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LARGE DEVIATIONS FOR A BROWNIAN MOTION IN A DRIFTED BROWNIAN POTENTIAL

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We derive a large deviation principle both quenched and annealed for a one-dimensional diffusion process in a drifted Brownian environment providing the continuous time analogue of what Comets, Gantert and Zeitouni recently establish for the random walk in random environment. A key-ingredient, Kotani's lemma, allows us to compute the corresponding rate functions. The results are more explicit than in the discrete-time setting.

1. Introduction. Large deviations for a one-dimensional random walk in random environment (RWRE) were first investigated by Greven and den Hollander [8], in 1994. They proved that if $\{S_n : n > 0\}$ denotes the RWRE, the distributions of S_n/n, at fixed environment, also called the *quenched* setting, satisfy a large deviation principle (hereafter abbreviated LDP) with speed n and explicit, deterministic rate function. We say that a sequence of probability measures $\{\mu_t : t > 0\}$ on a topological space satisfies a LDP with rate function I if I is non-negative, lower semicontinuous and for all measurable set A we have

$$-\inf_{x\in A^\circ} I(x) \le \liminf_{t\to\infty} \frac{1}{t}\log \mu_t(A) \le \limsup_{t\to\infty} \frac{1}{t}\log \mu_t(A) \le -\inf_{x\in \bar{A}} I(x)$$

where A° and $\bar{A}$ denote the interior and the closure of A respectively. For general background concerning large deviations we refer to [4].

The large variety of tail behavior has recently motivated a number of papers on refined LD estimates (see [7] for a review). Using the duality between the RWRE and its first hitting times process, Comets, Gantert and Zeitouni [3] recently proved an LDP for the one-dimensional random walk in the general ergodic environment, both quenched and annealed. By *annealed* we mean after averaging over the environment. Large deviations for a RWRE in higher dimension were initiated by Zerner [27]. Using powerful methods Sznitman [21] developed in the study of the Brownian motion in a Poissonian potential, he proved a quenched LDP for the multi-dimensional RWRE and expressed the associated rate function in terms of certain Lyapunov exponents.

This paper aims at studying the RWRE's continuous time analogue, the diffusion in random environment (also called Brownian motion in a random potential), that is, the solution of the *formal* stochastic differential equation

$$dX(t) = d\beta(t) - \tfrac{1}{2}W'(X(t))dt$$

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where β is a standard Brownian motion, B is a two-sided Brownian motion starting from 0, independent of β and

$$W(x) \stackrel{\text{def}}{=} W_\kappa(x) = B(x) - \frac{\kappa}{2}x, \qquad x \in \mathbb{R},$$

W serving as an *environment*. More rigorously, at frozen W, X is a diffusion process with generator

$$\mathscr{L}_W = \frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

Using martingale theory, X can be constructed from a Brownian motion via scale-transformation and time-change, namely

$$X(t) = S^{-1}\left(\mathscr{B}\left(T^{-1}(t)\right)\right) \tag{1.1}$$

where $\mathscr{B}$ is a standard Brownian motion starting from 0, independent of W, S and T are defined as

$$S(x) = \int_0^x e^{W(u)}\,du, \qquad x \in \mathbb{R},$$

$$T(t) = \int_0^t \exp\left(-2W\left(S^{-1}\left(\mathscr{B}(u)\right)\right)\right)\,du, \qquad t > 0$$

(S^{-1} and T^{-1} denoting the respective inverse functions of S and T).

Making use of the Brownian self-similarity, Brox [1] and Schumacher [19] established the weak convergence of $X(t)/(\log t)^2$, as $t \to \infty$, in the recurrent case, namely for $\kappa = 0$, and the limit distribution was explicitly identified by Kesten [14]. For $\kappa \neq 0$, the diffusion is transient. Its long-time behavior was investigated by Kawazu and Tanaka [12]. Let $v_\kappa = \lim_{t\to\infty} X(t)/t$ denote the a.s. speed of the diffusion (a.s. must be understood w.r.t $\mathbb{P}$; see the notation below). Using Krein's spectral theory and Kotani's lemma, they found two speed regimes, namely,

$$v_\kappa = \begin{cases} \operatorname{sgn}(\kappa)\dfrac{|\kappa|-1}{4}, & \text{for } |\kappa| > 1, \\ 0, & \text{for } |\kappa| \le 1 \end{cases}$$

[with sgn(x)=1 for $x \ge 0$ and -1 otherwise].

Their results have been recently revisited by Hu, Shi and Yor [9] who characterized all possible convergence rates via a Bessel process approach. Carmona [2] extended some of Tanaka's results to the Lévy environment.

In this paper we prove a LDP both quenched and annealed for the Brownian motion in a drifted Brownian environment. In the quenched case, our approach is akin to that of [3] and the study of the corresponding rate function is possible making use of a key-ingredient, Kotani's lemma. A variational formula relating the quenched and annealed rate functions enabled Comets, Gantert and Zeitouni [3] to derive annealed LDP's from quenched ones. As we shall see in Section 3, we do not find a similar variational formula. An obvious reason is that our quenched results deal only with specific environments $W(x) = B(x) - \kappa x/2$, not with general ones. In the annealed case, we proceed differently; our proof hinges upon a change of probability. If some of the properties of the rate functions can be favorably compared with those of their discrete counterparts, the computations are, thanks to Kotani's lemma, more explicit than in the discrete-time, discrete-space setting.

Throughout the sequel, Q will denote the Wiener measure, E_Q the expectation w.r.t. Q, P_x^W the law of X at fixed W, starting from x, $\mathbb{P}_x$ the averaged probability $E_Q[P_x^W[.]]$, E_x^W and $\mathbb{E}_x$ the expectations w.r.t. P_x^W and $\mathbb{P}_x$ respectively. For notational convenience, P_0^W, $\mathbb{P}_0$, E_0^W and $\mathbb{E}_0$ will be noted P^W, $\mathbb{P}$, E^W and $\mathbb{E}$ respectively. We denote by

$$\tau_r = \inf\{s > 0 : \quad X(s) = r\},$$

the first hitting time process associated with X, $(\mathscr{F}_t)$ the natural filtration of X, that is the σ-field $\sigma(X(s); s \le t))$ and Θ the shift operator defined as

$$\Theta_x W(y) = W(x+y) - W(x),$$

for all x and y. Then, for every bounded measurable F we have

$$E_x^W\left[F(X(t); t \ge 0)\right] = E^{\Theta_x W}\left[F(X(t) + x); t \ge 0)\right]. \tag{1.2}$$

The probability measure $\mathbb{P}$ is invariant and ergodic under the action of the group of transformations $\{\Theta_x; x \in \mathbb{R}\}$. For a full proof see, for instance, [2]. We say that the environment is *spatially homogeneous*.

On the other hand, reasoning on $-X$ amounts to flipping the sign of κ. Indeed, $-X$ is a diffusion moving in the environment $\{W_\kappa(-x)\}$ which has the same law as $\{W_{-\kappa}(x)\}$. This feature will be called *space reversal invariance* throughout.

We now state our results the main of which are Theorems 2 and 4. Following the approach of [3] we first prove LDP's for the hitting times which we transfer to the positions by duality. Let us start with the *quenched* results.

THEOREM 1. *For Q-almost all environment W, the distributions of $\{\tau_r/r; r > 0\}$ under P^W satisfy a (weak) LDP with deterministic, convex, rate function I_κ defined as follows*:

$$I_\kappa(u) = \sup_{\lambda \ge 0}(\Gamma_\kappa(\lambda) - \lambda u) \tag{1.3}$$

where

$$\Gamma_\kappa(\lambda) = -E_Q\left[\log E^W\left[e^{-\lambda\tau_1}1_{\tau_1<\infty}\right]\right]. \tag{1.4}$$

Moreover, for all $r > 0$

$$\Gamma_\kappa(\lambda) = -E_Q\left[\frac{1}{r}\log E^W\left[e^{-\lambda\tau_r}1_{\tau_r<\infty}\right]\right]. \tag{1.5}$$

By space reversal invariance, we deduce a LDP for $\{\tau_{-r}/r; r > 0\}$ under P^W, for Q almost all W, with rate function $I_{-\kappa}$ and translate the previous result into:

THEOREM 2. *For Q-almost all environment W, the distributions of $\{X(t)/t; t > 0\}$ under P^W satisfy a (strong) LDP with deterministic, convex, good rate function J_κ given by*

$$J_\kappa(v) = \begin{cases} vI_\kappa(1/v), & \text{for } v > 0, \\ |v|\, I_\kappa(1/|v|) + \kappa\,|v|/2, & \text{for } v < 0, \end{cases}$$

and $J_\kappa(0) = \lim_{v\to 0} J_\kappa(v) = 0$.

Before stating the *annealed* LDP we observe that the proof of Theorem 1, hence that of Theorem 2, only requires the independence of the environment's increments and the invariance of the law of W under the action of $\{\Theta_x; x \in \mathbb{R}\}$. Hence, the quenched LDP is still valid in the case of a Lévy process environment.

THEOREM 3. *The distributions of $\{\tau_r/r; r > 0\}$ under $\mathbb{P}$ satisfy a (weak) LDP with deterministic, convex, rate function I_κ^a*

$$I_\kappa^a(u) = \sup_{\lambda \ge 0} (\Gamma_\kappa^a(\lambda) - \lambda u)$$

where

$$\Gamma_\kappa^a(\lambda) = -\lim_{r\to\infty} \frac{1}{r} \log \mathbb{E}\left[e^{-\lambda\tau_r} 1_{\tau_r<\infty}\right]. \tag{1.6}$$

In the spirit of the quenched case, given the space reversal invariance, it follows that

THEOREM 4. *The distributions of $\{X(t)/t; t > 0\}$ under $\mathbb{P}$ satisfy a (strong) LDP with deterministic, convex good rate function J_κ^a given by*

$$J_\kappa^a(v) = \begin{cases} vI_\kappa^a(1/v), & \text{for } v > 0, \\ |v|\, I_{-\kappa}^a(1/|v|), & \text{for } v < 0, \end{cases}$$

and $J_\kappa^a(0) = \lim_{v\to 0} J_\kappa^a(v) = 0$.

Note that if we set $\alpha_\lambda^\kappa(v) = \Gamma_\kappa(\lambda)v$ [resp $\beta_\lambda^\kappa(v) = \Gamma_\kappa^a(\lambda)v$], for $v > 0$, α_λ^κ (resp. β_λ^κ) can be viewed as a quenched (resp. annealed) Lyapunov exponent. Hence, we have expressed our rate functions as Legendre transforms of Lyapunov exponents. Our continuous model corresponds to the "nestling case" in the terminology of [27]. The qualitative properties of the rate functions are listed in the following propositions.

PROPOSITION 1.1. *For all v and all κ*:

(a) $J_{-\kappa}(v) = J_\kappa(-v) = J_\kappa(v) + \kappa v/2$;
(b) $J_{1/2}(v) = v^2/2$ *for* $v > 0$, *and* $J_{1/2}(v) = v^2/2 - v/4$ *for* $v < 0$;
(c) J_κ *vanishes on* $[0, \operatorname{sgn}(\kappa)\frac{(|\kappa|-1)^+}{4}]$, *equals* $-\kappa v/2$ *on* $[0, -\operatorname{sgn}(\kappa)\frac{(|\kappa|-1)^+}{4}]$, *is strictly convex*, *analytic for* $|v| > \frac{(|\kappa|-1)^+}{4}$ *and non analytic at* 0;
(d) $J_\kappa(0) = 0$, $J'_\kappa(0^+) = 0$, $J'_\kappa(0^-) = -\kappa/2$, $J''_\kappa(0^+) = J''_\kappa(0^-)$, $J''_\kappa(0^+) = 1$, *for* $|\kappa| = 1/2$, $J''_\kappa(0^+) = 0$ *for* $|\kappa| > 1/2$, $J''_\kappa(0^+) = +\infty$ *for* $0 \le |\kappa| < 1/2$, [$x^+ = \sup(x, 0)$ *and*, *by abuse of notation*, $[0, x]$ *means* $[x, 0]$ *for negative* x's].

Somewhat surprisingly, at $\kappa_c = 1/2$, the quenched rate function for positive speeds is the one for the linear Brownian motion. Intuitively, a *neutral* random medium slows down the diffusion but the drift in our medium makes the diffusion transient to $+\infty$. The critical drift $\kappa_c = 1/2$ realizes a perfect balance between the two opposite effects. This agrees with the diffusive behavior found by Kawazu and Tanaka, that is, $X(t)/\sqrt{t}$ converges in law (under $\mathbb{P}$) to a non-degenerate random variable (a 1/2-stable variable more precisely).

