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ASYMPTOTIC DENSITY IN A THRESHOLD COALESCING AND ANNIHILATING RANDOM WALK[1]

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We consider an interacting random walk on $\mathbb{Z}^d$ where particles interact only at times when a particle jumps to a site at which there are at least $n-1$ other particles present. If there are $i \geq n-1$ particles present, then the particle coalesces (is removed from the system) with probability c_i and annihilates (is removed along with another particle) with probability a_i. We call this process the n-threshold randomly coalescing and annihilating random walk. We show that, for $n \geq 3$, if both a_i and $a_i + c_i$ are increasing in i and if the dimension d is at least $2n+4$, then

$$P\{\text{the origin is occupied at time } t\} \sim C(d,n)t^{-\frac{1}{n-1}},$$

$$E\{\text{number of particles at the origin at time } t\} \sim C(d,n)t^{-\frac{1}{n-1}}.$$

The constants $C(d,n)$ are explicitly identified. The proof is an extension of a result obtained by Kesten and van den Berg for the 2-threshold coalescing random walk and is based on an approximation for $dE(t)/dt$.

1. Introduction. In this paper, we study a broad class of systems which make up an extension of both the standard coalescing random walk and the standard annihilating random walk. First introduced as the dual of the voter model, the standard coalescing random walk is one of the more basic particle systems. In brief, it considers a system of particles on the space $\mathbb{Z}^d$, starting at time 0 with a particle at each site. Particles jump at times determined by Poisson processes, with displacements determined by some transition kernel $q(\cdot)$. One particle is removed at each collision of two particles. In the annihilating random walk, two particles (i.e., both particles) are removed at each collision. One of the main problems of interest, having applications to the dual voter model, is that of determining the asymptotic order of $P(t)$, the probability that the origin (or any other fixed site) is occupied at time t.

In 1980, Bramson and Griffeath applied a certain result of Sawyer's (1979) to show that, as $t \to \infty$ in the coalescing case, $P(t) \sim \log t/(\pi t)$ for $d = 2$ and $P(t) \sim 1/(\gamma_d t)$ for $d \geq 3$. Here γ_d denotes the probability that a d-dimensional simple random walk never returns to the origin. Technically, these results were established in the case of simple random walks only. Soon afterward, Arattia (1981) showed that $P(t)$ in the annihilating system is asymptotically one-half that of the coalescing system [see Arratia (1981), Theorem 3], so long as the

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random walks are multidimensional (i.e., not restricted to a one-dimensional subspace).

Recently, Kesten and van den Berg (2000) established a new technique for analyzing the asymptotic density, $E(t)$, of coalescing random walks using a differential equation for $E(t)$. In addition to providing the result of Bramson and Griffeath for general random walks satisfying certain moment conditions, their work also provides a rate of convergence to the limiting density. More importantly, it extends to the cases where particles coalesce only with probabilities strictly less than one (so long as these probabilities are increasing with the number of particles present during a collision). Their result requires the probabilities of coalescence to be strictly positive for any collision.

In this paper, we extend their result in several ways, considering systems where multiple particles may collide with zero probability of interaction and where collisions may result either in coalescence or in annihilation of particles (resulting in the loss of one or two particles, respectively).

The following is a heuristic description of the model studied in this paper:

Each system, ξ_t, is associated with a continuous time random walk, S_t, which has transition kernel $q(\cdot)$ and jumps at the jump-times of a rate one Poisson process. We assume that the support of $q(\cdot)$ contains d linearly independent vectors. We also assume that S_t has zero mean and finite second moment.

We start at time 0 with one particle at each site of $\mathbb{Z}^d$. For any $x \in \mathbb{Z}^d$, we let the particle starting at x move according to S_t^x, which is an independent copy of $S_t + x$. The particle may be removed, however, if it jumps to a site which already contains particles, or if another particle jumps onto the site that the first particle currently occupies.

Each system has a threshold n, which is the minimum number of particles which must be present at a site in order for interaction to occur. There are also positive constants a_i and c_i, with $a_i + c_i \leq 1$, representing the probabilities that a particle jumping onto a site occupied by $i \geq n-1$ particles will cause an annihilation or a coalescence (respectively). The proof of the main result assumes that both a_i and $a_i + c_i$ are increasing in i. To each jump, we attach a uniform $[0, 1]$ random variable U. If a particle jumps to a position y which is occupied by i particles and if the corresponding U satisfies $U \leq a_i$, then we remove two particles from y. If $a_i < U \leq a_i + c_i$, then we remove one particle only. No particles are removed if $U > a_i + c_i$.

Our main result is the following theorem.

THEOREM 1. *Assume that*

$$a_i = c_i = 0 \quad \textit{for } i = 0, \ldots, n-2;\ a_{n-1} + c_{n-1} > 0$$

and that

$$a_i \textit{ and } a_i + c_i \textit{ are increasing in } i. \tag{1}$$

Assume also that the particles move according to continuous time random walks which are distributed as translates of $\{S_t\}$, where S_t is a continuous *time random walk with transition kernel* $q(\cdot)$ *and jump-times dictated by a rate one Poisson process. Assume also that*

$$\text{the support of } q(\cdot) \text{ contains } d \text{ linearly independent vectors}, \tag{2}$$

$$q(\mathbf{0}) = 0, \ ES_t = \mathbf{0} \ \text{ and } \sum_{y \in \mathbb{Z}^d} |y|^2 q(y) < \infty. \tag{3}$$

Define

$$\gamma_{n,d} := P\big\{n \text{ independent copies of } S_{\cdot} \text{ never coincide after the first walk leaves } \mathbf{0}\big\}$$

and

$$C_1 = C_1(d,n) := \left[\frac{(n-1)(2a_{n-1} + c_{n-1})\gamma_{n,d}}{1 - (1 - a_{n-1} - c_{n-1})(1 - \gamma_{n,d})}\right]^{-\frac{1}{n-1}}.$$

Then, for $n \geq 3$ *and* $d \geq 2n+4$, *there exists a* $\zeta = \zeta(d,n)$ *such that, for large* t,

$$P\{\text{the origin is occupied at time } t\} = C_1 t^{-\frac{1}{n-1}} + O\left(t^{-\frac{1}{n-1}-\zeta}\right) \tag{4}$$

and

$$E\{\text{number of particles at the origin at time } t\} = C_1 t^{-\frac{1}{n-1}} + O\left(t^{-\frac{1}{n-1}-\zeta}\right). \tag{5}$$

Also, for $p \geq 2$,

$$\begin{aligned} &P\{\text{there are at least } p \text{ particles at the origin at time } t\} \\ &\quad = O\left(t^{-\frac{p}{n-1}} \vee t^{\frac{n-2}{n-1} - d(1-\varepsilon)/(2n-2)}\right), \end{aligned} \tag{6}$$

where $\varepsilon > 0$ *can be taken to be arbitrarily small.*

The condition $n \geq 3$ is necessary to achieve the lower bound in Lemma 11. However, the case where $n = 2$ and $a_i = 0$ for all i is the system studied by Kesten and van den Berg (2000).

2. Description and construction of the Markov process. Since our system is so similar to the system introduced in Kesten and van den Berg (2000), we will follow their construction in many places. The main difference between the construction of our system and that of the system in Kesten and van den Berg is that, because we allow for annihilation to occur, our system does not have the *monotonicity property* that increasing the number of particles present at time zero will increase the number of particles present at any site for any time t. We do, however, establish a certain relation between the two types of systems (Lemma 8).

The state space for the n-threshold randomly coalescing and annihilating random walk will be a subset of $\Xi_0 := \{\mathbb{Z}^+\}^{\mathbb{Z}^d}$. A point of Ξ_0 is denoted by

$\xi = \{\xi(x) : x \in \mathbb{Z}^d\}$. ξ_t is then the state of our system at time t, and $\xi_t(x)$ denotes the number of particles at position x.

