theorem 1, condition (ii)(a) is replaced by (ii)(b) with the roles of G and G^* interchanged.

THEOREM 3. *If $X_0 = X_0^* = x_0$ with probability* 1, *the process X_n^* is irreducible, conditions* (i) *and* (ii)(b) *are satisfied and there exists a nonnegative function $\pi_x^*(z)$ such that, for all x, y,*

$$(3.7)\quad G(x,y) \leq \sum_{z\in\mathbf{Z}^N} \pi_x^*(z)G^*(z,y) \quad \textit{and} \quad M_1^* := \sup_x \pi_x^*\{S\} < \infty,$$

then the conclusion of Theorem 2 *is valid.*

PROOF. Our proof of Theorem 2 was based only on inequalities (3.2) and (3.3). From (3.7) we get (3.5) with M_1^* in place of M and from the latter the inequality (3.2) with $C_1^* := \lambda M_1^* M_1$ in place of C_1. Instead of (3.3) it is enough to show that there exist constants C_2 and $\gamma_k \geq 0$ such that

$$(3.8)\qquad h(x_0, A_k) \leq C_2 h^*(x_0, A_k) + \gamma_k \quad \text{with } \sum \gamma_k < \infty.$$

For the proof of (3.8) let e and e^* be the equilibrium charges of A_k relative to X and X^*, respectively. By Lemma 2,

$$(3.9)\qquad \begin{aligned} h(x_0, A_k) &= Ge(x_0) \leq \sum_z \pi_{x_0}^*(z) G^*e(z) \\ &\leq M_1 \sum_z \pi_{x_0}^*(z) h^*(z, A_k). \end{aligned}$$

From the irreducibility of X^* it follows that the ratio $G^*(x,y)/G^*(x_0,y)$ is bounded as y ranges over S for each x fixed. For $r > 0$ set

$$\beta_r := \sup_{x\in B_r(x_0)} \sup_{y\in S} \frac{G^*(x,y)}{G^*(x_0,y)} < \infty.$$

Then

$$(3.10)\qquad h^*(x, A_k) \leq \beta_r h^*(x_0, A_k) \quad \text{for } x \in B_r(x_0),$$

and, by (3.2),

$$\begin{aligned} &\sum_z \pi_{x_0}^*(z) h^*(z, A_k) \\ &\qquad \leq \pi_{x_0}^*\{B_r(x_0)\}\beta_r h^*(x_0, A_k) + C_1^* \pi_{x_0}^*\{S \setminus B_r(x_0)\} h(x_0, A_k) + \gamma_k', \end{aligned}$$

where $\gamma_k' = M_1^* \sum_z \pi_{x_0}^*(z)\pi_z\{S_k'\}$. Now, choosing the number r so that

$$\pi_{x_0}^*\{S \setminus B_r(x_0)\} < (2M_1C_1^*)^{-1},$$

we infer from (3.9) that

$$\tfrac{1}{2}h(x_0, A_k) \le \left[M_1 \pi^*_{x_0}\{B_r(x_0)\}\beta_r \right] h^*(x_0, A_k) + M_1\gamma'_k.$$

Thus (3.8) holds with $\gamma_k = 2M_1\gamma'_k$ and $C_2 = 2M_1^*\pi^*_{x_0}\{B_r(x_0)\}\beta_r$. The proof of Theorem 3 is complete. □

REMARK 1. The irreducibility of X^* in Theorem 3 is not crucial at all: it is used only to show (3.10), which may further be replaced by a rather milder condition that, for each $r > 0$,

$$h^*(x, A_k) \le \alpha_r h^*(x_0, A_k) + \gamma_{r,k} \quad \text{for } x \in B_r(x_0) \tag{3.11}$$

with $\sum_k \gamma_{r,k} < \infty$ and $\alpha_r < \infty$. Space−time random walks are the only reducible chains for which the present author knows Wiener's test is studied and interesting. Every pair of two one-dimensional space−time random walks having zero mean and the same variance satisfies conditions (ii)(b), (3.7) and (3.11). Although Theorem 3 is not applicable since condition (i) fails to hold, combination of Lemma 2 and a result of [2], in which Wiener's test for the space−time walk is proved, implies that its conclusion is true for such pairs under $\sum p(x)x^2[\ln(|x| \vee 1)]^{1+\delta} < \infty$ (and false under merely $\sum p(x)x^2 < \infty$ according to a result of [1]).

REMARK 2. If the variational formula (1.3) holds for both Cap and Cap*, capacities relative to X and X^*, respectively, then they are comparable under condition (ii): in fact $M^{-1}\,\mathrm{Cap}^*(F) \le \mathrm{Cap}(F) \le M_1\,\mathrm{Cap}^*(F)$ for all F, as easily shown by using (3.1) and its dual $Ge^* \le M$. For the validity of (1.3) (for X) it is sufficient that the uniform measure is excessive, that is, $\sum_x P(x, \cdot) \le 1$. In general (1.3) may fail to hold.