Observe that J_0 is symmetric and that for $\kappa \neq 0$, J_κ presents a change of slope at the origin.

Part (c) reveals both flat and linear pieces in J_κ for non-zero speeds which is compatible with the flat and linear pieces found in [8]. On the other hand, we will see in Section 3 that the $\mathbb{P}$ a.s. speed of the diffusion is linked to $\mathbb{E}[\tau_1]$ in the following way:

$$\mathbb{E}[\tau_1] = \frac{4}{(\kappa-1)^+} = v_\kappa^{-1}, \qquad (\kappa \ge 0).$$

The interest in analyticity is reminiscent of phase transitions in statistical mechanics (see [8]). Part (d) tells us that $J''_0(0^+) = +\infty$, result we have already encountered in [8], which is in accordance with fluctuations results. Indeed, we already know (see [1]) that in the recurrent case, that is, for $\kappa = 0$, $X(t)$ is of order $(\log t)^2$. This is slower than central limit behavior which typically corresponds to J_κ having finite curvature at 0.

PROPOSITION 1.2. *For all* $\kappa \in \mathbb{R}$:

(a) $J^a_\kappa(v) \le J_\kappa(v)$ *for all* v;
(b) $J^a_{-\kappa}(v) = \frac{\kappa-1}{2}v + J^a_{\kappa-2}(v)$, *for all* $v > 0$;
(c) $J^a_{-\kappa}(v) = J^a_\kappa(-v) = \frac{\kappa+1}{2}v + J^a_{\kappa+2}(v)$, *for all* $v < 0$;
(d) $J^a_\kappa(0) = 0$, $(J^a_\kappa)'(0^+) = 0$, $(J^a_\kappa)'(0^-) = (1-\kappa)/2$.
For non-negative κ *we have*:
(e) J^a_κ *vanishes on* $[0, \frac{(\kappa-1)^+}{4}]$, *equals* $(1-\kappa)v/2$ *on* $[-\frac{(\kappa-3)^+}{4}, 0]$, *is convex on* $\mathbb{R} \setminus [-\frac{(\kappa-3)^+}{4}, \frac{(\kappa-1)^+}{4}]$.

Part (e) can be easily completed for $\kappa < 0$ in the light of the space reversal invariance conveyed by the first equality of (c). As in the quenched case, J^a_0 is symmetric and J^a_κ presents a change of slope at the origin for $\kappa \neq 0$.

Unlike J_κ, the annealed rate function presents two different shapes for nonzero speeds (i.e., for $|\kappa| > 1$), depending on whether or not $|\kappa| > 3$. We find both flat and linear pieces in the case where $|\kappa| > 3$ whereas only flat pieces occur for $1 < |\kappa| \le 3$. It is worth noting that we completely identify the linear pieces in the annealed framework which is not the case in [3].

The outline of the paper is as follows: in Section 2 we introduce Kotani's lemma and give a thorough study of Γ_κ and its annealed counterpart Γ_κ^a. Section 3 is devoted to the properties of the rate functions. In Section 4 we prove a quenched LDP for the hitting times which we transfer to the diffusion in Section 5. We end up by proving the annealed LDP in Section 6.

2. Computing Γ_κ and Γ_κ^a. The function Γ_κ and its annealed counterpart Γ_κ^a play a key role in this paper. Some of their properties are provided by the following propositions.

PROPOSITION 2.1. *For all* $\kappa \in \mathbb{R}$:

(a) *For all* $\lambda > 0$,

$$\Gamma_\kappa(\lambda) = 2\lambda \frac{F_{\kappa-1}(\lambda)}{F_\kappa(\lambda)}, \tag{2.1}$$

where

$$F_\kappa(\lambda) = \int_0^\infty x^{-\kappa-1} \exp\left(-\frac{2}{x} - 4\lambda x\right) dx; \tag{2.2}$$

(b) $\Gamma_\kappa(\lambda) = \Gamma_{-\kappa}(\lambda) - \kappa/2$ *for all* $\lambda \ge 0$;

(c) $\Gamma_\kappa(\lambda)\,\Gamma_{1-\kappa}(\lambda) = 2\lambda$, *in particular* $\Gamma_{1/2}(\lambda) = \sqrt{2\lambda}$, *for all* $\lambda \ge 0$;

(d) Γ_κ *solves* $xy' - 2y^2 - \kappa y = -4x$ *on* $(0,\infty)$ *with* $\Gamma_\kappa(0) = \sup(0, -\kappa/2)$, *and can be expanded as a continued fraction*, *that is*,

$$2\Gamma_\kappa(\lambda) = \frac{8\lambda|}{\kappa - 1} + \frac{8\lambda|}{\kappa - 2} + \frac{8\lambda|}{\kappa - 3} + \cdots;$$

(e) $\Gamma_\kappa(\lambda) = -\infty$ *for all* $\lambda < 0$.

REMARKS. Note that $F_\kappa(\lambda) = 2(4\lambda)^{\kappa/2} K_\kappa(4\sqrt{\lambda})$, K_κ being the modified Bessel function. On the other hand, (d) illustrates the well-known fact that solutions of Ricatti's equations can be expanded in terms of continued fractions (see, e.g., [15]). □

Calculations are possible thanks to:

LEMMA 2.1 (see [12], page 191). *For* $\lambda > 0$ *and* $r \ge 0$,

$$E^W\left[e^{-\lambda\tau_r} 1_{\tau_r < \infty}\right] = \exp\left(-\int_0^r U_\lambda(s)\,ds\right), \qquad Q\ a.s. \tag{2.3}$$

where $U_\lambda(t)$ is the unique stationary, positive, solution of

$$dU_\lambda(t) = U_\lambda(t)dB_t + \left(2\lambda + \frac{1-\kappa}{2}U_\lambda(t) - U_\lambda^2(t)\right)dt.$$

As in [12], it is convenient to consider $X_\lambda(t) = U_\lambda(t)/2\lambda$. According to Kotani's lemma, $X_\lambda(t)$ is the unique stationary, positive, solution of

$$dX_\lambda(t) = X_\lambda(t)dB_t + \left(1 + \frac{1-\kappa}{2}X_\lambda(t) - 2\lambda X_\lambda^2(t)\right)dt. \tag{2.4}$$

Its scale function and speed measure are given by

$$S_\lambda(x) = \int_1^x y^{\kappa-1}\exp\left(\frac{2}{y} + 4\lambda y\right)dy,$$
$$n_\lambda(dx) = 2x^{-\kappa-1}\exp\left(-\frac{2}{x} - 4\lambda x\right)dx, \qquad x > 0.$$

Let m_κ denote the invariant probability measure for X_λ and h_κ its density w.r.t. the Lebesgue measure. In other words,

$$h_\kappa(x) \stackrel{\text{def}}{=} h_{\kappa,\lambda}(x) = \frac{x^{-\kappa-1}}{F_\kappa(\lambda)}\exp\left(-\frac{2}{x} - 4\lambda x\right). \tag{2.5}$$

As it is pointed in [25], (2.4) is still valid for $\lambda = 0$.

PROPOSITION 2.2. *Γ_κ^a satisfies*:

(a) $\Gamma_\kappa^a(\lambda) = -\infty$ *for all* $\lambda < 0$ *and all* κ;
(b) $\Gamma_\kappa^a(0) = 0$ *for* $\kappa \ge 0$, $\Gamma_\kappa^a(0) = \kappa^2/8$ *for* $-2 < \kappa \le 0$, *and* $\Gamma_\kappa^a(0) = -(1+\kappa)/2$ *for* $\kappa \le -2$;
(c) *For all* $\lambda > 0$ *and all* κ, Γ_κ^a *is finite and equals*

$$\Gamma_\kappa^a(\lambda) = \inf_{f\ge 0:\int_0^\infty fh_\kappa = 1}\left(2\lambda\int_0^\infty xf(x)h_\kappa(x)\,dx + \frac{1}{8}\int_0^\infty x^2\frac{f'^2(x)}{f(x)}h_\kappa(x)\,dx\right), \tag{2.6}$$

where $h_\kappa = h_{\kappa,\lambda}$ [*see* (2.5)];
(d) $\Gamma_{-\kappa}^a(\lambda) = \frac{\kappa-1}{2} + \Gamma_{\kappa-2}^a(\lambda)$ *for all* λ *and all* κ.

REMARK. The infimum in (2.6) runs over $f \ge 0$ satisfying $\int_0^\infty fh_\kappa = 1$ and $\sqrt{f} \in H^1([0,\infty), x^2h_\kappa(x)dx)$. We shall omit the last constraint in all similar infimum encountered throughout so as not to burden the notation.

2.1. *Proof of Proposition* 2.1.

PROOF OF (a). Recalling (1.4), plugging X_λ into (2.3) and making use of the stationarity of X_λ, we get

$$\Gamma_\kappa(\lambda) = 2\lambda E_Q\left[X_\lambda(0)\right] = 2\lambda \int_0^\infty x m_\kappa(dx).$$

This together with (2.5) and (2.2) delivers (2.1), as desired. □

PROOF OF (b). Clearly, $F_\kappa(\lambda)$ is differentiable on $(0,\infty)$ and its derivative satisfies

$$F'_\kappa(\lambda) = -4F_{\kappa-1}(\lambda), \tag{2.7}$$

giving that

$$\Gamma_\kappa(\lambda) = -\frac{\lambda}{2}\frac{\partial}{\partial\lambda}\left(\log F_\kappa(\lambda)\right). \tag{2.8}$$

Accordingly, relating Γ_κ and $\Gamma_{-\kappa}$ reduces to relating F_κ and $F_{-\kappa}$. By the change of variables $z = (2\lambda x)^{-1}$, we have that

$$F_\kappa(\lambda) = (2\lambda)^\kappa F_{-\kappa}(\lambda), \tag{2.9}$$

which in conjunction with (2.8) delivers (b) for $\lambda > 0$.

There remains the computation of $\Gamma_\kappa(0) = -E_Q\left[\log P^W\left[\tau_1 < \infty\right]\right]$. Whenever $\kappa \geq 0$, X is either recurrent ($\kappa = 0$) or transient to the right (i.e., $X_t \to +\infty$ a.s.), implying that $\Gamma_\kappa(0) = 0$.

Now assume that $\kappa < 0$. We know that for all $r > 0$ (see, e.g., [18]),

$$\begin{aligned} P^W\left[\tau_1 < \tau_{-r}\right] &= \frac{\int_0^{-r} e^{W(x)}\,dx}{\int_1^{-r} e^{W(x)}\,dx}, \\ &= e^{-W(1)}\frac{\int_0^{-r} e^{B(x)-\kappa x/2}\,dx}{\int_0^{-r-1} e^{\hat{B}(x)-\kappa x/2}\,dx}, \end{aligned}$$

where $\hat{B}(x) = B(x+1) - B(1)$ is another Brownian motion. We take logarithms, then expectations w.r.t. Q and finally the non-decreasing limit in r to find $\Gamma_\kappa(0) = -\kappa/2$.

Note that in the light of part (e), which is the most delicate part to prove in this proposition and which we haven't proved yet, (b) also holds for $\lambda < 0$. □

PROOFS OF (c) AND (d). Taking logarithmic derivative of (2.1) and making use of (2.7) lead to

$$\lambda\Gamma'_\kappa(\lambda) = \Gamma_\kappa(\lambda)\left(1 + 2\left(\Gamma_\kappa(\lambda) - \Gamma_{\kappa-1}(\lambda)\right)\right). \tag{2.10}$$

Integrating both sides of (2.4) over the interval $[0, r]$, taking expectations w.r.t. Q and using the stationarity of X_λ we get

$$4\lambda E_Q\left[X_\lambda^2(0)\right] + (\kappa - 1)\,E_Q\left[X_\lambda(0)\right] = 2$$

where

$$E_Q\left[X_\lambda^2(0)\right] = \frac{F_{\kappa-2}(\lambda)}{F_\kappa(\lambda)} = \frac{\Gamma_\kappa \Gamma_{\kappa-1}(\lambda)}{4\lambda^2}.$$

Combining the last two equalities yields

$$\Gamma_\kappa(\lambda)\left(\Gamma_{\kappa-1}(\lambda) + \frac{\kappa-1}{2}\right) = 2\lambda, \tag{2.11}$$

which in conjunction with (b) delivers the first part of (c). Clearly, making $\kappa = 1/2$ proves the second one. Now, substituting $\Gamma_{\kappa-1}$ into (2.10) shows that Γ_κ solves

$$xy' - 2y^2 - \kappa y = -4x \tag{2.12}$$

on $(0, \infty)$. As for the continued fraction expansion, it follows from (b) and (c). We write

$$2\Gamma_\kappa(\lambda) = \frac{4\lambda}{\Gamma_{1-\kappa}(\lambda)} = \frac{8\lambda}{\kappa - 1 + 2\Gamma_{\kappa-1}(\lambda)},$$

and follow by iteration. □

PROOF OF (e). Let us first prove that (1.5) still holds for $\lambda < 0$, at least along the integers. Indeed, the strong Markov property together with (1.2) imply that for all r, $s > 0$

$$E^W\left[e^{-\lambda\tau_{r+s}} 1_{\tau_{r+s}<\infty}\right] = E^W\left[e^{-\lambda\tau_r} 1_{\tau_r<\infty}\right] E^{\Theta_r W}\left[e^{-\lambda\tau_s} 1_{\tau_s<\infty}\right].$$

The second term of the r.h.s. has the same law as $E^W\left[e^{-\lambda\tau_s} 1_{\tau_s<\infty}\right]$ by the invariance of Q under the action of $\{\Theta_x; x \in \mathbb{R}\}$. Thus, taking logarithms, then expectations w.r.t. Q entail (1.5) for all r, *s rational*.