Define $\xi_{\mathbf{N},\mathbf{t}}$ to be the process with initial state $\xi_0(y)\mathbb{1}[|y| \le N]$. We start by constructing the processes $\xi_{N,t}$ for any $N > 0$ and then use a limiting procedure (Lemma 2) to arrive at the system ξ_t.

For each $x \in \mathbb{Z}^d$ and $k \ge 1$, we construct an independent Poisson process $\{N_t(x, k)\}_{t \ge 0}$ with jump-times $\tau_1(x, k) < \tau_2(x, k) < \cdots$ and $N_0(x, k) = \tau_0(x, k) = 0$. To each $\tau_n(x, k)$, we attach a displacement $y_n(x, k)$ and a collection of random variables $X(n, x, k, i)$, $i \ge 0$. Particles perform continuous-time random walks dictated by these random variables. At time $\tau_n(x, k)$, if $\xi_{N,t}(x) \ge k$, then one particle jumps to $x + y_n(x, k)$. $X(n, x, k, i)$ takes the values 2, 1 and 0, depending, as described below, on the number of particles i at the position $x + y_n(x, k)$. $X(n, x, k, i)$ specifies whether a particle which jumps from x at time $\tau_n(x, k)$ annihilates, coalesces or stays in the system, respectively, so that

$$\xi_{\tau_n(x,k)}(x) = \xi_{\tau_n(x,k)^-}(x) - 1$$

and

$$\xi_{\tau_n(x,k)}(x + y_n(x, k)) = \xi_{\tau_n(x,k)^-}(x + y_n(x, k)) + 1 - X(n, x, k, i).$$

Assume that all $y_n(x, k)$ and $X(n, x, k, \cdot)$ for different (n, x, k) are independent of each other and of all Poisson processes and that, for fixed (n, x, k), the $y_n(x, k)$ and $X(n, x, k, \cdot)$ are independent. The $X(n, x, k, i)$ for different i are dependent, as described below.

Let $U(n, x, k)$, $x \in \mathbb{Z}^d$, $n, k \ge 1$, be a family of uniform random variables on $[0, 1]$ which are independent of all y's and of all Poisson processes $N_t(x, k)$. We define the joint distribution of $y_n(x, k)$ and $U(n, x, k)$ by

$$P\{y_n(x, k) = y, U(n, x, k) \le \lambda\} = q(y)\lambda, \quad 0 \le \lambda \le 1.$$

We then set

$$X(n, x, k, i) = \begin{cases} 2, & \text{if } U(n, x, k) \le a_i, \\ 1, & \text{if } a_i < U(n, x, k) \le a_i + c_i, \\ 0, & \text{if } a_i + c_i < U(n, x, k). \end{cases}$$

Define

$\mathscr{F}_s :=$$\sigma$-field generated by all $N_u(x, k)$ for $u \le s$ and all $y_n(x, k)$ and $U(n, x, k)$ attached to some $\tau_n(x, k) \le s$.

LEMMA 1. *Assume* (1). *Let ξ_0' and ξ_0'' be initial states with finite numbers of particles, and define ξ_t' and ξ_t'' to be the processes with initial states ξ_0' and ξ_0'', respectively. If we run these processes using the same random variables $N_u(x, k)$, $y_n(x, k)$ and $U(n, x, k)$, then, for all $t \ge 0$, we have that*

$$\sum_x |\xi_t'(x) - \xi_t''(x)| \le \sum_x |\xi_0'(x) - \xi_0''(x)|. \tag{7}$$

In addition, define

$$\eta_t(x) := \xi''_t(x) - \xi'_t(x)$$

and

$$\mathscr{I} := \{x_1, \ldots, x_k\},$$

where $k = \sum_x |\xi'_0(x) - \xi''_0(x)|$ *and each* $x \in \mathbb{Z}^d$ *is listed* ℓ *times if* $|\eta_0(x)| = \ell$. *Then there exist independent random walks* $\{S^y_\cdot\}_{y\in\mathscr{I}}$ *which are translates of* $S_\cdot$ *but with* $S^y_0 = y$ *and such that at all times* t *the following inclusion of events holds*:

$$\{\eta_t(x) \neq 0\} \quad \subseteq \quad \{S^y_t = x \text{ for some } y \in \mathscr{I}\}. \tag{8}$$

Here each S^y_t *is completely determined by the* $N_u(x, k)$ *for* $u \leq t$ *and the* $y_n(x, k)$ *and* $U(n, x, k)$ *attached to some* $\tau_n(x, k) \leq t$.

Note that (8) essentially states that locations of discrepancy between ξ'_t and ξ''_t move as independent random walks but may vanish under certain conditions. Intuitively, this follows by noting that the particles not common to both systems follow independent random walks up until collision with a particle. At the point of collision, a particle either remains unaffected, coalesces or annihilates. The second case creates proper inclusion in (8), and the other two cases represent preservation of the discrepancy between the systems at the place and time of collision. By the Strong Markov property for the finite system, the location of the non-common particle continues to move as a random walk (although the non-common particle may now be in the other system). Although the preservation of non-common particles is not independent of other particles in the system, the positions of these non-common particles depend only on independent random jumps.

Proof of Lemma 1. We couple the processes ξ'_t and ξ''_t to create another process ζ_t, where $\zeta_0(x)$ has $\xi'_0(x) \wedge \xi''_0(x)$ white particles, $\xi'_0(x) - [\xi'_0(x) \wedge \xi''_0(x)]$ blue particles and $\xi''_0(x) - [\xi'_0(x) \wedge \xi''_0(x)]$ green particles. As will be seen, blue and green particles never occupy the same sites. Blue and green particles move normally but are always considered to be *above* the white particles at each site (i.e., jump according to the processes $N_t(x, i+1) \cdots N_t(x, i+j)$ if there are i white particles and j blue or green particles at a site at time t). We will also introduce black particles, which will move normally but not interact at all. The black particles will move above all other particles.

We set up the coupling so that at all times the white particles represent particles common to both ξ'_t and ξ''_t, the blue particles represent particles present in only the ξ'-process, and the green particles represent particles present in only the ξ''-processes. The black particles represent discrepancies which have disappeared due to extra particles coalescing or annihilating. The following rules govern the interactions of particles in the process ζ_t:

Suppose, for some $y \in \mathbb{Z}^d$, $n > 0$ and time $\tau_n(y, k)^-$, that a site x has i white particles and j blue/green particles. (As we will see, at no time will a site have both blue and green particles.) Suppose one of the following three jumps occurs:

(i) A white particle jumps from some y to x at time $\tau_n(y,k)$. (This is a jump that occurs in both the ξ' and the ξ''-systems.)

(a) If $U(n,y,k) \leq a_i$, then the particle is removed along with another white particle from x. (An annihilation occurs in both system.)

(b) If $U(n,y,k) \in (a_i, a_{i+j}]$ and $U(n,y,k) \leq (a_i + c_i)$, then the white particle is removed and one blue or green particle (whichever is present) at x is colored black. (An annihilation occurs at x in the system with extra particles at x, but a coalescence occurs in the other system. This jump eliminates one discrepancy between the systems.)

(c) If $U(n,y,k) \in (a_i, a_{i+j}]$ and $U(n,y,k) > (a_i + c_i)$, then, if $j \geq 2$, we color two blue or green particles (whichever color is present) black, and the white particle remains. If $j < 2$, then the white particle is removed and the one blue or green particle is colored green or blue, respectively. (An annihilation occurs at x in the system with extra particles at x, but nothing happens in the other system. The jump eliminates two particles at the site: either two extra particles or one extra particle and one common particle. The latter case preserves the discrepancy between the systems, the extra particle now being in the other system.)