4. Proof of Theorem 1. Let $\{p_k(y - x),\ x, y \in \mathbf{Z}^N\}$, $k = 0, 1, \dots,$ be the k-step transition probability of an aperiodic random walk on $\mathbf{Z}^N$ with mean 0 and a finite variance, and let $\{g(y - x),\ x, y \in \mathbf{Z}^N\}$ be its Green function,

$$g(x) = \sum_{k=0}^{\infty} p_k(x).$$

We are to apply Theorem 2 with $S = \mathbf{Z}^N$, $G^*(x, y) = g(y - x)$ and with X_n being the standard simple random walk. Green's function $G(x, y)$ of X_n is known to have the asymptotic form

$$G(x, y) = \lambda_N |x - y|^{-(N-2)}(1 + o(1)) \quad \text{as } |x - y| \to \infty, \tag{4.1}$$

where λ_N is a positive constant (cf. [3]). Let Q^{-1} be the inverse matrix of the matrix whose (i, j)-entry is the second moment $Q_{i,j} := \sum p(x)x_i x_j$, $1 \le i, j \le N$, and define the norm $\|x\| = \sqrt{x \cdot Q^{-1}x}$. It is shown by Uchiyama ([6], Theorems 3 and 4) that if $N = 4$, then, for each $\varepsilon \in (0, 1/2)$,

$$g(x) = \frac{\lambda}{\|x\|^2} + a \sum_{y:\,|y-x|\le \varepsilon r} p(y) \ln \frac{|x|}{|y-x|+1} + o(|x|^{-2}) \tag{4.2}$$

as $r := |x| \to \infty$ (a and λ are positive constants); and if $N \ge 5$, then

$$\liminf_{|x|\to\infty} \|x\|^{N-2} g(x) > 0, \tag{4.3a}$$

$$g(x) \le \frac{C}{|x|^2} \max_{\{+,-\}^{N-4}} E\left\{ |X_1| \prod_{j=1}^{N-4} \frac{|X_j|}{|X_1 \pm \cdots \pm X_j \pm x| \vee 1} \right\}, \tag{4.3b}$$

where $X_1, X_2, \dots$ are independent $\mathbf{Z}^N$-valued random variables having the law p and the maximum is taken over all the $(N-4)$-tuples of $+$ and $-$.

Notice that $g(x) > 0$ for all x since, by (4.2) and (4.3a), g is positive outside a ball and that $G(x, y) \le G(0,0) < \infty$, and you will see that condition (ii)(a) for Theorem 2 follows immediately from (4.2) and (4.3a). Condition (i) is easy to check in view of (4.1). Let us show that (ii)(b) is also fulfilled. When $N = 4$, noticing that $\ln[|x|/(|y-x|+1)] \le \text{const}\,|y|^2/[|y-x|+1]^2$ if $|y-x| \le \frac{1}{2}|x|$ and that $g(x) \le g(0) < \infty$, we obtain from (4.2) that

$$g(x) \le 2\lambda \|x\|^{-2} + \text{const} \sum_y p(y)|y|^2 [|y-x|+1]^{-2};$$

hence (ii)(b) with $\pi_x(y) = \text{const}\; p(y-x)|y-x|^2$.

Now let $N \ge 5$ and set

$$q(y) = \max_{1\le j\le N-4} \max_{\{+,-\}^j} E\left\{ |X_1| \prod_{i=1}^{N-4} |X_i|;\; X_1 \pm \cdots \pm X_j = \pm y \right\}.$$

Then the expectation in (4.3b) is bounded by

$$\sum_y q(y) \frac{1}{[|x-y| \vee 1]^{N-4}}.$$

Here we have used the inequality $E\{\xi \prod_{i=1}^n \eta_i\} \le \prod_{i=1}^n (E\{\xi \eta_i^n\})^{1/n}$, a variant of Hölder's valid for any nonnegative random variables ξ and η_i. Observing

$$\sum q(y) \le \text{const}\; E\left\{ |X_1| \prod_{i=1}^{N-4} |X_i| \right\} < \infty,$$

we deduce

$$g(x) \le C_1 \frac{1}{|x|^{N-2}} + C_2 \sum_y q(y) \frac{1}{[|x-y| \vee 1]^{N-2}}.$$

Thus (ii)(b) holds with $\pi_x(z) = C_1 \delta_0(z-x) + C_2 q(z-x)$. The proof of Theorem 1 is complete. □

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REFERENCES

[1] FELLER, W. (1946). The law of the iterated logarithm for identically distributed random variables. *Ann. Math.* **47** 631–638.
[2] FUKAI, Y. and UCHIYAMA, K. (1996). Wiener's test for space-time random walks and its applications. *Trans. Amer. Math. Soc.* **348** 4131–4152.
[3] ITÔ, K. and MCKEAN, H. P. (1960). Potentials and random walks. *Illinois J. Math.* **4** 119–132.
[4] LAMPERTI, J. (1963). Wiener's test and Markov chains. *J. Math. Anal. Appl.* **6** 58–66.
[5] SPITZER, F. (1986). *Principles of Random Walk*, 2nd ed. Springer, New York.
[6] UCHIYAMA, K. (1996). Green's function for random walks on $\mathbf{Z}^N$. *Proc. London Math. Soc.* To appear.

DEPARTMENT OF APPLIED PHYSICS
TOKYO INSTITUTE OF TECHNOLOGY
TOKYO 152
JAPAN
E-MAIL: uchiyama@neptune.ap.titech.ac.jp