We first prove (e) for $\kappa \geq 0$, in which case we may drop the indicators in (1.4). Recall that *r is rational throughout*.

Let $\sigma_r = \inf\{t > 0 : \mathscr{B}(t) = r\}$. Thanks to Itô–McKean's representation [see (1.1)] and the occupation times formula, τ_r can be written as

$$\tau_r = T(\sigma_{S(r)}) = \int_{-\infty}^{r} e^{-W(y)} L\left(\sigma_{S(r)}; S(y)\right) dy$$

where $\{L(t; x);\ t > 0,\ x \in \mathbb{R}\}$ is the local time of $\mathscr{B}$.

Let

$$\tau_r^{ref} \stackrel{\text{def}}{=} \int_0^r e^{-W(y)} L\left(\sigma_{S(r)}; S(y)\right) dy,$$

and set

$$R^2(x) \stackrel{\text{def}}{=} \frac{1}{S(r)} L\left(\sigma_{S(r)}; S(r)(1-x)\right), \qquad 0 \leq x \leq 1.$$

Combining the (first) Ray-Knight theorem (see [18]) and a scaling argument tells us that $\{R^2(x), 0 \le x \le 1\}$ is a two-dimensional squared Bessel process, starting from zero. Accordingly,

$$\tau_r^{ref} = S(r) \int_0^r e^{-W(y)} R^2 \left(\frac{S(r) - S(y)}{S(r)} \right) dy.$$

The process R is independent of W and the superscript *ref* refers to *reflected*. Indeed, if we assume $W(x) = +\infty$ for $x < 0$ (a reflection at the origin) τ_r reduces to τ_r^{ref}.

Now, τ_r being stochastically greater than τ_r^{ref}, we get that for all $\lambda < 0$ and all $r > 0$,

$$\begin{aligned} \Gamma_\kappa(\lambda) &= -E_Q \left[\frac{1}{r} \log E^W \left[e^{-\lambda \tau_r} \right] \right] \\ &\le -E_Q \left[\frac{1}{r} \log E^W \left[e^{-\lambda \tau_r^{ref}} \right] \right]. \end{aligned}$$

By virtue of the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \tau_r^{ref} &\ge \left(\int_0^r R \left(\frac{S(r) - S(y)}{S(r)} \right) dy \right)^2 \\ &\overset{\text{stoch}}{\ge} \left(\int_0^r \mathscr{B} \left(\frac{S(r) - S(y)}{S(r)} \right) dy \right)^2, \end{aligned}$$

since R is stochastically greater than $|\mathscr{B}|$.

At fixed environment, $\int_0^r \mathscr{B}(\frac{S(r)-S(y)}{S(r)})\, dy$ is a centered Gaussian variable with variance $\sigma_W^2(r)$ given by

$$\sigma_W^2(r) = 2 \int_0^r y \left(1 - \frac{S(y)}{S(r)} \right) dy.$$

As a result,

$$\Gamma_\kappa(\lambda) \le -E_Q \left[\frac{1}{r} \log E^W \left[e^{-\lambda \sigma_W^2(r) Z(\frac{1}{2}, \frac{1}{2})} \right] \right], \tag{2.13}$$

$Z(a, b)$ being a gamma variable with density $\frac{1}{\Gamma(b)} e^{-ax} a^b x^{b-1} 1_{x>0}\, dx$.

Clearly, whenever $\mu \ge 1/2$, $E[e^{\mu Z(\frac{1}{2}, \frac{1}{2})}]$ is infinite. Thus, making r depend on λ in such a way that

$$Q\left[-\lambda \sigma_W^2(r) \ge 1/2 \right] > 0, \tag{2.14}$$

gives $\Gamma_\kappa(\lambda) = -\infty$ for all $\lambda < 0$ (and $\kappa \ge 0$).

Let us now seek a suitable choice of r for which (2.14) is valid:

$$\begin{aligned} Q\left[-\lambda \sigma_W^2(r) \ge 1/2 \right] &\ge Q\left[\int_0^{r/2} y \left(1 - \frac{S(y)}{S(r)} \right) dy \ge -1/4\lambda \right], \\ &\ge Q\left[\frac{S(r/2)}{S(r) - S(r/2)} \le 1 + 2/\lambda r^2 \right]. \end{aligned}$$

We have to choose r such that $-\lambda r^2 > 2$. Now, the last probability can be rewritten as

$$Q\left[\frac{S(r/2)\,e^{-B(r/2)}}{\int_{r/2}^{r} e^{B(x)-B(r/2)-\frac{\kappa}{2}x}\,dx} \le 1+2/\lambda r^2\right]$$

$$\begin{aligned}(2.15)\qquad &\ge Q\left[S(r/2)e^{-B(r/2)} \le 2r(1+2/\lambda r^2)\right]\\ &\times Q\left[\int_{r/2}^{r} e^{B(x)-B(r/2)-\frac{\kappa}{2}x}\,dx \;\ge\; 2r\right]\end{aligned}$$

since $e^{-B(r/2)}(S(r)-S(r/2))$ is independent of $\{B(x); 0 \le x \le r/2\}$.

The first term of (2.15) is greater than or equal to

$$Q\left[\exp\left(\overline{B}(r/2) - B(r/2)\right) \le 4(1+\frac{2}{\lambda r^2})\right]$$

which is positive if r is chosen such that $-\lambda r^2 > 8/3$ (> 2). As for the second term of (2.15), it equals

$$Q\left[\int_0^{r/2} \exp\left(B(x+r/2) - B(r/2) - \frac{\kappa}{2}x\right)\,dx \;\ge 2re^{\kappa r/4}\right]$$

which is nothing but $Q\left[S(r/2) \ge 2re^{\kappa r/4}\right]$. On the other hand, according to Jensen's inequality,

$$\frac{2}{r}S(r/2) \ge \exp\left(\frac{2}{r}\int_0^{r/2}(B(x)-\frac{\kappa}{2}x)\,dx\right)$$

the r.h.s. of which is a log-normal variable. As a result, the second term of (2.15) is positive, proving (e) for $\kappa \ge 0$.

Now take $\kappa < 0$ and suppose that there exists $\lambda_0 < 0$ such that $\Gamma_\kappa(\lambda_0) > -\infty$. Note that for all $\lambda < 0$, $\Gamma_\kappa(\lambda) < +\infty$ since it is less than or equal to $-E_Q[\log P^W[\tau_1 < \infty]] = \Gamma_\kappa(0) = -\kappa/2$.

Let $n > 1$, *integer*, be given. We have

$$\begin{aligned}E^W\left[e^{-\lambda_0\tau_n}1_{\tau_n<\infty}\right] &\ge E^W\left[e^{-\lambda_0\tau_n}1_{\tau_{-1}<\tau_n<\infty}\right]\\ &= E^W\left[e^{-\lambda_0\tau_{-1}}1_{\tau_{-1}<\tau_n}\right]E^{\Theta_{-1}W}\left[e^{-\lambda_0\tau_{n+1}}1_{\tau_{n+1}<\infty}\right]\end{aligned}$$

where we have used the fact that the event $\tau_{-1} < \tau_n$ belongs to the σ-field $\mathscr{F}_{\tau_{-1}}$ and that on $\tau_{-1} < \tau_n$, $\tau_n = \tau_{-1} + \tau_n \circ \theta_{\tau_{-1}}$.

Taking logarithms then expectations w.r.t. Q and using (1.5) we get

$$E_Q\left[\log E^W\left[e^{-\lambda_0\tau_{-1}}1_{\tau_{-1}<\tau_n}\right]\right] \le \Gamma_\kappa(\lambda_0). \tag{2.16}$$

Now set $f_n^W = \log E^W\left[e^{-\lambda_0\tau_{-1}}1_{\tau_{-1}<\tau_n}\right]$ and $a^W = \log P^W[\tau_{-1} < \tau_1]$. Plainly, the sequence $\{f_n^W - a^W; n > 0\}$ is both non-decreasing and non-negative. Moreover, since

$$P^W[\tau_{-1} < \tau_1] = \frac{\int_0^1 e^{W(x)}\,dx}{\int_{-1}^1 e^{W(x)}\,dx},$$

(see [18], page 278) and since for all $b > 0$,

$$\log 2b - \frac{\kappa}{2} \le E_Q\left[\log \int_{-b}^{b} e^{W(x)}\,dx\right] \le \log \int_{-b}^{b} E_Q\left[e^{W(x)}\right] dx$$

by Jensen's inequality, $E_Q[a^W]$ is bounded. Thus, the monotone convergence theorem applies and the l.h.s of (2.16) approaches $-\Gamma_{-\kappa}(\lambda_0)$ thanks to space reversal invariance. But the first part of the proof of (e) ($-\kappa > 0$ now!) tells us that $-\Gamma_{-\kappa}(\lambda_0) = +\infty$. In the light of (2.16), this contradicts the fact that $\Gamma_\kappa(\lambda_0) < \infty$ which completes the proof of (e). □

2.2. *Proof of Proposition* 2.2.

PROOFS OF (a) AND (b). By Jensen's inequality, we get that for all λ,

$$\Gamma_\kappa^a(\lambda) \le \Gamma_\kappa(\lambda).$$

Since $\Gamma_\kappa(\lambda) = -\infty$ whenever $\lambda < 0$ so is $\Gamma_\kappa^a(\lambda)$. In addition, $\Gamma_\kappa(0) = 0$ for $\kappa \ge 0$ implying that $\Gamma_\kappa^a(0) = 0$ for $\kappa \ge 0$.

Now for $\kappa < 0$, Kawazu and Tanaka [11] investigated the asymptotic behavior of the tail of the distribution of the maximum of X. They proved that $\mathbb{P}[\max_{s\ge 0} X(s) > r]$, which is nothing but $\mathbb{P}[\tau_r < \infty]$, decays exponentially fast to zero as r tends to infinity in the following way.

THEOREM. *As r tends to infinity, if $\kappa < -2$,*

$$\mathbb{P}[\tau_r < \infty] \sim \frac{\kappa+2}{\kappa+1} \exp\left(\frac{\kappa+1}{2} r\right),$$

if $\kappa = -2$,

$$\mathbb{P}[\tau_r < \infty] \sim \sqrt{2/\pi}\sqrt{r} \exp\left(-\frac{r}{2}\right),$$

if $0 < \kappa < -2$,

$$\mathbb{P}[\tau_r < \infty] \sim const.\ r^{-3/2} \exp\left(-\frac{\kappa^2}{8} r\right).$$

The rate of decay being $-\Gamma_\kappa^a(0)$, we have proved (b). □

PROOF OF (c). We shall prove that the limit in

$$\Gamma_\kappa^a(\lambda) = -\lim_{r\to\infty} \frac{1}{r} \log \mathbb{E}\left[e^{-\lambda \tau_r} 1_{\tau_r < \infty}\right]$$

exists for $\lambda > 0$ and compute its value. According to Kotani's lemma, the previous limit can be written as

$$\Gamma_\kappa^a(\lambda) = -\lim_{r\to\infty} \frac{1}{r} \log E_{m_\kappa}\left[\exp\left\{-2\lambda \int_0^r X_\lambda(s)\,ds\right\}\right],$$

m_κ being the invariant probability measure for X_λ. Define $Z_\lambda = \log X_\lambda$. Recalling (2.4), Z_λ solves the following (Smoluchowski) SDE:

$$dZ_\lambda(t) = dB(t) - \tfrac{1}{2}V'(Z_\lambda(t))\,dt$$

with $V(z) = 2\left(2\lambda e^z + e^{-z} + \frac{\kappa}{2}z\right)$. Its infinitesimal generator is nothing but $\mathscr{L}_V$ [see the display above (1.1)]. On the other hand, the process Z_λ is stationary (as X_λ), and its invariant probability measure is given by

$$m^*_\kappa(dz) \stackrel{\text{def}}{=} m^*_{\kappa,\lambda}(dz) = \frac{e^{-V(z)}}{F_\kappa(\lambda)}\,dz, \qquad z \in \mathbb{R}$$

since $\int_{\mathbb{R}} e^{-V(z)}\,dz = F_\kappa(\lambda)$, by the change of variables $y = e^z$. Accordingly, Z_λ is a symmetric diffusion and m^*_κ is reversing for the corresponding transition probability function.