(d) If $U(n,y,k) \in (a_{i+j}, a_{i+j} + c_{i+j}]$ and $a_i < U(n,y,k) \leq (a_i + c_i)$, then the white particle is removed. (Coalescence occurs in either system.)

(e) If $U(n,y,k) \in (a_i + c_i, a_{i+j} + c_{i+j}]$ and $U(n,y,k) > (a_i + c_i)$, then a blue/green particle at x is colored black and the white particle remains. (Coalescence occurs only in the system with the extra particles.)

(f) If $U(n,y,k) > a_{i+j} + c_{i+j}$, then all particles remain unchanged.

(ii) A blue particle jumps from y to x at time $\tau_n(y,k)$: (This is a jump by a particle that is only in the ξ'-system.)

(a) If there is a green particle at x and $U(n,y,k) \leq a_i$, then the blue particle is colored green and one of the white particles at x is colored black. (Here there is an annihilation in the ξ'-system only, eliminating both the extra blue particle and one of the common particles. Note that a_i is nonzero only if $i > 0$, so the existence of a white particle is guaranteed in this case.)

(b) If there is a green particle at x and $a_i \leq U(n,y,k) \leq a_i + c_i$, then the blue particle is colored black. (Here the extra blue particle has coalesced and been removed from the ξ'-system.)

(c) If there is a green particle at x and $U(n,y,k) \geq a_i + c_i$, then the blue and green particles are colored black and a white particle is inserted at x. (Here two particles which were in separate systems have occupied the same site, resulting in one particle common to both systems.)

(d) If there are no green particles at x and if $U(n,y,k) \leq a_{i+j}$, then the blue particle is colored black along with another blue particle from x, provided there is another blue particle present at x. If there are no blue particles at x, then one white particle is removed from x and the blue particle which jumped is colored green. (An annihilation occurs in the ξ'-system, and the result is that two of the extra particles at x are removed, or, if the new particle is the only extra particle at x, it is removed along with one common particle from x. In this last case, one particle is no longer common to both systems and the blue particle is colored green because the ξ''-system now contains the extra particle. An equivalent, more intuitive description would be that the blue particle is removed and the white particle is colored green. However, we want to keep white particles separate from colored particles for clarity in tracing paths.)

(e) If there are no green particles at x and if $U(n, y, k) \in (a_{i+j}, a_{i+j} + c_{i+j}]$, then the blue particle is colored black. (The extra particle has coalesced and been removed.)

(f) If there are no green particles at x and if $U(n, y, k) > a_{i+j} + c_{i+j}$, then all particles remain unchanged.

(iii) A green particle jumps from y to x at time $\tau_n(y, k)$: This is a jump by a particle that is only in the ξ''-system and is treated analogously to case (ii).

The black particles move normally but do not interact at all. Note that, at any time t, the white and blue particles together make up the ξ'_t-system and the white and green particles make up the ξ''_t-system. We then have

$$\begin{aligned}\sum_x \big|\xi'_t(x) - \xi''_t(x)\big| &= \sum_x \big[\text{number of blue and green particles at } x \text{ at time } t\big] \\ &\le \sum_x |\xi'_0(x) - \xi''_0(x)|,\end{aligned}$$

since the total sum of blue and green particles never increases. Equation (7) follows.

The inclusion (8) follows in a similar way by noting that the position of any particle not shared by both ξ'_t and ξ''_t must lie at either a blue or a green particle, and that, although the color (indicating preservation of the discrepancy) is affected by other particles, the distribution of the path of a blue or green particle (possibly having been changed to black) is not affected. The random walks $\{S_t^y\}$ are then these paths followed by the blue, green, or black particles. The paths $\{S_t^y\}$ are independent because the particles jump according to independent Poisson processes, so that the distribution of the exit time of a particle from a site does not change even if the movement of another particle onto or off of a site causes the particle to jump according to a different Poisson process. □

LEMMA 2. *For all* $x \in \mathbb{Z}^d$ *and* $t > 0$, *it holds, with probability one, that* $\xi_{N,t}(x)$ *converges to a finite limit as* $N \to \infty$. *We call this limit* $\xi_t(x)$.

PROOF. For any positive integer N, let $\{u_1, u_2, \dots\}$ be the lattice positions satisfying $|u_i| > N$. In addition, let $\xi_{\Lambda_k,t}$ be the process $\xi_{N,t}$ with the addition of the particles starting at $u_1, \ldots, u_k$. Then

$$\begin{aligned}
&P\{\xi_{N,t}(x) \neq \xi_{M,t}(x) \text{ for any } M > N\} \\
&\quad \le E\left\{\sup_{M>N} |\xi_{N,t}(x) - \xi_{M,t}(x)|\right\} \\
&\quad \le \sum_{k=0}^{\infty} E|\xi_{\Lambda_{k+1},t}(x) - \xi_{\Lambda_k,t}(x)| \\
&\quad = \sum_{k=0}^{\infty} P\left\{\xi_{\Lambda_{k+1},t}(x) \neq \xi_{\Lambda_k,t}(x)\right\} \quad \left[\text{since } |\xi_{\Lambda_{k+1},t}(x) - \xi_{\Lambda_k,t}(x)| = 0 \text{ or } 1\right] \\
&\quad \le \sum_{k=0}^{\infty} P\left\{S_t^{u_{k+1}} = x\right\} \quad [\text{from equation (8)}] \\
&\quad \to 0 \qquad \text{as } N \to \infty,
\end{aligned}$$

since

$$\sum_{|u|>N} P\{S_t^u = x\} \le P\{|S_t^x| > N\} \to 0 \quad \text{as } N \to \infty. \qquad \square$$

The limiting process ξ_t is our version of the infinite process.
We define the norm

$$N_t(\xi) := \sum_{x\in\mathbb{Z}^d} \xi(x)\alpha_t(x),$$

where

$$\alpha_t(x) := P\{S_t = -x\}.$$

We then take

$$\Xi := \{\xi \in \Xi_0 : N_t(\xi) < \infty \text{ for all } t > 0\}$$

to be the state space for our process. For any function $f : \Xi \to \mathbb{R}$, define

$$L_t(f) := \sup_{x\in\mathbb{Z}^d} \sup_{\xi\in\Xi} \frac{|f(\xi + e_x) - f(\xi)|}{\alpha_t(x)}.$$

Note that, for $\xi', \xi'' \in \Xi$,

$$|f(\xi') - f(\xi'')| \le L_t(f) \sum_x \alpha_t(x)|\xi'(x) - \xi''(x)| = L_t(f) N_t(|\xi' - \xi''|). \tag{9}$$

Define

$$\mathscr{G}_n := \sigma\text{-field of subsets of } \Xi \text{ generated by the coordinate functions } \xi(x) \text{ with } |x| \le n$$

and

$$\mathscr{G} := \bigvee \mathscr{G}_n.$$

We can now define the class of functions

$$\mathscr{L} := \{f : f \text{ is } \mathscr{G}\text{-measurable and } L_t(f) < \infty \text{ for some } t > 0\}.$$

LEMMA 3. *Assume* (1). *For any* $\xi_0', \xi_0'' \in \Xi$,

$$EN_t(|\xi_s' - \xi_s''|) \le N_{t+s}(|\xi_0' - \xi_0''|).$$

PROOF. We define initial states $\xi_0^{(1)}, \xi_0^{(2)}, \ldots \in \Xi$, where $\xi_0^{(1)} = \xi_0'$ and each subsequent $\xi_0^{(i)}$ differs from $\xi_0^{(i-1)}$ by at most one particle and in such a way that, for each $y \in \mathbb{Z}^d$, there exists $M(y) > 0$ such that $n \ge M(y) \Rightarrow \xi_0^{(n)}(y) = \xi_0''(y)$. Then

$$\begin{aligned}
&EN_t(|\xi_s' - \xi_s''|)\\
&\quad = \sum_x \alpha_t(x) E\big|\xi_s'(x) - \xi_s''(x)\big|\\
&\quad \le \sum_x \alpha_t(x) \liminf_{N\to\infty} E\big|\xi_{N,s}'(x) - \xi_{N,s}''(x)\big|\\
&\quad \le \sum_x \alpha_t(x) \liminf_{N\to\infty} \sum_{i>1} E\big|\xi_{N,s}^{(i)}(x) - \xi_{N,s}^{(i-1)}(x)\big|\\
&\quad \le \sum_x \alpha_t(x) \liminf_{N\to\infty} \sum_{i>1}\sum_y \big|\xi_{N,0}^{(i)}(y) - \xi_{N,0}^{(i-1)}(y)\big|\alpha_s(y-x) \quad \text{(by Lemma 1)}\\
&\quad = \sum_x \alpha_t(x) \sum_y \big|\xi_0'(y) - \xi_0''(y)\big|\alpha_s(y-x)\\
&\quad = N_{t+s}(|\xi_0' - \xi_0''|).
\end{aligned}$$

□

For $\xi_0 \in \Xi$ and $B \in \mathscr{G}$, define

$$K_t(\xi_0, B) := P\{\xi_t \in B\}.$$

For fixed ξ_0 and t, $K_t(\xi_0, \cdot)$ is a probability measure on $\mathscr{G}$. In addition, it can be shown, from Lemma 2 and a monotone class argument, that $\xi \to K_t(\xi, B)$ is $\mathscr{G}$-measurable for all $B \in \mathscr{G}$ and $t \ge 0$.

Lastly, define

$$K_t f(\xi) := \int_\Xi K_t(\xi, d\eta) f(\eta), \quad \xi \in \Xi,$$

where f is any $\mathscr{G}$-measurable function on Ξ such that the above integral converges absolutely.

The next two lemmas show that K_t preserves $\mathscr{L}$ and has the semigroup property when applied to $\mathscr{L}$.

LEMMA 4. *Assume* (1). *If* $f \in \mathscr{L}$, *then*

$$L_u\big(K_s f\big) \le e^{u-t-s} L_t(f) \quad (u \ge t+s); \tag{10}$$

$$K_s f(\cdot) = \int_{\Xi} K_s(\cdot, d\eta) f(\eta) \in \mathscr{L}; \quad \textit{and} \tag{11}$$

$$K_s f(\xi_0) = \lim_{N\to\infty} K_s f(\xi_{N,0}), \quad \xi_0 \in \Xi. \tag{12}$$

PROOF. First note that, for $s \le u$,

$$\alpha_u(x) \ge \alpha_s(x) P\left\{S^{\mathbf{0}}_{u-s} = \mathbf{0}\right\} \ge \alpha_s(x) e^{-u+s},$$

so that, for any ξ and $s \le u$,

$$N_s(\xi) \le e^{u-s} N_u(\xi).$$

We then have

$$\begin{aligned} &\big|K_s f(\xi_0') - K_s f(\xi_0'')\big| \\ &\quad = \big|E\big[f\big(\xi_s'\big) - f\big(\xi_s''\big)\big]\big| \\ &\quad \le L_t(f) E N_t\big(|\xi_s' - \xi_s''|\big) \quad [\text{by } (9)] \\ &\quad \le L_t(f) N_{t+s}(|\xi_0' - \xi_0''|) \quad (\text{by Lemma 3}) \\ &\quad \le e^{u-t-s} L_t(f) N_u\big(|\xi_0' - \xi_0''|\big) \quad (\text{for } u \ge t+s). \end{aligned} \tag{13}$$

This proves (10), since

$$\begin{aligned} L_u(K_s f) &= \sup_{x,\xi} \frac{|K_s f(\xi + e_x) - K_s f(\xi)|}{\alpha_u(x)} \\ &\le \sup_{x,\xi} \frac{e^{u-t-s}}{\alpha_u(x)} L_t(f) N_u(|\xi + e_x - \xi|) = e^{u-t-s} L_t(f). \end{aligned}$$

Equation (11) follows from (10). For (12), we apply (13) with $\xi_0' = \xi_0$ and $\xi_0'' = \xi_{N,0}$. Choose $t > 0$ such that $L_t(f) < \infty$ and then set $u = t+s$. This gives

$$\big|K_s f(\xi_0) - K_s f(\xi_{N,0})\big| \le L_t(f) N_{t+s}(|\xi_0 - \xi_{N,0}|) \to 0 \quad \text{as } N \to \infty. \qquad \square$$

LEMMA 5. *Assume* (1). *For* $f \in \mathscr{L}$, $\xi_0 \in \Xi$,

$$K_{s+t} f(\xi_0) = K_s(K_t f)(\xi_0).$$

PROOF.

$$\begin{aligned} K_{s+t} f(\xi_0) &= \lim_{N\to\infty} K_{s+t} f(\xi_{N,0}) \quad (\text{by Lemma 4}) \\ &= \lim_{N\to\infty} \int K_s(\xi_{N,0}, d\eta) K_t f(\eta) \\ &= \lim_{N\to\infty} K_s(K_t f)(\xi_{N,0}). \end{aligned}$$

By (11), we see that $K_t f \in \mathscr{L}$, which allows us to apply (12). This then gives

$$K_{s+t} f(\xi_0) = K_s(K_t f)(\xi_0). \qquad \square$$

For $\xi \in \Xi$ and $f \in \mathscr{L}$, define

$$\Omega f(\xi) := \sum_x \xi(x) \sum_y q(y-x) \Big\{(1 - a_{\xi(y)} - c_{\xi(y)})[f(\xi - e_x + e_y) - f(\xi)] + a_{\xi(y)}[f(\xi - e_x - e_y) - f(\xi)] + c_{\xi(y)}[f(\xi - e_x) - f(\xi)]\Big\}. \tag{14}$$

Here e_x is the vector in Ξ with one at x and zeros at all other positions. The following lemma is a portion of Proposition 6 in Kesten and van den Berg, and is analogous to Lemma 2.16 in Liggett and Spitzer (1981) and Theorem IX.1.14 in Liggett (1985).

LEMMA 6. *Assume* (1) *holds and that* $f \in \mathscr{L}$ *and* $\xi_0 \in \Xi$. *Then* $\Omega(K_t f)(\xi_0)$ *is well defined and*

$$K_t f(\xi_0) = f(\xi_0) + \int_0^t \Omega(K_s f)(\xi_0)\, ds.$$

In addition,

$$\frac{d}{dt} K_t f(\xi_0) = \Omega(K_t f)(\xi_0) = K_t(\Omega f)(\xi_0) = E\{(\Omega f)(\xi_t)\}.$$

The proof is identical to that in Kesten and van den Berg (2000) and demonstrates, among other things, that the right hand side of (14) converges for all $f \in \mathscr{L}$ and $\xi \in \Xi$. We apply Lemma 6 to get a formula which will form the cornerstone of our proof of Theorem 1.