Let $M_1(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$ and

$$L_r(Z_\lambda) = \frac{1}{r}\int_0^r \delta_{Z_\lambda(s)}\,ds.$$

Let $V^* = \frac{1}{4}V'^2 - \frac{1}{2}V''$. The level sets $\{V^* \le R\}$, $R \in (0,\infty)$ are compact. Hence, Theorem 6.2.21 of [5] applies to $\{L_r(Z_\lambda); r > 0\}$ with $\Sigma = \mathbb{R}$ endowed with the standard Euclidean structure and $U = V$. Namely, $\{L_r(Z_\lambda); r > 0\}$ satisfies a LDP on $M_1(\mathbb{R})$ with *good* rate function $\mathscr{J}$ given through the Dirichlet form $\mathscr{E}$ by

$$\mathscr{J}(\nu) = \mathscr{E}(|g|, |g|)$$

with

$$\mathscr{E}(g,g) = \tfrac{1}{2}\int_{\mathbb{R}} g'^2\,dm^*_\kappa$$

if $d\nu = g^2 dm^*_\kappa$ (or also $\nu \ll m^*_\kappa$) and infinite otherwise.

Varadhan's theorem (see [5], page 43) tells us that for all continuous, bounded functional Ψ, we have

$$\lim_{r\to\infty}\frac{1}{r}\log E_{m^*_\kappa}\left[e^{r\Psi(L_r(Z_\lambda))}\right] = \sup_{\nu \ll m^*_\kappa}(\Psi(\nu) - \mathscr{J}(\nu)),$$

or equivalently, performing the change of variables $y = e^x$,

$$\lim_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Xi(L_r(X_\lambda))}\right] = \sup_{\mu \ll m_\kappa}(\Xi(\mu) - \mathscr{I}(\mu)),$$

for all continuous, bounded functional Ξ, $\mathscr{I}(\mu)$ being nothing but $\mathscr{J}(\mu \circ \exp)$.

Actually, proving a LDP for $\{L_r(X_\lambda); r > 0\}$ can be carried out *directly* upon checking Varadhan's hypotheses (see [26]). Reasoning on $\{L_r(Z_\lambda); r > 0\}$ merely shortens the proof the rest of which will only deal with X_λ. Throughout the sequel, for notational convenience, L_r will denote $L_r(X_\lambda)$.

Define $\Phi : M_1(\mathbb{R}) \mapsto \mathbb{R}$ as

$$\Phi(\mu) = -2\lambda\int x\,d\mu(x).$$

Accordingly, the quantity of interest $E_{m_\kappa}\left[\exp\left(-2\lambda\int_0^r X_\lambda(s)ds\right)\right]$ can be written as $E_{m_\kappa}\left[e^{r\Phi(L_r)}\right]$. Although $\Phi(\mu)$ is not weakly continuous, we shall prove that

$$\lim_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Phi(L_r)}\right]=\sup_{\mu\ll m_\kappa}\left(\Phi(\mu)-\mathscr{I}(\mu)\right).$$

A monotone convergence argument shows that $\Phi(\mu)$ can be written as the infimum of continuous functions, $\Phi_n(\mu)$, namely

$$\Phi(\mu)=\inf_{n\geq 0}\Phi_n(\mu)=\inf_{n\geq 0}\left(-2\lambda\int(x\wedge n)\,d\mu(x)\right).$$

As a result, Φ is upper semi-continuous and it follows from [6] that

$$\limsup_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Phi(L_r)}\right]\leq\sup_{\mu\ll m_\kappa}(\Phi(\mu)-\mathscr{I}(\mu)).$$

Let us now prove

$$\liminf_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Phi(L_r)}\right]\geq\sup_{\mu\ll m_\kappa}(\Phi(\mu)-\mathscr{I}(\mu)). \tag{2.17}$$

Set

$$\nu_x=\inf\{s>0:X_\lambda(s)=x\}.$$

By the strong Markov property,

$$\begin{aligned}&E_{m_\kappa}\left[\exp\left\{-2\lambda\int_0^r X_\lambda(s)\,ds\right\}\right]\\&\quad\geq E_{m_\kappa}\left[\exp\left\{-2\lambda\int_0^{\nu_0} X_\lambda(s)\,ds\right\}\right]E_0\left[\exp\left\{-2\lambda\int_0^r X_\lambda(s)\,ds\right\}\right].\end{aligned}$$

The first term of the r.h.s. does not depend on r, thus taking logarithms, dividing by r and taking the liminf leads to

$$\liminf_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Phi(L_r)}\right]\geq\liminf_{r\to\infty}\frac{1}{r}\log E_0\left[e^{r\Phi(L_r)}\right]. \tag{2.18}$$

On the other hand, although the drift coefficient of (2.4) is not Lipschitz-continuous, it is easy to prove that if X_λ^x denotes the diffusion X_λ started at x, the mapping $x\mapsto X_\lambda^x$ is both continuous and non-decreasing. Indeed, one can write

$$X_\lambda^x(t)-X_\lambda^y(t)=(x-y)\exp U(t)$$

where

$$U(t)=B(t)-\frac{\kappa t}{2}-2\lambda\int_0^t(X_\lambda^x(s)+X_\lambda^y(s))\,ds. \tag{2.19}$$

As a result, $E_0\left[e^{r\Phi(L_r)}\right]=\sup_{x\geq 0}E_x\left[e^{r\Phi(L_r)}\right]$ so that the liminf in the r.h.s. of (2.18) is a non-increasing *limit* due to the subadditive theorem.

Now, since Φ_n is non-increasing in n and approaches Φ as n tends to infinity, we have

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{r}\log E_0\left[e^{r\Phi(L_r)}\right] &= \lim_{r\to\infty}\lim_{n\to\infty}\frac{1}{r}\log E_0\left[e^{r\Phi_n(L_r)}\right]\\ &= \lim_{n\to\infty}\lim_{r\to\infty}\frac{1}{r}\log E_0\left[e^{r\Phi_n(L_r)}\right]\\ &\geq \lim_{n\to\infty}\lim_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Phi_n(L_r)}\right].\end{aligned}$$

(Interchanging the limits in n and r is possible since we are dealing with non-increasing limits.) Now since Φ_n is continuous, non-positive and bounded from below, Varadhan's theorem entails

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{r}\log E_{m_\kappa}\left[e^{r\Phi_n(L_r)}\right] &= \sup_{\mu\ll m_\kappa}(\Phi_n(\mu)-\mathscr{I}(\mu))\\ &\geq \sup_{\mu\ll m_\kappa}(\Phi(\mu)-\mathscr{I}(\mu)),\end{aligned}$$

Φ_n being non-increasing in n. This completes the proof of (2.17).

We have proved that

$$\Gamma^a_\kappa(\lambda) = -\sup_{g\geq 0:\int_{-\infty}^{\infty} g^2\,dm^*_\kappa(x)=1}\left(-2\lambda\int_{-\infty}^{\infty} e^x g^2(x)\,dm^*_\kappa(x) - \tfrac{1}{2}\int_{-\infty}^{\infty} g^{'2}(x)\,dm^*_\kappa(x)\right),$$

which is nothing but (2.6) upon the change of variables $y=e^z$, the definition of h_κ and $f=g^2\circ\log$. $\square$

PROOF OF (d). Recalling (2.5) we have

$$h_\kappa\left(\frac{1}{2\lambda x}\right) = 2\lambda x^2 h_{-\kappa}(x), \tag{2.20}$$

$$\frac{h'_\kappa}{h_\kappa}(x) = -4\lambda + \frac{2}{x^2} - \frac{\kappa+1}{x}. \tag{2.21}$$

In this light,

$$\begin{aligned}\Gamma^a_{-\kappa}(\lambda) &= \inf_{f\geq 0:\int_0^\infty f h_{-\kappa}=1}\left(2\lambda\int_0^\infty x f(x) h_{-\kappa}(x)dx + \frac{1}{8}\int_0^\infty x^2\frac{f^{'2}(x)}{f(x)}h_{-\kappa}(x)\,dx\right),\\ &= \inf_{g\geq 0:\int_0^\infty g h_\kappa=1}\left(\int_0^\infty \frac{g(y)}{y}h_\kappa(y)\,dy + \frac{1}{8}\int_0^\infty y^2\frac{g^{'2}(y)}{g(y)}h_\kappa(y)\,dy\right),\\ &= \frac{\kappa}{2} + \inf_{g\geq 0:\int_0^\infty g h_\kappa=1}\left(2\lambda\int_0^\infty y g(y) h_\kappa(y)\,dy + \cdots\right.\\ &\qquad\left. + \frac{1}{8}\int_0^\infty y^2\frac{g^{'2}(y)}{g(y)}h_\kappa(y)dy - \frac{1}{2}\int_0^\infty y g'(y) h_\kappa(y)\,dy\right).\end{aligned}$$

We have made the change of variables $x \mapsto 1/2x\lambda$ and used (2.20) in deriving the second equality above. On the other hand, mutiplying both sides of (2.21) by $yg/2$, integrating over $\mathbb{R}_+$, then integrating by parts lead to the last equality.

Now set

$$f(x) = \frac{h_\kappa(x)}{h_{\kappa-2}(x)} g(x) = \frac{F_{\kappa-2}(\lambda)}{F_\kappa(\lambda)} \frac{g(x)}{x^2}, \tag{2.22}$$

the last equality following from (2.5). We express g'/g in terms of f'/f, which easily follows from taking logarithmic derivative of (2.22), then substitute gh_κ by $fh_{\kappa-2}$ to get (d) for $\lambda > 0$. Actually, in light of (a) and (b) we easily see that (d) is valid for *all* λ. $\square$

3. Properties of the rate functions.

We start this section with additional properties of both Γ_κ and Γ_κ^a which will be of constant use in this paper.

3.1. *Further preliminaries.*

PROPOSITION 3.1. *Γ_κ is*:

(a) *increasing, strictly concave, analytic on* $(0, +\infty)$ *and the range of* Γ_κ' *is* $(0, \frac{4}{(|\kappa|-1)^+})$;

(b) $\lim_{\lambda\to 0} \Gamma_\kappa(\lambda)/\sqrt{\lambda} = +\infty$, *for* $0 \le \kappa < 1/2$, $\lim_{\lambda\to 0} \Gamma_\kappa(\lambda)/\sqrt{\lambda} = 0$ *for* $1/2 < \kappa < 1$. [*Recall that* $\Gamma_{1/2}(\lambda) = \sqrt{2\lambda}$.]

PROOF OF (a). Thanks to Proposition 2.1 (b), $\Gamma_\kappa' = \Gamma_{-\kappa}'$ thus we may assume $\kappa \ge 0$ throughout and drop the indicator in the definition of Γ_κ, (1.4). It is plain that for all $\lambda > 0$,

$$\Gamma_\kappa'(\lambda) = E_Q\left[\frac{E^W\left[\tau_1 e^{-\lambda\tau_1}\right]}{E^W\left[e^{-\lambda\tau_1}\right]}\right].$$

On the other hand, following [12], page 209, we have

$$E^W[\tau_1] = \int_0^1 dS(x) \int_{-\infty}^x m(dy)$$

m being the speed measure of X, that is $m(dy) = 2e^{-W(y)}\, dy$.