LEMMA 7. *$E(t)$ is differentiable and*

$$\frac{d}{dt} E(t) = -\sum_{y \in \mathbb{Z}^d} E\left\{\xi_t(\mathbf{0}) q(y)(2a_{\xi_t(y)} + c_{\xi_t(y)})\right\}. \tag{15}$$

PROOF. We define the function $f(\xi)$, $\xi \in \Xi$, as

$$f(\xi) := \xi(\mathbf{0}).$$

Then

$$K_t f(\xi_0) = E\{\xi_t(\mathbf{0})\}.$$

We apply Lemma 6 to get

$$\frac{d}{dt} K_t f(\xi_0) \;=\; \Omega(K_t f)(\xi_0) \;=\; K_t(\Omega f)(\xi_0) \;=\; E\{(\Omega f)(\xi_t)\}.$$

For our choice of f,

$$\frac{d}{dt}K_t f(\xi_0) \ = \ \frac{d}{dt}E(t).$$

For any $\xi_0 \in \Xi$, Lemma 3 in Kesten and van den Berg states that, with probability one, $\xi_t \in \Xi$ for all $t \geq 0$ (although their proof uses monotonicity, we use Lemma 8 below to extend the result to our system). Thus, for any $\xi_0 \in \Xi$, $\Omega f(\xi_t)$ converges with probability one for all $t > 0$. Substituting our choice of f into (14), we now have

$$E\{(\Omega f)(\xi_t)\} = -\sum_{y\in\mathbb{Z}^d} E\left\{\xi_t(\mathbf{0})q(y)(2a_{\xi_t(y)} + c_{\xi_t(y)})\right\}. \qquad \square$$

One immediate consequence of this lemma is that $E(t)$ is decreasing in t.

The following lemma provides a relation between systems with and without the possibility of annihilation.

LEMMA 8. *Let ξ_t be the system with annihilation and coalescence probabilities $\{a_i\}$ and $\{c_i\}$, respectively, satisfying* (1). *Define ξ_t^* to be the system where $\xi_0^* = \xi_0$ and*

$$a_i^* = 0 \ \ \textit{and} \ \ c_i^* = a_i + c_i \quad \textit{for all } i.$$

If we run ξ_t^* using the same random variables $N_u(x, k)$, $y_n(x, k)$ and $U(n, x, k)$ as used for ξ_t, then, for all $t \geq 0$ and $x \in \mathbb{Z}^d$, we have $\xi_t(x) \leq \xi_t^*(x)$.

PROOF. We perform the following coupling:

Start at time zero with all particles colored white. White particles will represent particles present in both systems, while black particles will represent those present only in the ξ_t^*-system, in which annihilation is replaced by coalescence. White particles at a site will move according to the Poisson processes with the lower numbers, so that, as we will see, the white particles will exactly correspond to the particles in the ξ_t-system and the white and black together to those in the ξ_t^*-system. Notice that each time we remove a white particle, it will be when a removal occurs in both systems, and each time we color a white particle black it will be when a removal occurs in the ξ_t-system only.

> When a white particle jumps at time $\tau_n(y, k)$ from a site y to a site x with i white particles and j black particles, if $U(n, y, k) \leq a_i$, then the particle is removed and a white particle at x is colored black (note that $a_i > 0$ implies the existence of another white particle at x). However, if $a_i < U(n, y, k) \leq a_i + c_i$, then only one white particle is removed. If $a_i + c_i < U(n, y, k) \leq a_{i+j} + c_{i+j}$, then one of the black particles is removed and the white particle remains. If $U(n, y, k) > a_{i+j} + c_{i+j}$, then no particles are removed or recolored.
>
> When a black particle jumps to a site x with i white particles and j black particles, we remove the black particle if $U(n, y, k) \leq a_{i+j} + c_{i+j}$. If $U(n, y, k) > a_{i+j} + c_{i+j}$, then no particles are removed or recolored.

The result is that the white particles make up the ξ_t-system, while the white and black together make up the ξ_t^*-system. $\square$

The next lemma shows that the system ξ_t^* has the desired monotonicity property described in Section 2.

LEMMA 9. *Let ξ_t^* be the system defined in Lemma 8 and, for any set $A \subset \mathbb{Z}^d$, let ξ_t^A be the ξ_t^* system run with only those particles which begin in the set A. If $B \cap A = \varnothing$, then there exist versions of $\xi_t^{A\cup B}$, ξ_t^A and ξ_t^B such that, for any $t \geq 0$, it holds with probability one that $\xi_t^{A\cup B}(x) \leq \xi_t^A(x) + \xi_t^B(x)$ for all $x \in \mathbb{Z}^d$.*

PROOF. Initially, we assume that both A and B are bounded. We use a different coupling, where particles start colored and may be recolored. At time $t = 0$, color all particles in A blue and all particles in B green. We will also introduce blue striped and green striped particles, corresponding to particles which are not present in $\xi_t^{A\cup B}$, but only in ξ_t^A or ξ_t^B, respectively. If there are i blue particles at a site x, then these blue particles will jump according to $N_t(x, 1), \ldots, N_t(x, i)$. If, in addition, there are j green particles at x, the green particles will jump according to $N_t(x, i+1), \ldots, N_t(x, i+j)$. k blue striped particles will then jump according to $N_t(x, i+j+1), \ldots, N_t(x, i+j+k)$, and ℓ green striped particles according to $N_t(x, i+j+k+1), \ldots, N_t(x, i+j+k+\ell)$.

Defining the variables i, j, k and ℓ as above and recalling the definition of the random variable U on page 140, we handle interactions as follows:

When a blue particle jumps from a site y to a site x:

> If $U(n, y, k) \leq \min\{c^*_{i+j}, c^*_{i+k}\}$, then the blue particle is removed. This would be a removal in either system.
>
> If $c^*_{i+k} \leq U(n, y, k) \leq c^*_{i+j}$, then the blue particle is made into a blue striped particle. In this case, the blue particle would be removed only in the system $\xi_t^{A\cup B}$.
>
> If $c^*_{i+j} < U(n, y, k) \leq c^*_{i+k}$, then one of the k blue striped particles at x is removed and the blue particle remains at x.
>
> If $U(n, y, k) > \max\{c^*_{i+j}, c^*_{i+k}\}$, then all particles remain unchanged.

Green particle jumps are treated in a corresponding manner.

When a blue striped particle jumps to a site x occupied by i blue particles and k blue striped particles, it is removed if $U(n, y, k) \leq c^*_{i+k}$. Otherwise, all particles remain unchanged. Green striped particle jumps are treated analogously. The result is that the solid particles make up $\xi_t^{A\cup B}$, while the solid and striped together make up $\xi_t^A + \xi_t^B$.

We remove the boundedness condition on A and B by using Lemma 2 to take the limit as $N \to \infty$ of the inequality $\xi_t^{A_N\cup B_N}(x) \leq \xi_t^{A_N}(x) + \xi_t^{B_N}(x)$.

Here the sets $A_N, B_N \subset \mathbb{Z}^d$ are defined as $A_N(x) := A(x)\mathbb{1}[|x| \le N]$ and $B_N(x) := B(x)\mathbb{1}[|x| \le N]$. $\square$

3. An upper bound for $E(t)$. For $t \le u$, we define the system $\xi^*_{\Lambda,t,u}$ to be the ξ^*_u-system (from Lemma 8) with all particles not in the set Λ at time t removed at time t.

LEMMA 10. *For $x_1, \dots, x_n \in \mathbb{Z}^d$, let $S^{x_1}_t, \dots, S^{x_n}_t$ be independent random walks beginning at $x_1, \dots, x_n$ (respectively). Define*

$$H_s(x_1, \dots, x_n) := P\{S^{x_1}_r = S^{x_2}_r = \cdots = S^{x_n}_r \text{ for some } r < s\}.$$

Then, for any $\Lambda \subset \mathbb{Z}^d$,

$$\sum_{x\in\Lambda} \xi^*_t(x) - E\left\{\sum_x \xi^*_{\Lambda,t,u}(x) \mid \mathcal{F}_t\right\} \tag{16}$$
$$\ge c^*_{n-1}\left[\left(\sum_{x\in\Lambda} \xi^*_t(x)\right) - (n-1)\right] \min_{x_1,\dots,x_n\in\Lambda} H_{u-t}(x_1, \dots, x_n).$$

PROOF.

$$\sum_{x\in\Lambda} = \xi^*_t(x) - = E\left\{\sum_x \xi^*_{\Lambda,t,u}(x) \mid \mathcal{F}_t\right\}$$
$$= E\{\text{number of particles from } \xi^*_t \text{ located in } \Lambda \text{ at time } t \text{ which would coalesce by time } u \text{ in the absence of particles outside of } \Lambda \text{ at time t} \mid \mathcal{F}_t\}$$
$$\ge E\{\text{number of particles located in } \Lambda \text{ at time } t \text{ which coalesce, by time } u\text{, with any given } (n-1) \text{ particles also located in } \Lambda \text{ at time } t\text{, in the absence of particles outside of } \Lambda \text{ at time t} \mid \mathcal{F}_t\}$$
$$\ge c^*_{n-1}\left[\left(\sum_{x\in\Lambda} \xi^*_t(x)\right) - (n-1)\right] \min_{x_1,\dots,x_n\in\Lambda} H_{u-t}(x_1, \dots, x_n). \qquad \square$$

The interested reader may refer to Lemma 3 in Bramson and Griffeath (1980), where a similar result is proven using the underlying percolation substructure.