Therefore,

$$\begin{aligned}
\mathbb{E}[\tau_1] &= 2\int_0^1 dx \int_{-\infty}^x E_Q\left[e^{W(x)-W(y)}\right] dy \\
&= 2\int_0^1 dx \int_{-\infty}^x e^{\frac{1-\kappa}{2}(x-y)}\, dy \\
&= \frac{4}{(\kappa-1)^+}.
\end{aligned}$$

Accordingly, whenever $\kappa \leq 1$, $\mathbb{E}[\tau_1]$ is infinite implying that $\Gamma_\kappa^{'}(0^+) = +\infty$ by virtue of Fatou's lemma. Now, for $\kappa > 1$ $E^W[\tau_1]$ is integrable w.r.t. Q and Lebesgue's theorem proves that $\Gamma_\kappa^{'}(0^+) = \mathbb{E}[\tau_1]$.

Since $\Gamma_\kappa < \Gamma_{\kappa-1}$, (2.10) together with Proposition 2.1 (c) imply that

$$\Gamma_\kappa^{'}(\lambda) \leq \frac{\Gamma_\kappa(\lambda)}{\lambda} = \frac{2}{\Gamma_{1-\kappa}(\lambda)}.$$

Clearly, $\Gamma_\kappa(\lambda)$ tends to infinity as $\lambda \to \infty$ giving that $\lim_{\lambda\to\infty} \Gamma_\kappa^{'}(\lambda) = 0$. We have proved that the range of $\Gamma_\kappa^{'}$ is $(0, \frac{4}{(|\kappa|-1)^+})$ for $\kappa \geq 0$.

That Γ_κ is increasing and concave is plain. Suppose that Γ_κ is not strictly concave. It follows that it is linear on an interval, say (λ_1, λ_2) with $\lambda_1 > 0$. Since it solves the ODE in (d) on (λ_1, λ_2), replacing $y^{'}$ by a positive constant, say c, leads to $4\Gamma_\kappa(\lambda) = -\kappa + \sqrt{\kappa^2 + 8(4+c)\lambda}$ for all $\lambda \in (\lambda_1, \lambda_2)$ (recall that $\Gamma_\kappa \geq 0$). This contradicts the fact that Γ_κ is linear on (λ_1, λ_2). Thus, Γ_κ is strictly concave, as desired.

Now recalling the definition of F_κ, (2.2), one easily sees that F_κ is analytic on $(0, \infty)$. Indeed, for all $\lambda_0 > 0$ we have

$$F_\kappa(\lambda) = \sum_{n\geq 0} \frac{(4(\lambda - \lambda_0))^n}{n!} F_{\kappa-n}(\lambda_0),$$

$F_\kappa(\lambda)$ being positive on $(0, \infty)$, (2.1) implies that Γ_κ is analytic on $(0, \infty)$. □

PROOF OF (b). First note that for all $0 < \kappa < 1$,

$$\lim_{\lambda\to 0} \frac{F_{1-\kappa}(\lambda)}{F_\kappa(\lambda)} = c_\kappa,$$

where c_κ is a positive, finite constant. Accordingly, combining (2.1) and (2.9) gives that $\Gamma_\kappa(\lambda)/\sqrt{\lambda}$ behaves like $\lambda^{\kappa-1/2}$ as λ tends to zero. This proves (b). We shall also need the following.

PROPOSITION 3.2. *Γ_κ^a is:*

(a) *increasing, concave,* $\lim_{\lambda\to\infty}(\Gamma_\kappa^a)^{'}(\lambda^+) = 0$;

(b) $(\Gamma_\kappa^a)^{'}(0^+) = +\infty$ *for* $-3 \leq \kappa \leq 1$, $(\Gamma_\kappa^a)^{'}(0^+) = 4/(\kappa - 1)$ *for* $\kappa > 1$ *and* $(\Gamma_\kappa^a)^{'}(0^+) = 4/(-\kappa - 3)$ *for* $\kappa < -3$. *In other words,*

$$(\Gamma_\kappa^a)^{'}(0^+) = \frac{4}{(|\kappa+1| - 2)^+}. \tag{3.1}$$

[We denote by $f^{'}(x^+)$ (resp. $f^{'}(x^-)$) the right-hand (resp. left-hand) derivative of f in x.]

PROOF OF (a). That Γ_κ^a is increasing and concave is plain. By concavity, we get that for all $\lambda > 0$.

$$(\Gamma_\kappa^a)^{'}(\lambda^+) \leq \frac{\Gamma_\kappa^a(\lambda) - \Gamma_\kappa^a(0)}{\lambda} \leq \frac{\Gamma_\kappa(\lambda)}{\lambda},$$

since $0 \le \Gamma_\kappa^a \le \Gamma_\kappa$. Having in mind part (c) of Proposition 2.1, we are done by sending λ to infinity.

PROOF OF (b). We first show that for all $\kappa \ge 0$,

$$(\Gamma_\kappa^a)'(0^+) = \mathbb{E}[\tau_1] = \frac{4}{(\kappa-1)^+}.$$

By virtue of Jensen's inequality, we have

$$-\frac{1}{r\lambda}\log \mathbb{E}\left[e^{-\lambda\tau_r}\right] \le \frac{\mathbb{E}[\tau_r]}{r} = \mathbb{E}[\tau_1] \qquad \forall r,\ \lambda > 0,$$

giving that $(\Gamma_\kappa^a)'(0^+) \le \mathbb{E}[\tau_1]$.

On the other hand, the quantity of interest is

$$\lim_{\lambda\to 0}\frac{\Gamma_\kappa^a(\lambda)}{\lambda} = \lim_{\lambda\to 0}\lim_{r\to\infty} -\frac{1}{r\lambda}\log E_{m_\kappa}\left[\exp\left\{-2\lambda\int_0^r X_\lambda(s)\,ds\right\}\right],$$

thanks to Kotani's lemma. Following the reasoning in (2.19), one can prove that the (positive) stationary solution of (2.4) is greater than or equal to $(X_\lambda^0)^+$. Thus, the last quantity above is greater than or equal to

$$\begin{aligned}\lim_{\lambda\to 0}\lim_{r\to\infty} &-\frac{1}{r\lambda}\log E_0\left[\exp\left\{-2\lambda\int_0^r (X_\lambda)^+(s)\,ds\right\}\right]\\ &= \lim_{r\to\infty}\lim_{\lambda\to 0} -\frac{1}{r\lambda}\log E_0\left[\exp\left\{-2\lambda\int_0^r (X_\lambda)^+(s)\,ds\right\}\right],\end{aligned}$$

since both limits are non-decreasing. [This is true due to the concavity (in λ) and to the subadditive theorem.] Just as in (2.19) once again, $X_\lambda^0 \ge X_{\lambda_0}^0$ for $\lambda_0 > \lambda$. This leads us to

$$\begin{aligned}(\Gamma_\kappa^a)'(0^+) &\ge \lim_{r\to\infty}\lim_{\lambda\to 0} -\frac{1}{r\lambda}\log E_0\left[\exp\left\{-2\lambda\int_0^r (X_{\lambda_0})^+(s)\,ds\right\}\right]\\ &= \lim_{r\to\infty}\frac{2}{r}E_0\left(\int_0^r X_{\lambda_0}^+(s)\,ds\right).\end{aligned} \tag{3.2}$$

Now, the process X_{λ_0} being stationary, an ergodic result we learned from [16] ensures that for all continuous bounded function!f,

$$\frac{1}{r}\int_0^r f(X_{\lambda_0}(s))\,ds \to \int f(x)\,h_{\kappa,\lambda_0}(x)\,dx \qquad \text{as } r\to\infty,$$

P_x-a.s., thus in $L^1(P_x)$, for all $x \ge 0$. Hence, setting $f(x) = x^+ \wedge M$ for $M > 0$, taking the non-decreasing limit in M, and having in mind that the limit in r is non-decreasing, we get that (3.2) equals $\Gamma_\kappa(\lambda_0)/\lambda_0$. We are done by sending λ_0 to 0 to get $\mathbb{E}[\tau_1]$.

We have proved (3.1) for $\kappa \ge 0$. In particular, $(\Gamma_\kappa^a)'(0^+) = +\infty$ for $0 \le \kappa \le 1$ hence for $-3 \le \kappa \le -2$, by the symmetry w.r.t. $\kappa = -1$ conveyed by Proposition 2.2(d). All is left to prove is that $(\Gamma_\kappa^a)'(0^+) = +\infty$ for $-2 < \kappa < 0$.

Making $f = gh_{-\kappa}$ in (2.6) and using (2.21) together with the fact that

$$\frac{h'_{-\kappa}}{h_{-\kappa}}(x) = \frac{\kappa}{x} + \frac{h'_0}{h_0}(x)$$

entails

$$\begin{aligned}\Gamma^a_{-\kappa}(\lambda) = \inf_{f\geq 0:\int_0^\infty f=1}&\bigg(2\lambda\int_0^\infty xf(x)dx\\ &+\frac{1}{8}\int_0^\infty x^2 f(x)\bigg(\frac{f'}{f}(x)-\frac{h'_0}{h_0}(x)-\frac{\kappa}{x}\bigg)^2 dx\bigg),\\ \geq \frac{\kappa^2}{8} + \inf_{f\geq 0:\int_0^\infty f=1}&\Big(1-\frac{\kappa}{2}\Big)\\ &\times\bigg(2\lambda\int_0^\infty xf(x)dx+\frac{1}{8}\int_0^\infty x^2 f(x)\bigg(\frac{f'}{f}(x)-\frac{h'_0}{h_0}(x)\bigg)^2 dx\bigg).\end{aligned} \tag{3.3}$$

Whenever $0 < \kappa < 2$, $\Gamma^a_{-\kappa}(0) = \kappa^2/8$ and

$$\Gamma^a_{-\kappa}(\lambda) - \frac{\kappa^2}{8} \geq \Big(1-\frac{\kappa}{2}\Big)\Gamma^a_0(\lambda).$$

Having in mind that $(\Gamma^a_0)'(0^+) = +\infty$, dividing by $\lambda > 0$ then sending λ to zero reveals that $(\Gamma^a_{-\kappa})'(0^+) = +\infty$ for $-2 < -\kappa < 0$, as required. □

3.2. *Proof of Proposition* 1.1.

PROOFS OF (a) AND (b). That $J_\kappa(v) = J_{-\kappa}(-v)$ for all κ and all v is a consequence of the space reversal invariance. In this light, we may take u, v and κ *non-negative* throughout. Proposition 2.1(b) together with (1.3) imply that

$$I_{-\kappa}(u) = I_\kappa(u) + \kappa/2, \tag{3.4}$$

which, according to the definition of J_κ (see Theorem 2), gives the second equality in (a).

On the other hand, part (c) of Proposition 2.1 tells us that $\Gamma_{1/2}(\lambda) = \sqrt{2\lambda}$. As a result, $I_{1/2}(u) = 1/2u$ which in turn implies that $J_{1/2}(v) = v^2/2$. This delivers (b). □

PROOFS OF (c) AND (d). Suppose first that $\kappa > 1$. According to Proposition 3.1(a), the range of Γ'_κ is $[0, 4/(\kappa-1)]$. Hence, the mapping $\lambda \mapsto \Gamma_\kappa(\lambda) - \lambda u$ is decreasing for $u > 4/(\kappa-1)$, in which case $I_\kappa(u) = \Gamma_\kappa(0) = 0$. Consequently, $J_\kappa(v) = 0$ for all $0 \leq v \leq (\kappa-1)/4$ and

$$J_\kappa(0^+) = J'_\kappa(0^+) = J''_\kappa(0^+) = 0 \qquad \forall\, \kappa > 1.$$

For $0 \leq \kappa \leq 1$, the range of Γ'_κ is $\mathbb{R}^+$. Let $\lambda_\kappa(u)$ denote the maximizer of $\lambda \to \Gamma_\kappa(\lambda) - \lambda u$. Clearly, $\lambda_\kappa(u) = (\Gamma'_\kappa)^{-1}(u)$ and may we write

$$J_\kappa(v) = v\Gamma_\kappa(\lambda_\kappa(1/v)) - \lambda_\kappa(1/v),$$

due to the definition of J_κ. Since both Γ_κ and $v \mapsto 1/v$ are analytic on $(0,\infty)$, the analyticity of J_κ on $\{v > (\kappa-1)^+/4\}$ is guaranteed.