The following proposition generalizes Theorem 1 of Bramson and Griffeath (1980).

PROPOSITION 1. *Assume* (2) *and* (3). *For the n-threshold randomly coalescing and annihilating random walk in dimension $d \ge 3$, there exists a constant $C_2 = C_2(d, n)$ such that*

$$E(t) \le C_2 t^{-\frac{1}{n-1}}. \tag{17}$$

PROOF. We prove the proposition in the system ξ_t^* with coalescing probabilities

$$c_i^* = a_i + c_i \text{ and } a_i^* = 0 \quad \text{for all } i.$$

By Lemma 8, this suffices to prove the proposition.

Define

$$f_t := t^{\frac{1}{n-1}};$$
$$g_t := f_t E(t).$$

Our goal is to show that g_t is bounded uniformly in t. To this end, let Λ be any bounded subset of $\mathbb{Z}^d$ and define

$$E(t, \Lambda) := E\left\{\text{number of particles in } \Lambda \text{ at time } t\right\}.$$

By translation invariance, $E(t, \Lambda) = |\Lambda| E(t)$. Taking $0 \le t \le u$, we see that

$$\begin{aligned} E(u) &= E(u, \Lambda)/|\Lambda|; \\ E(t) &= E(t, \Lambda)/|\Lambda|; \\ E(u) &= E(t)\left[1 - \frac{E(t, \Lambda) - E(u, \Lambda)}{E(t, \Lambda)}\right]. \end{aligned} \tag{18}$$

For a given integer $R > 0$, let Λ be the set of lattice points in the half open box $[-R, R)^d$ and define

$$V := \{2zR : z \in \mathbb{Z}^d\}.$$

Note that $\{\Lambda - v : v \in V\} = \{\Lambda + v : v \in V\} = \mathbb{Z}^d$. By Lemma 9, we know that

$$E\left\{\sum_{x\in\Lambda} \xi_u^*(x)\right\} \le E\left\{\sum_{x\in\Lambda}\sum_{v\in V} \xi^*_{\Lambda+v,t,u}(x)\right\}.$$

We then have

$$\begin{aligned} E(u, \Lambda) &\le E\left\{\sum_{x\in\Lambda}\sum_{v\in V} \xi^*_{\Lambda+v,t,u}(x)\right\} \\ &= E\left\{\sum_{x\in\Lambda}\sum_{v\in V} \xi^*_{\Lambda,t,u}(x-v)\right\} \\ &= E\left\{\sum_{x\in\mathbb{Z}^d} \xi^*_{\Lambda,t,u}(x)\right\}. \end{aligned} \tag{19}$$

Now define

$$\Delta_{t,u}(\Lambda) := E(t, \Lambda) - E\left\{\sum_x \xi^*_{\Lambda,t,u}(x)\right\}.$$

We apply (19) and (18) to get

$$E(u) \le E(t)\left[1 - \frac{E(t,\Lambda) - E\left\{\sum_x \xi^*_{\Lambda,t,u}(x)\right\}}{E(t,\Lambda)}\right] \tag{20}$$
$$= E(t)\left[1 - \frac{\Delta_{t,u}(\Lambda)}{E(t,\Lambda)}\right].$$

We wish to bound the growth rate of g_t. Taking the expectation of (16), we have

$$\Delta_{t,u}(\Lambda) \ge c^*_{n-1}\left[E(t,\Lambda) - (n-1)\right] \min_{x_1,\dots,x_n\in\Lambda} H_{u-t}(x_1,\dots,x_n).$$

For ease of notation, define

$$h_s := \min_{x_1,\dots,x_n\in\Lambda} H_s(x_1,\dots,x_n).$$

When we set s:=u-t, (20) becomes

$$E(t+s) \le E(t)\left[1 - \frac{c^*_{n-1}\left[E(t,\Lambda)-(n-1)\right]h_s}{E(t,\Lambda)}\right] \tag{21}$$
$$= E(t)\left[1 - c^*_{n-1}\left(1-(n-1)\left[E(t,\Lambda)\right]^{-1}\right)h_s\right].$$

Choose $\Lambda = \Lambda_t$ to have side width $2R$, with

$$R = R_t := \left\lceil \left(\frac{n}{2E(t)}\right)^{1/d}\right\rceil,$$

so that

$$E(t,\Lambda) = (2R)^d E(t) \ge n.$$

Applying (21), we have

$$E(t+s) \le E(t)\left[1 - c^*_{n-1}h_s/n\right].$$

Adding $\lfloor t/s \rfloor$ multiples of s gives us

$$E(2t) \le E(t + s\lfloor t/s\rfloor) \le E(t)\left(1 - c^*_{n-1}h_s/n\right)^{\lfloor t/s\rfloor},$$

where the first inequality holds because $E(t)$ is decreasing in t (see Lemma 7). Since $1 - x \le \exp\{-x\}$, we may now write

$$E(2t) \le E(t)\exp\left\{-c^*_{n-1}h_s\lfloor t/s\rfloor/n\right\} \quad \text{for all } s,t \ge 0. \tag{22}$$

Note that $f_{2t} = 2^{1/(n-1)}f_t$, so that multiplying (22) by f_{2t} gives

$$g_{2t} \le g_t \exp\left\{(\log 2)/(n-1) - c^*_{n-1}h_s\lfloor t/s\rfloor/n\right\} \quad \text{for all } s,t \ge 0. \tag{23}$$

We now show that there exists a set of positive real numbers $\{s_t\}_{t\in\mathbb{R}^+}$ such that

$$\liminf_{t\to\infty} g_t^{-(n-1)} h_{s_t}\left\lfloor t s_t^{-1}\right\rfloor > 0. \tag{24}$$

To this end, set

$$s_t := (n/2)^{-(n-1)} R_t^2/4.$$

Given n independent random walks on $\mathbb{Z}^d$ with starting points $x_1, \ldots, x_n$, respectively, we can define

$$D_1(t), \ldots, D_{n-1}(t) \in \mathbb{Z}^d$$

as the relative displacements at time t of the last $n-1$ random walks with respect to the first walk. $(D_1(t), \ldots, D_{n-1}(t))$ is then a random walk on $\mathbb{Z}^{(n-1)d}$ which hits the origin at exactly those times when the n random walks coincide. We let $\tilde{x} \in \mathbb{Z}^{(n-1)d}$ be the direct sum of the differences $(x_2 - x_1), \ldots, (x_n - x_1)$ and let $\eta_t^{\tilde{x}}$ denote the random walk $(D_1(t), \ldots, D_{n-1}(t))$ as just described.