For notational convenience, we will deal with

$$\mu_\kappa(u) = \lambda_\kappa(1/u) = \left(\Gamma_\kappa^{'}\right)^{-1}(1/u)$$

throughout. Accordingly,

$$J_\kappa(v) = v\Gamma_\kappa(\mu_\kappa(v)) - \mu_\kappa(v). \tag{3.5}$$

Straightforward calculations tell us that

$$J_\kappa^{'}(v) = \Gamma_\kappa(\mu_\kappa(v)), \tag{3.6}$$

$$J_\kappa^{''}(v) = \frac{\mu_\kappa^{'}(v)}{v} = -\frac{1}{v^3\Gamma_\kappa^{''}(\mu_\kappa(v))} = -\frac{\Gamma_\kappa^{'3}}{\Gamma_\kappa^{''}}(\mu_\kappa(v)). \tag{3.7}$$

Since $\mu_\kappa(v)$ tends to zero as $v \to 0$, (3.5) together with (3.6) entails $J_\kappa(0) = 0$, $J_\kappa^{'}(0^+) = 0$. Moreover, (3.7) implies that J_κ is strictly convex on $(0,\infty)$. Now differentiating (2.12) w.r.t. x then taking $x = \mu_\kappa(v)$, (3.7) becomes

$$J_\kappa^{''}(v) = \frac{\mu_\kappa(v)}{v^2(1-\kappa+4(v-\Gamma_\kappa\mu_\kappa(v))))}. \tag{3.8}$$

For $0<\kappa<1$,

$$J_\kappa^{''}(v) \sim \frac{1}{1-\kappa}\frac{\mu_\kappa(v)}{v^2},$$

as $v \to 0$. Using (2.10) at $\lambda = \mu_\kappa(v)$, it follows that as $v \to 0$,

$$\frac{\mu_\kappa(v)}{u^2} \sim \kappa\frac{\Gamma_\kappa(\mu_\kappa(v))}{v}$$

and

$$\Gamma_\kappa\Gamma_\kappa^{'}(\mu_\kappa(v)) \sim \kappa\frac{\Gamma_\kappa^2(\mu_\kappa(v))}{\mu_\kappa(v)}.$$

Accordingly,

$$J_\kappa^{''}(v) \sim_{\lambda\to 0} \frac{\kappa^2}{1-\kappa}\frac{\Gamma_\kappa^2(\mu_\kappa(v))}{\mu_\kappa(v)}. \tag{3.9}$$

Proposition 3.1(b) provides us with the behavior of $\Gamma_\kappa(\lambda)/\sqrt{\lambda}$ as λ approaches zero. Accordingly, $J_\kappa^{''}(0^+) = +\infty$ for $0<\kappa<1/2$ and $J_\kappa^{''}(0^+) = 0$ for $1>\kappa>1/2$.

The remaining cases $\kappa=0$ and $\kappa=1$ are related. Indeed, since $J_0^{'}(0) = 0$, we have

$$J_0^{''}(0^+) = \lim_{v\downarrow 0}\frac{\Gamma_0(\mu_0(v))}{v}\left(=\Gamma_0\Gamma_0^{'}(\mu_0(v))\right).$$

On the other hand, parts (c) and (d) of Proposition 2.1, (2.10) and a few lines of elementary calculations yield

$$\frac{\Gamma_1''(\lambda)}{\left(\Gamma_1'(\lambda)\right)^3} \sim_{\lambda\to 0} -\frac{1}{4}\Gamma_0\Gamma_0'(\lambda).$$

The last two equalities in conjunction with

$$J_1''(0^+) = -\lim_{v\to 0}\frac{\left(\Gamma_1'(\mu_1(v))\right)^3}{\Gamma_1''(\mu_1(v))}$$

prove the connection. Now, (2.12), (3.6) together with (3.8) may be rewritten as

$$J_0''(v)\,(1+4(v-\Gamma_0(\mu_0(v)))) = \frac{J_0'(v)}{v}\,(1+2(\Gamma_0(\mu_0(v))-\Gamma_{-1}(\mu_0(v)))).$$

Accordingly, if $J_0''(0^+)$ exists it equals either zero or $+\infty$. Assume $J_0''(0^+)$ to be zero. It follows that $\{\Gamma_0\Gamma_0'(v)\}$, and hence $\{\Gamma_0^2(u)/v\}$, are bounded in the neighborhood of zero which contradicts the fact that

$$2\frac{\Gamma_0^2(\lambda)}{\lambda} = 4+\Gamma_0'(\lambda)$$

increases to infinity as $\lambda \downarrow 0$. Furthermore, it is easy to see that $\{\Gamma_0\Gamma_0'(v)\}$ tends to infinity as $u \downarrow 0$. Thus, $J_0''(0^+) = +\infty$ in which case $J_1''(0^+) = 0$, and the proof of Proposition 1.1. is complete. The proof of Proposition 1.2 is now clear. Parts (b) and (c) follow from Proposition 2.2(d), the definition of J_κ^a together with the space reversal argument. On the other hand, we saw in the proof of Propositioin 2.2(a) that $\Gamma_\kappa^a \le \Gamma_\kappa$. This implies (a) for $v > 0$ hence for $v < 0$ thanks to space reversal invariance once again. As for the proofs of (c) and (d), they follow from (a), (b) and Proposition 1.2(c). Finally, (e) follows from (b) and (c). □

We close this section with an important remark.

REMARK. A variational formula linking annealed and quenched rate functions was obtained in [3] via a min-max theorem. Recalling the definition of I_κ^a, we have

$$\begin{aligned} I_\kappa^a(u) &= \sup_{\lambda\ge 0}(\Gamma_\kappa^a(\lambda)-\lambda u) \\ &= \sup_{\lambda\ge 0}\Bigg(\inf_{g\ge 0:\int_0^{+\infty} g=1}\Bigg(2\lambda\int_0^{+\infty} xg(x)\,dx \\ &\qquad\qquad + \frac{1}{8}\int_0^{\infty} x^2 g(x)\left(\frac{g'}{g}-\frac{h_\kappa'}{h_\kappa}\right)^2(x)\,dx - \lambda u\Bigg)\Bigg), \end{aligned}$$

as in (3.3). Recalling (2.21), one can not exchange the sup and the inf in the above expression since it gives $+\infty$. An interesting reason why a min-max theorem can *not* apply is that [3] assumed that the local drifts are bounded away from -1 and $+1$, which is not a-priori the case in our framework.

4. Proof of Theorem 1.

The upper bound. Kotani's lemma tells us that

$$-\frac{1}{r}\log E^W\left[e^{-\lambda\tau_r}\,1_{\tau_r<\infty}\right]=\frac{1}{r}\int_0^r U_\lambda(s)ds,$$

Q-a.s., for all $\lambda>0$ and $r\geq 0$. The diffusion U_λ being positive-recurrent, the r.h.s. converges both Q a.s. and in $L^1(Q)$ to $\Gamma_\kappa(\lambda)$. Since every compact set of $\mathbb{R}_+$ is nested in a closed interval $[0,a]$ for a certain $a>0$, it suffices to prove the upper bound on $[0,a]$. For all arbitrary $\lambda>0$, we have

$$\limsup_{r\to\infty}\frac{1}{r}\log P^W\left[\frac{\tau_r}{r}<a\right]\leq-(\Gamma_\kappa(\lambda)-\lambda a)\leq -I_\kappa(a),$$

using Chebychev's inequality, (1.3), together with the fact that I_κ is non-increasing. This delivers the upper bound on compact sets of $\mathbb{R}_+$. We now turn to:

The lower bound. The proof will be given in two steps. We first prove the lower bound along the integers, and then fill in the gaps. Let

$$T_0=0,$$
$$T_n=\tau_n-\tau_{n-1}\qquad\forall\, n>0,$$

with the convention that $\infty-\infty=\infty$ in this definition, and

$$\Upsilon_{n,\delta}=\left\{\frac{\tau_n}{n}\in(u-\delta,u+\delta)\right\}. \tag{4.1}$$

By conditioning on the event

$$A_n^M=\bigcap_{1\leq i\leq n}\{T_i\leq M\},$$

and calling $P^{W,M}$ the conditional probability given A_n^M we get

$$\frac{1}{n}\log P^W\left[\Upsilon_{n,\delta}\right]=\frac{1}{n}\log P^{W,M}\left[\Upsilon_{n,\delta}\right]+\frac{1}{n}\log P^W\left[A_n^M\right], \tag{4.2}$$

for all $\delta>0$. For M large enough, define $\lambda_\kappa^M(u)$ as the unique λ solving

$$E_Q\left[\frac{E^{W,M}\left[\tau_1 e^{-\lambda\tau_1}\right]}{E^{W,M}\left[e^{-\lambda\tau_1}\right]}\right]=u.$$

We follow next a change of probability by setting

$$\left.\frac{d\hat{P}^{W,M}}{dP^{W,M}}\right|_{\mathscr{F}_{\tau_n}} = \frac{e^{-\lambda_\kappa^M(u)\tau_n}}{E^{W,M}[e^{-\lambda_\kappa^M(u)\tau_n}]}$$

for each n. According to Proposition 3.1, the range of $\Gamma_\kappa^{'}$ is $(0,\infty)$ whenever $|\kappa| \le 1$. In this case, for all $u > 0$ there exists a unique $\lambda_\kappa(u) = \left(\Gamma_\kappa^{'}\right)^{-1}(u)$ and we need not truncate by M. It suffices to perform the previous change of probability with P^W and $\lambda_\kappa(u)$ replacing $P^{W,M}$ and $\lambda_\kappa^M(u)$ respectively. This is no longer true for $|\kappa| > 1$ and $u > 4/(|\kappa|-1)$, and we are done by *truncating*. Note that $\lambda_\kappa^M(u) < 0$. The first term of the r.h.s of (4.2) is greater than

$$\frac{1}{n}\log E^{W,M}\left[e^{-\lambda_\kappa^M(u)\tau_n}\right] + \lambda_\kappa^M(u)(u+\delta) + \frac{1}{n}\log \hat{P}^{W,M}\left[\Upsilon_{n,\delta}\right]. \tag{4.3}$$

Set

$$\Gamma_\kappa^M(\lambda) = -E_Q\left[\log E^W\left[e^{-\lambda\tau_1}1_{\tau_1\le M}\right]\right].$$

The T_i 's remaining independent under $P^{W,M}$, the first term of (4.3) may be written as

$$\frac{1}{n}\sum_{1\le i\le n}\log E^W\left[e^{-\lambda_\kappa^M(u)T_i}|T_i \le M\right]. \tag{4.4}$$

The law of W being ergodic under the action of Θ, (4.4) approaches

$$E_Q\left[\log E^W\left[e^{-\lambda_\kappa^M(u)\tau_1}|\tau_1\le M\right]\right] = -\Gamma_\kappa^M(\lambda_\kappa^M(u)) - E_Q\left[\log P^W\left[\tau_1 \le M\right]\right]$$

as $n\to\infty$. Following the same pattern, we have

$$P^W\left[A_n^M\right] = \prod_{1\le i\le n} P^W\left[T_i\le M\right] = \prod_{1\le i\le n-1} P^{\Theta_i W}\left[\tau_1\le M\right],$$

and the ergodic theorem implies that Q-a.s.,

$$\lim_{n\to\infty}\frac{1}{n}\log P^W\left[A_n^M\right] = E_Q\left[\log P^W\left[\tau_1\le M\right]\right].$$

Next we prove that Q a.s.,

$$\lim_{n\to\infty}\hat{P}^{W,M}\left[\Upsilon_{n,\delta}\right] = 1 \qquad \forall\,\delta > 0. \tag{4.5}$$

The ergodic theorem once again implies that

$$\frac{1}{n}E^{\hat{P}^{W,M}}\left[\tau_n\right] = \frac{1}{n}\sum_{1\le i\le n} E^{\hat{P}^{W,M}}\left[T_i\right] = \frac{1}{n}\sum_{1\le i\le n-1} E^{\hat{P}^{\Theta_i W}}\left[\tau_1\right]$$

which tends to $E_Q\left[E_{\hat{P}^{W,M}}\left[\tau_1\right]\right] = u$ by the definitions of $\hat{P}^{W,M}$ and $\lambda_\kappa^M(u)$. Moreover,

$$E^{\hat{P}^{W,M}}\left[\left(\frac{\tau_n}{n} - E^{\hat{P}^{W,M}}\left[\frac{\tau_n}{n}\right]\right)^4\right] = E^{\hat{P}^{W,M}}\left[\left(\frac{1}{n}\sum_{1\le i\le n} T_i - E^{\hat{P}^{W,M}}[T_i]\right)^4\right] \le \frac{3M^4}{n^2},$$

where we have used the T_i 's independence under $\hat{P}^{W,M}$ so that (4.5) follows from Borel-Cantelli lemma.