It is simple to show that

$$H_s(x_1, \ldots, x_n) \geq \frac{\int_0^s P\{\eta_t^{\tilde{\mathbf{0}}} = \tilde{x}\}\, dt}{\int_0^s P\{\eta_t^{\tilde{\mathbf{0}}} = \tilde{\mathbf{0}}\}\, dt}. \tag{25}$$

Since (2) and (3) hold, we may apply the local central limit theorem to estimate $P\{\eta_t^{\tilde{\mathbf{0}}} = \tilde{x}\}$ and $P\{\eta_t^{\tilde{\mathbf{0}}} = \tilde{\mathbf{0}}\}$ [see, e.g., Proposition 7.9 and the proof of Proposition 26.1 of Spitzer (1976), as well as the comments at the beginning of Section 2 and at the end of the proof of Lemma 8 in Kesten and van den Berg with regard to weakening the assumptions of aperiodicity and strong aperiodicity of $S_\cdot$]. With a little work, the local central limit theorem estimates yield

$$h_{s_t} \geq C_{d,n} \, [R_t]^{2-(n-1)d}. \tag{26}$$

[For this calculation, we first decompose the probabilities according to the number of jump events. Note that all of the coordinates of $\tilde{x}$ lie within the interval $[-R_t, R_t)$ and that s_t is a constant multiple of R_t^2. This allows us to bound the exponential term in the estimate of the numerator of (25).]

Let p_t denote the probability that there is a particle at the origin at time t in the system where $c_2 = 1$ [see Bramson and Griffeath (1980)]. We have

$$t\,[E(t)]^{2/d} \geq t p_t^{2/d} \to \infty, \qquad t \to \infty$$

[cf. Kesten and van den Berg (2000), Lemma 2, with regard to the inequality], so that

$$\lfloor t s_t^{-1} \rfloor \sim t s_t^{-1}, \qquad t \to \infty. \tag{27}$$

Lastly, note that $E(t) \leq 1$ for all t [recall from Lemma 7 that $E(t)$ is decreasing in t]. Thus

$$R_t = \left\lceil \left(\frac{n}{2E(t)} \right)^{1/d} \right\rceil \leq 2 \left(\frac{n}{2E(t)} \right)^{1/d}$$

and hence

$$t/s_t \geq \frac{t}{(n/2)^{-(n-1)} \left(\frac{n}{2E(t)} \right)^{2/d}}. \tag{28}$$

Applying (26), (27) and (28) and substituting for g_t and R_t give

$$\begin{aligned}
&\liminf_{t\to\infty} g_t^{-(n-1)} h_{s_t} \lfloor t s_t^{-1} \rfloor \\
&\quad \geq C_{d,n} \liminf_{t\to\infty} t^{-1} [E(t)]^{-(n-1)} \left(\frac{n}{2E(t)}\right)^{2/d-(n-1)} \lfloor t s_t^{-1} \rfloor \\
&\quad \geq C_{d,n} \liminf_{t\to\infty} t^{-1} [E(t)]^{-(n-1)} \left(\frac{n}{2E(t)}\right)^{2/d-(n-1)} \\
&\qquad \times t \left[(n/2)^{-(n-1)} \left(\frac{n}{2E(t)}\right)^{2/d} \right]^{-1} \\
&\quad = C_{d,n} > 0.
\end{aligned}$$

This proves (24), which we we now apply in the following form:

(29) For some $\varepsilon > 0$, there exists a t_0 such that $t > t_0 \Rightarrow h_{s_t} \lfloor t s_t^{-1} \rfloor > \varepsilon g_t^{(n-1)}$.

For $t \geq t_0$, (23) becomes

$$\begin{aligned}
g_{2t} &\leq g_t \exp\left\{(\log 2)/(n-1) - \varepsilon c_{n-1}^* g_t^{(n-1)}/n\right\} \\
&\leq 2^{1/(n-1)} C(n, \varepsilon, c_{n-1}^*),
\end{aligned}$$

since $x \exp(-k x^{(n-1)}) \leq C(k)$.

This completes the proof of Proposition 1. □

4. Crucial variance and expectation bounds. Note that in proofs utilizing a large number of constants we will occasionally employ numbered superscripts, as in $C^{(j)}$, to reference constants not referred to outside of the proof.

We start with a crude lower bound for $P(t)$.

LEMMA 11. *Define*

$$P(t) := P\{\textit{the origin is occupied at time } t\}.$$

Then, *for* $n \geq 3$,

$$P(t) \geq C_3/t \quad \textit{for some constant } C_3 > 0. \tag{30}$$

PROOF. The standard annihilating random walk, η_t, has asymptotic lower bound C_3/t, so long as the random walk is multidimensional (not restricted to a one-dimensional subspace) and satisfies (3) [see Arratia (1981), Theorem 3, along with the asymptotic for *general* coalescing random walks as established in Kesten and van den Berg (2000)]. We set up the following coupling to show that, so long as $n \geq 3$, we have $\eta_t(x) \leq \xi_t(x)$ for all $t \geq 0$ and $x \in \mathbb{Z}^d$.

At time zero we color all particles black except at those sites x which have an odd number of particles. At such x, we color one particle white and the other particles black. White particles will correspond to those particles present in both η_t and ξ_t, while black particles will correspond to those particles which would be removed in η_t but not in ξ_t. As we will see, a site will have at most one white particle at any time t, so that, for $n \geq 3$, there will always be at least two black particles present whenever a removal occurs. Whenever either a white particle jumps to a site occupied only by black particles or a black particle jumps to any site, we remove black particles from the site according to the normal procedure for ξ, with both white and black particles counted in determining coalescence and annihilation events. This corresponds to a removal of particles which are present only in the ξ-system. Whenever a white particle jumps to a site containing another white particle, we color both black, corresponding to an annihilation in the η-system. We then proceed to remove black particles from the site according to the normal procedure for ξ.

We set up the coupling so that, if there are i white particles and j black particles at a site x at time $\tau_n(x,k)^-$, then a white particle jumps at time $\tau_n(x,k)$ if $k \leq i$ but a black particle jumps if $i < k \leq i+j$. In this way, white particles make up η_t, the standard annihilating random walk, while the white and black together make up our system, ξ_t. $\square$

LEMMA 12. *Assume* (1), (2) *and* (3). *If $\beta(x) \in \mathbb{R}$ satisfy $\sum_x |\beta(x)| < \infty$, then there exists a constant $C_4 = C_4(n,d)$, independent of β and t, such that*

$$\operatorname{Var}\left\{\sum_x \beta(x)\xi_t(x)\right\} \leq C_4 t^{\frac{n-2}{n-1}} \sum_{x\in\mathbb{Z}^d} \beta^2(x) \qquad (n \geq 3).$$

This is the n-threshold analogue of Proposition 7 in Kesten and van den Berg. Our Lemma 1 is used in place of their Lemma 1, and, otherwise, the proof is essentially identical. We will apply this result in the proof of the following lemma to get a type of correlation bound.

As in Kesten and van den Berg (2000), define $\Lambda_t(u_1, u_2, \ldots, u_p)$ to be the number of ordered p-tuples of distinct particles which we can select from the particles present at the sites $u_1, u_2, \ldots, u_p$ at time t. Note that these sites are not necessarily distinct, so, as demonstrated in Lemma 14, we can use the bound on $E\Lambda_t(u_1, u_2, \ldots, u_p)$ to bound the probability that at least p particles occupy one site x. The following is similar to Lemma 10 of Kesten and van den Berg (2000).