Having proved (4.5), it follows that Q a.s.,

$$\begin{aligned}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P^W\left[\Upsilon_{n,\delta}\right] &\geq \lambda_\kappa^M(u)u-\Gamma_M(\lambda_\kappa^M(u))\\ &\geq -\sup_\lambda\left(\Gamma_\kappa^M(\lambda)-\lambda u\right)=: -I_M(u).\end{aligned}$$

Let $I^*(u)=\limsup_{M\to\infty} I_M(u)$. By definition, $I_M(u)\geq \Gamma_\kappa^M(0)\geq 0$ giving that $I^*(u)\geq 0$. Moreover, $I_M(u)<\infty$ for large M, so is $I^*(u)$. Hence, the level sets $\{\lambda : \Gamma_\kappa^M(\lambda)-\lambda u\geq I^*(u)\}$ are non-empty, compact, nested sets implying that their intersection contain some $\lambda^*<\infty$. By Lebesgue's monotone convergence, we get

$$\log\Gamma_\kappa(\lambda^*)=\lim_{M\to\infty}\log\Gamma_\kappa^M(\lambda^*)\geq \lambda^* u+I^*(u),$$

giving that $-I^*(u)\geq -\sup_\lambda(\Gamma_\kappa(\lambda)-\lambda u)=-I_\kappa(u)$ by Proposition 2.1(e) and (1.3).

All that is left to do is fill in the gaps. For the r's lying in the gap, that is, for $n\leq r<n+1$,

$$\begin{aligned}&P^W\left[\tau_r\in(r(u-\delta);r(u+\delta))\right]\\ &\quad\geq P^W\left[r(u-\delta)<\tau_n<\tau_{n+1}<r(u+\delta)\right]\\ &\quad\geq P^W\left[r(u-\delta/2)<\tau_n<r(u+\delta/2)\right]P^W\left[0<\tau_{n+1}-\tau_n<r\delta/2\right]\\ &\quad\geq P^W\left[n(u-\delta/4)<\tau_n<n(u+\delta/2)\right]P^W\left[0<T_{n+1}<n\delta/2\right],\end{aligned}$$

for n large enough, and the statement follows from

$$\lim_{n\to\infty}\frac{1}{n}\log P^W\left[0<T_n<n\delta\right]=0. \tag{4.6}$$

For all $\lambda>0$, we split $\left\{e^{-\lambda T_n}1_{T_n<\infty}\right\}$ into two parts depending on whether or not $T_n>n\delta$, so that

$$E^W\left[e^{-\lambda T_n}1_{T_n<\infty}\right]\leq P^W\left[T_n\leq n\delta\right]+e^{-\lambda n\delta}P^W\left[\infty>T_n>n\delta\right].$$

Thus,

$$P^W\left[T_n\leq n\delta\right]\geq E^W\left[e^{-\lambda T_n}1_{T_n<\infty}\right]-e^{-\lambda n\delta}.$$

On the other hand, the ergodic theorem yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{1\leq i\leq n}\log E^{\Theta_n W}\left[e^{-\lambda\tau_1}1_{\tau_1<\infty}\right]=E_Q\left[\log E^W\left[e^{-\lambda\tau_1}1_{\tau_1<\infty}\right]\right]>-\infty,$$

Q-a.s. Since

$$E^W\left[e^{-\lambda T_n}1_{T_n<\infty}\right]=E^{\Theta_{n-1}W}\left[e^{-\lambda\tau_1}1_{\tau_1<\infty}\right],$$

we have

$$\lim_{n\to\infty} \frac{1}{n} \log E^W \left[e^{-\lambda T_n} 1_{T_n<\infty}\right] = 0,$$

Q-a.s. In other words, Q-a.s., for all $\varepsilon > 0$, and n large enough $E^W\left[e^{-\lambda T_n}\right] \geq e^{-\varepsilon n}$ giving that for $\varepsilon < \lambda\delta$,

$$0 \geq \liminf_{n\to\infty} \frac{1}{n} \log P^W\left[T_n \leq n\delta\right] \geq \liminf_{n\to\infty} \frac{1}{n} \log\left(e^{-\varepsilon n} - e^{-\lambda\delta n}\right) \geq -\lambda\delta,$$

which implies straightforwardly (4.6) by sending λ to zero. □

5. Proof of Theorem 2. First note that, Q a.s., at frozen environment W, the family of the distributions of $X(t)/t$ is exponentially tight. Indeed, thanks to space reversal invariance, all we need to check is

$$\lim_{u\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log P^W\left[X(t) > ut\right] = -\infty.$$

Pick $u > 0$. Clearly, $\{X(t) > ut\} \subset \{\tau_{ut} < t\}$ so that

$$\limsup_{t\to\infty} \frac{1}{t} \log P^W\left[X(t) > ut\right] \leq -uI_\kappa(1/u), \tag{5.1}$$

where we have used the upper bound for τ_r/r. Recalling the definition of I_κ, (1.3), we have

$$-uI_\kappa(1/u) \leq -u\Gamma_\kappa(\lambda_0) + \lambda_0,$$

for all $\lambda_0 > 0$. Since $\Gamma_\kappa(\lambda_0) > 0$, the r.h.s. of (5.1) tends to $-\infty$ as $u \to \infty$, as required. Section 3 guarantees that J_κ is convex and that $J_\kappa(0) = 0$. By virtue of Theorem 1, this delivers the upper bound on compact sets of $\mathbb{R}_+$ ($u > 0$), hence on *all* compact sets thanks to the space reversal invariance hence on closed sets thanks to the exponential tightness.

Let $\varepsilon > 0$ and $\delta > 0$ be given. By splitting the event $\{t(1-\varepsilon) < \tau_{tu} < t\}$ into two parts depending on whether or not $\left|X(t) - tu\right| \leq t\delta$ we get

$$P^W\left[t(1-\varepsilon) < \tau_{tu} < t\right] \leq P^W\left[\left|X(t) - tu\right| \leq t\delta\right] + \mu_{t,\varepsilon}$$

where

$$\mu_{t,\varepsilon} = P^W\left[\left|X(t) - tu\right| > t\delta; t(1-\varepsilon) < \tau_{tu} < t\right].$$

We have

$$\begin{aligned}\mu_{t,\varepsilon} &\leq P^W\left[\sup_{0<s-\tau_{tu}<t\varepsilon} \left|X(s) - tu\right| > t\delta\right] = P^{\Theta_{tu}W}\left[\sup_{0<s<t\varepsilon} \left|X(s)\right| > t\delta\right] \\ &= P^{\Theta_{tu}W}\left[\tau_{t\delta} \wedge \tau_{-t\delta} < t\varepsilon\right] \leq P^{\Theta_{tu}W}\left[\tau_{t\delta} < t\varepsilon\right] + P^{\Theta_{tu}W}\left[\tau_{-t\delta} < t\varepsilon\right].\end{aligned}$$

The lower bound follows from Theorem 1 so long as we prove

$$\lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log \mu_{t,\varepsilon} = -\infty,$$

which reduces to

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log P^{\Theta_{tu}W}\left[\tau_{t\delta}<t\varepsilon\right]=-\infty, \tag{5.2}$$

thanks to space reversal invariance. For all $\lambda>0$,

$$\limsup_{t\to\infty}\frac{1}{t}\log P^{\Theta_{tu}W}\left[\tau_{t\delta}<t\varepsilon\right]\le\lambda\varepsilon+\limsup_{t\to\infty}\frac{1}{t}\log E^{W}_{tu}\left[\exp\left(-\lambda\tau_{t(u+\delta)}\right)\right]$$

due to Chebyshev's inequality. The last formula together with the identity

$$E^{W}\left[e^{-\lambda\tau_{t(u+\delta)}}\right]=E^{W}\left[e^{-\lambda\tau_{tu}}\right]E^{W}_{tu}\left[e^{-\lambda\tau_{t(u+\delta)}}\right],$$

(where we have dropped the indicators, for brevity) give

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log P^{\Theta_{tu}W}\left[\tau_{t\delta}<t\varepsilon\right]\le-\delta\Gamma_{\kappa}(\lambda).$$

At fixed $\delta>0$, the latter bound tends to $-\infty$ as λ goes to $+\infty$. □

6. Proofs of Theorems 3 and 4. We saw how one can transfer LDP for the hitting times to the positions in the quenched framework. The same is true in the annealed case and the argument differs very little from that used in the proof of Theorem 2. We begin this section by showing how Theorem 4 follows from Theorem 3.

PROOF OF THEOREM 4. A glance at the proof of Theorem 2 shows that we are done by rewriting the same lines with $\mathbb{P}$, Γ^{a}_{κ} and I^{a}_{κ} replacing $P^{\Theta_{tu}W}$, Γ_{κ} and I_{κ} respectively. The only difference comes from the proof of (5.2), with $\mathbb{P}$ instead of $P^{\Theta_{tu}W}$, which is even simpler in the annealed case: all we need is Chebychev's inequality and the definition of Γ^{a}_{κ}. □

PROOF OF THEOREM 3. *Upper bound.* The upper bound follows straightforwardly from Chebychev's inequality as in the proof of Theorem 1.

Lower bound. For all $u>0$, set

$$\hat{\lambda}_{\kappa}(u)=\inf\{\lambda>0;(\Gamma^{a}_{\kappa})'(\lambda^{+})<u\}.$$

Having in mind Proposition 3.2(a), $(\Gamma^{a}_{\kappa})'(\lambda^{+})$ approaches 0 as λ tends to infinity and the above definition makes sense. Now, since Γ^{a}_{κ} is concave, its right-hand derivative is right-continuous so that

$$\hat{\lambda}_{\kappa}(u)=0\Leftrightarrow(\Gamma^{a}_{\kappa})'(0^{+})\le u.$$

Thus, $\hat{\lambda}_{\kappa}(u)>0$ iff $u<(\Gamma^{a}_{\kappa})'(0^{+})$. For $|\kappa+1|\le 2$, the latter holds for all u since, in this case, according to Proposition 3.2(b), $(\Gamma^{a}_{\kappa})'(0^{+})$ is infinite. Concavity once again, shows that $\lambda\mapsto\Gamma^{a}_{\kappa}(\lambda)-\lambda u$ is non-decreasing for $0\le\lambda\le\hat{\lambda}_{\kappa}(u)$ and non-increasing for $\lambda>\hat{\lambda}_{\kappa}(u)$ so that

$$I^{a}_{\kappa}(u)=\Gamma^{a}_{\kappa}(\hat{\lambda}_{\kappa}(u))-\hat{\lambda}_{\kappa}(u)u.$$

The proof of the lower bound is organized as follows: in the case where $\hat{\lambda}_\kappa(u) > 0$ we prove the lower bound by performing a change of probability under which τ_r/r approaches u as r tends to infinity. Otherwise, the proof will be carried out *directly*.

Suppose first that $|\kappa+1| \le 2$, or that $u < (\Gamma_\kappa^a)'(0^+)$. In both situations $\hat{\lambda}_\kappa(u) > 0$ and we define $\hat{\mathbb{P}}$ by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathscr{F}_{\tau_r}} = \frac{e^{-\hat{\lambda}_\kappa(u)\tau_r}}{\mathbb{E}[e^{-\hat{\lambda}_\kappa(u)\tau_r}1_{\tau_r<\infty}]}1_{\tau_r<\infty}.$$

Keeping with the notation (4.1) of Section 4, we write

$$\mathbb{P}\left[\Upsilon_{r,\delta}\right] \ge e^{\hat{\lambda}_\kappa(u)r(u-\delta)}\mathbb{E}\left[e^{-\hat{\lambda}_\kappa(u)\tau_r}1_{\tau_r<\infty}\right]\hat{\mathbb{P}}\left[\Upsilon_{r,\delta}\right].$$

As in the quenched case, all we need to prove is that

$$\lim_{\delta\to 0}\lim_{r\to\infty}\hat{\mathbb{P}}\left[\Upsilon_{r,\delta}\right] = 1. \tag{6.1}$$

Making use of Chebychev's inequality coupled with the concavity of Γ_κ^a, we get that for all $\lambda, \delta, \varepsilon > 0$ and r sufficiently large,

$$\hat{\mathbb{P}}\left[\tau_r > r(u+\delta)\right] \le \exp -r\left[\lambda\left(u+\delta-(\Gamma_\kappa^a)'\left((\hat{\lambda}_\kappa(u)-\lambda)^-\right)-2\varepsilon\right)\right],$$
$$\hat{\mathbb{P}}\left[\tau_r < r(u-\delta)\right] \le \exp -r\left[\lambda\left(-u+\delta+(\Gamma_\kappa^a)'\left((\hat{\lambda}_\kappa(u)+\lambda)^+\right)-2\varepsilon\right)\right].$$

If we manage to render the above exponents negative for ε small enough, (6.1) follows. From the definition of $\hat{\lambda}_\kappa(u)$ we get

$$(\Gamma_\kappa^a)'(\hat{\lambda}_\kappa(u)^+) \le u \le (\Gamma_\kappa^a)'(\hat{\lambda}_\kappa(u)^-).$$

Moreover, Γ_κ^a being strictly concave, its right-hand (resp. left-hand) derivative is right-continuous (resp. left-continuous) and decreasing. It follows that for all $\delta > 0$, one can pick λ in the vicinity of 0 in such a way that

$$u-\delta \le (\Gamma_\kappa^a)'\left((\hat{\lambda}_\kappa(u)+\lambda)^+\right) \le (\Gamma_\kappa^a)'\left((\hat{\lambda}_\kappa(u))^+\right) \le u,$$
$$u \le (\Gamma_\kappa^a)'\left((\hat{\lambda}_\kappa(u))^-\right) \le (\Gamma_\kappa^a)'\left((\hat{\lambda}_\kappa(u)-\lambda)^-\right) \le u+\delta.$$

Accordingly, taking ε small enough delivers (6.1) and hence the lower bound in this case.

For $|\kappa+1| > 2$ and $u > (\Gamma_\kappa^a)'(0^+)$, $\hat{\lambda}_\kappa(u) = 0$ and the above reasoning no longer applies. In this case, $I_\kappa^a(u) = \Gamma_\kappa^a(0)$ and we shall *directly* prove that

$$\lim_{\delta\to 0}\liminf_{r\to\infty}\frac{1}{r}\log\mathbb{P}\left[\frac{\tau_r}{r}\in(u-\delta,u+\delta)\right] \ge -\Gamma_\kappa^a(0),$$

$\mathbb{P}$ being the averaged probability $Q[P^W[.]]$.

Suppose first that $\kappa > 1$ and take $u > (\Gamma_\kappa^a)'(0^+) = 4/(\kappa-1)$. For all $\varepsilon, \delta > 0$, we have

$$E_Q\left[P^W\left[\Upsilon_{r,\delta}\right]\right] \geq e^{-\varepsilon r} Q\left[P^W\left[\Upsilon_{r,\delta}\right] \geq e^{-\varepsilon r}\right], \tag{6.2}$$

due to Chebychev's inequality. According to Theorem 1, Q a.s.,

$$\liminf_{r\to\infty} \frac{1}{r} \log P^W\left[\Upsilon_{r,\delta}\right] \geq -I_\kappa(u) = 0$$

for all $u > 4/(\kappa-1)$ and all $\delta > 0$. More accurately, the non-increasing $\lim_{\delta\downarrow 0}$ of the l.h.s. is ≥ 0 so that the above inequality holds for all $\delta > 0$. Moreover, the lim inf is a limit. As a result, for all $\delta > 0$, the probability term in the r.h.s of (6.2) approaches 1 as $r \uparrow \infty$. Since $\Gamma_\kappa^a(0) = 0$, and ε, δ are as small as we please, the lower bound is proved for $\kappa > 1$ and $u > 4/(\kappa-1)$.

All is left to treat is the case where $\kappa < -3$ and $u > 4/(-\kappa-3)$. Once again, the proof hinges upon a change of probability which, this time, modifies the environment's drift. Recall that $W(x) = B(x) - \kappa x/2$ where B is a Q-Brownian motion and let $\hat{Q}$ denote the probability under which B is a Brownian motion with drift -1. Accordingly,

$$W(x) = \hat{B}(x) - \frac{\kappa+2}{2}x$$

under $\hat{Q}$ where $\hat{B}$ is a $\hat{Q}$-Brownian motion. Before moving to the proof of the lower bound, we shall need the following.

CLAIM. *For all* $\varepsilon, \delta > 0$,

$$P^W\left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right] \quad \textit{only depends on} \quad \{W(x); -\varepsilon r < x < r\}.$$

PROOF. Given the quantity between brackets above, it suffices to show that $\{X(t\wedge\tau_r\wedge\tau_{-\varepsilon r})\,; t\geq 0\}$ depends only on $\{W(x); -\varepsilon r < x < r\}$. Using the Itô–Mc-Kean's representation, (1.1), we may write

$$\begin{aligned} X(t\wedge\tau_r\wedge\tau_{-\varepsilon r}) &= S^{-1}\left(\mathscr{B}\left(T^{-1}(t)\wedge T^{-1}(\tau_r)\wedge T^{-1}(\tau_{-\varepsilon r})\right)\right) \\ &= S^{-1}\left(\mathscr{B}\left(T^{-1}(t)\wedge \sigma_{S(r)}\wedge\sigma_{S(-\varepsilon r)}\right)\right), \end{aligned}$$

since $\tau_r = T(\sigma_{S(r)})$. Proving the claim amounts to proving that if we replace W by another continuous potential, say W^0, which is equal to W on $[-\varepsilon r, r]$ and constant elsewhere,

$$X(t\wedge\tau_r\wedge\tau_{-\varepsilon r}) = S^{0-1}\left(\mathscr{B}\left(T^{0\,-1}(t)\wedge\sigma_{S^0(r)}\wedge\sigma_{S^0(-\varepsilon r)}\right)\right), \tag{6.3}$$

where S^0 and T^0 denote the scale function and the time-change defined in Section 1 with W^0 replacing W. This is valid for the following reason.

For all $s \leq \sigma_{S(r)}\wedge\sigma_{S(-\varepsilon r)}$, $\mathscr{B}$ belongs to $[S(-\varepsilon r), S(r)]$ on $[0, s]$. Since $W \equiv W^0$ on $[-\varepsilon r, r]$, $S \equiv S^0$ on $[-\varepsilon r, r]$. Thus, for all $0 \leq u \leq s$, $S^{-1}(\mathscr{B}_u) = S^{0\,-1}(\mathscr{B}_u)$ giving that $T(u) = T^{0\,-1}(u)$. Taking $s = T^{-1}(t)$ delivers (6.3) and ends the proof of the claim. $\square$

We are now ready to prove the lower bound. For all $\varepsilon > 0$ and all $\delta > 0$, we write

$$\begin{aligned}
E_Q\left[P^W\left[\Upsilon_{r,\delta}\right]\right] &\geq E_Q\left[P^W\left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right]\right],\\
&\geq E_{\hat{Q}}\left[\left(\frac{d\hat{Q}}{dQ}\right)^{-1} P^W\left[\Upsilon_{r,\delta};\ \tau_r < \tau_{-r\varepsilon}\right]\right],\\
&= E_{\hat{Q}}\left[\exp\left(B(r) - B(-\varepsilon r) + \frac{1+\varepsilon}{2} r\right) P^W\left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right]\right],
\end{aligned}$$

where we have used the fact that $P^W[\Upsilon_{r,\delta};\ \tau_r < \tau_{-r\varepsilon}]$ only involves $\{W(x), -\varepsilon r < x < r\}$ so that

$$\frac{d\hat{Q}}{dQ} = \exp\left\{-\int_{-\varepsilon r}^{r} dB(s) - \frac{1}{2}\int_{-\varepsilon r}^{r} ds\right\}.$$

The expression $P[V; A]$ denotes the P-probability of V on the event A.

For all $\eta > 0$, set

$$\begin{aligned}
\Omega_r^W &\stackrel{\text{def}}{=} \Omega_{r,\delta,\varepsilon,\eta}^W\\
&= \left\{B(r) - B(-\varepsilon r) \geq -r(1+2\varepsilon); P^W\left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right] \geq e^{-r(I_{\kappa+2}(u)+\eta)}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{P}\left[\Upsilon_{r,\delta}\right] &\geq \exp r\left(-\frac{1+3\varepsilon}{2} - I_{\kappa+2}(u) - \eta\right)\hat{Q}\left[\Omega_r^W\right]\\
&= \exp r\left(\frac{1+\kappa}{2} - \frac{3}{2}\varepsilon - \eta - I_{-\kappa-2}(u)\right)\hat{Q}\left[\Omega_r^W\right],
\end{aligned}$$

thanks to (3.4). Now, observing that

$$I_{-\kappa-2}(u) = 0 \quad \textit{for} \quad u > 4/(-\kappa-3),$$

and recalling that for $\kappa < -3$, $-\Gamma_\kappa^a(0) = (1+\kappa)/2$, the lower bound is guaranteed if we prove that

$$\lim_{r\to\infty} \hat{Q}\left[\Omega_r^W\right] = 1,$$

for $\eta, \delta, \varepsilon > 0$ small enough. Clearly,

$$\hat{Q}\left[\Omega_r^W\right] \geq \hat{Q}\left[P^W\left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right] \geq e^{-r(I_{\kappa+2}(u)+\eta)}\right] - \hat{Q}\left[\hat{B}(r) - \hat{B}(-\varepsilon r) < -\varepsilon r\right].$$

Since the last probability term converges exponentially fast to zero as r tends to infinity, it suffices to prove that

$$\lim_{r\to\infty} \hat{Q}\left[P^W\left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right] \geq e^{-r(I_{\kappa+2}(u)+\eta)}\right] = 1,$$

for all ε, η, $\delta > 0$ small enough. This is indeed the case if we show that $\hat{Q}$ a.s,

$$\liminf_{r\to\infty} \frac{1}{r} \log P^W \left[\Upsilon_{r,\delta}; \tau_r < \tau_{-r\varepsilon}\right] = -I_{\kappa+2}(u) = \frac{\kappa+2}{2}, \tag{6.4}$$

for all ε and $\delta > 0$ small enough. Let us check (6.4).

By virtue of the strong Markov property, we have

$$\begin{aligned} P^W & \left[\frac{\tau_r}{r} \in (u-\delta, u+\delta); \tau_r < \tau_{-\varepsilon r}\right] \\ & \geq P^W \left[r(u-\delta) < \tau_r < r(u+\delta) < \tau_{-\varepsilon r}\right] \\ & \geq P^W \left[r(u-\delta) < \tau_r < r(u+\delta)\right] P_r^W \left[\tau_{-\varepsilon r} > 2r\delta\right]. \end{aligned}$$

Theorem 1 tells us that for all $\delta > 0$, Q a.s.,

$$\liminf_{r\to\infty} \frac{1}{r} \log P^W \left[\Upsilon_{r,\delta}\right] \geq -I_\kappa(u),$$

or equivalently since under $\hat{Q}$, W is a Brownian motion with drift $-(\kappa+2)/2$,

$$\liminf_{r\to\infty} \frac{1}{r} \log P^W \left[\Upsilon_{r,\delta}\right] \geq -I_{\kappa+2}(u),$$

$\hat{Q}$ a.s. Note that the lim inf in (6.4) is less than or equal to

$$\lim_{r\to\infty} \frac{1}{r} \log P^W \left[\tau_r < \infty\right] = -\Gamma_{\kappa+2}(0) = \frac{\kappa+2}{2}.$$

Thus, the lim inf is actually a limit. Accordingly, we are done so long as we prove that

$$\lim_{r\to\infty} P_r^W[\tau_{-\varepsilon r} > 2r\delta] = 1, \tag{6.5}$$

for all $\varepsilon, \delta > 0$ small enough.

Indeed, for all $\lambda > 0$ and all $\mu > 0$, Chebychev's inequality yields

$$\begin{aligned} P_r^W \left[\tau_{-\varepsilon r} < 2r\delta\right] & \leq e^{2\lambda r\delta} E_r^W \left[e^{-\lambda\tau_{-\varepsilon r}}\right] \\ & \leq e^{2\lambda r\delta} E^W \left[e^{-\lambda\tau_{-\varepsilon r}}\right] \\ & \leq e^{r(2\lambda\delta - \varepsilon(\Gamma_{-\kappa}(\lambda)-\mu))}, \qquad Q \text{ a.s.} \end{aligned}$$

for r large enough. We have used Kotani's lemma together with the space reversal invariance in deriving the last inequality. Taking μ and λ such that $\mu < \Gamma_{-\kappa}(\lambda)$, the last quantity decays to zero if δ and ε are such that

$$\delta \leq \varepsilon \frac{\Gamma_{-\kappa}(\lambda) - \mu}{2\lambda},$$

which delivers (6.5). The proof of the lower bound is complete upon first sending δ to zero, so that the above constraint is satisfied, then ε and finally η. □

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