LEMMA 13. *For any $d \geq 3$, $p \geq 2$, $n \geq 3$, $u_1, \ldots, u_p \in \mathbb{Z}^d$, $\Gamma < t$ and $\varepsilon < 1$, there exists a positive constant $C_5 = C_5(n,d,\varepsilon,p)$ such that, for $t \geq 1$,*

$$E\Lambda_t(u_1,\ldots,u_p) \leq C_5\left[E^p(t-\Gamma) \vee (t-\Gamma)^{\frac{(n-2)(1-\varepsilon)}{n-1}}\Gamma^{-\frac{d(1-\varepsilon)}{2}}\right]. \tag{31}$$

PROOF. Without loss of generality, we may take the u_i to be distinct, since, for example, $\Lambda_t(u,u) \leq \Lambda_t(u,v)$ for any v. We use the notation $\Lambda_{N,t}$ for Λ_t in the system where ξ_t is replaced by $\xi_{N,t}$. For a designated particle in $\xi_{N,t}$ with location x at time $s < t$, we also define

$$\bar{\xi}_{N,s,t}(x)(y) := \mathbb{1}[\text{the particle has moved from } x \text{ at time } s \text{ to position } y \text{ at time } t \text{ in the system where interaction is suspended from time } s].$$

Let $z_1, \dots, z_{r_s}$ be the positions at time s of the particles present in $\xi_{N,s}$. Here each position occurs with the proper multiplicity; if $\xi_{N,s} = k$ for some x, then k of the z_i equal x. Hence $r_s = \sum_x \xi_{N,s}(x)$. We note that

$$\xi_{N,t}(x) \le \sum_{i=1}^{r_s} \bar{\xi}_{N,s,t}(z_i)(x), \tag{32}$$

which can be seen by coupling the system with that system in which particles are colored black rather than removed. More specifically, all particles are colored white at time s, and any particle which would be removed at a time $u > s$ is colored black at time u and ceases to interact with other particles. The white particles continue to interact normally with other white particles. At time t, the white particles make up the left hand side of the above inequality, while the white and black together make up the right hand side.

Using (32) with $s = t - \Gamma$, we now have

$$\begin{aligned}
&\Lambda_{N,t}(u_1, \dots, u_p) \\
&\quad \le \left(\sum_{i=1}^{r} \sum_{j=1}^{p} [\bar{\xi}_{N,t-\Gamma,t}(z_i)(u_j)] \right) \times \left(\sum_{i=1}^{r} \sum_{j=1}^{p} [\bar{\xi}_{N,t-\Gamma,t}(z_i)(u_j)] - 1 \right) \\
&\qquad \times \cdots \times \left(\sum_{i=1}^{r} \sum_{j=1}^{p} [\bar{\xi}_{N,t-\Gamma,t}(z_i)(u_j)] - (p-1) \right) \\
&\quad = (\text{number of ordered } p-\text{tuples } z_1, \dots, z_p \text{ of positions of particles at time } t-\Gamma \text{ such that each of the corresponding particles ends at one of the positions } u_i \text{ at time } t) \\
&\quad = \sum_{i_1=1}^{r} \sum_{\substack{i_2=1 \\ i_2 \ne i_1}}^{r} \sum_{\substack{i_3=1 \\ i_3 \ne i_1, i_2}}^{r} \cdots \sum_{\substack{i_p=1 \\ i_p \ne i_1, \dots, i_{p-1}}}^{r} \mathbb{1}[\text{each of the } p \text{ particles corresponding to the } z_{i_k} \text{ ends at one of the } u_i] \\
&\quad = \sum_{i_1=1}^{r} \sum_{\substack{i_2=1 \\ i_2 \ne i_1}}^{r} \sum_{\substack{i_3=1 \\ i_3 \ne i_1, i_2}}^{r} \cdots \sum_{\substack{i_p=1 \\ i_p \ne i_1, \dots, i_{p-1}}}^{r} \prod_{k=1}^{p} \left(\sum_{j=1}^{p} [\bar{\xi}_{N,t-\Gamma,t}(z_{i_k})(u_j)] \right).
\end{aligned}$$

Noting that the particles in the $\bar{\xi}_{N,t-\Gamma,t}$ system move as independent random walks, we can apply $E\{\quad | \mathscr{F}_{t-\Gamma}\}$ to both sides and sum over all possible values of z_{i_k} $(k = 1, \dots, p)$ to get

$$\begin{aligned} &E\{\Lambda_{N,t}(u_1, \dots, u_p) | \mathscr{F}_{t-\Gamma}\} \\ &\quad \le \sum_{i_1=1}^{r} \sum_{\substack{i_2=1 \\ i_2 \ne i_1}}^{r} \sum_{\substack{i_3=1 \\ i_3 \ne i_1, i_2}}^{r} \cdots \sum_{\substack{i_p=1 \\ i_p \ne i_1, \dots, i_{p-1}}}^{r} \prod_{k=1}^{p} \biggl(\sum_{j=1}^{p} [\alpha_\Gamma(z_{i_k} - u_j)] \biggr) \\ &\quad \le \Biggl[\sum_{z \in \mathbb{Z}^d} \xi_{N,t-\Gamma}(z) \biggl[\sum_{j=1}^{p} \alpha_\Gamma(z - u_j) \biggr] \Biggr]^p . \end{aligned}$$

Taking the expectation of both sides, we have

$$E\Lambda_{N,t}(u_1, \dots, u_p) \le E \Biggl[\sum_{z \in \mathbb{Z}^d} \xi_{N,t-\Gamma}(z) \biggl[\sum_{j=1}^{p} \alpha_\Gamma(z - u_j) \biggr] \Biggr]^p .$$

We show that

$$E\Lambda_t(u_1, \dots, u_p) \le C^{(7)} \Bigl[E^p(t - \Gamma) \vee (t - \Gamma)^{\frac{(n-2)(1-\varepsilon)}{n-1}} \Gamma^{-\frac{d(1-\varepsilon)}{2}} \Bigr] \tag{33}$$

uniformly in $u_1, \dots, u_p \in \mathbb{Z}^d$.

To prove (33), we start by introducing the abbreviation

$$U_N = U_N(u_1, \dots, u_p) := \sum_z \xi_{N,t-\Gamma}(z) \sum_{j=1}^{p} \alpha_\Gamma(z - u_j).$$

In order to apply the dominated convergence theorem, we note that

$$\xi_{N,t-\Gamma}(x) \le \bar{\xi}_{t-\Gamma}(x) \quad \text{for all } N > 0,\ x \in \mathbb{Z}^d,$$

where $\bar{\xi}_t$ is the system in which particles move freely with no interaction. Also,

$$E\bar{\xi}_{t-\Gamma}(x) = \sum_y P\{S^y_{t-\Gamma} = x\} = 1.$$

We then have

$$\lim_{N \to \infty} EU_N = pE(t - \Gamma). \tag{34}$$

From the proof of Lemma 12, we also know that, for $n \ge 3$,

$$\begin{aligned} \operatorname{Var}(U_N) = E\{(U_N - EU_N)^2\} &\le C_4 (t - \Gamma)^{\frac{n-2}{n-1}} \sum_z \Biggl[\sum_{j=1}^{p} \alpha_\Gamma(z - u_j) \Biggr]^2 \\ &\le C^{(1)}(n, d, p) \Gamma^{-d/2} (t - \Gamma)^{\frac{n-2}{n-1}} . \end{aligned} \tag{35}$$

We now use

$$\begin{aligned} U_N^p &\le C^{(2)}(p) [|U_N - EU_N|^p + (EU_N)^p] \\ &\le C^{(2)}(p) |U_N - EU_N|^{2-\varepsilon} |U_N - EU_N|^{p-2+\varepsilon} + C^{(2)}(p) [EU_N]^p . \end{aligned}$$

By Hölder's inequality, this shows that

$$E\left\{U_N^p\right\} \le C^{(2)}(p)\left[E\left\{(U_N - EU_N)^2\right\}\right]^{1-\varepsilon/2} \times \left[E\left\{|U_N - EU_N|^{2(p-2+\varepsilon)/\varepsilon}\right\}\right]^{\varepsilon/2} + C^{(2)}(p)[EU_N]^p. \tag{36}$$

Note that $\sum_z \alpha_\Gamma(z-u) = 1$, and hence, by Jensen's inequality,

$$U_N^p \le \sum_x \alpha_\Gamma(x-u)\xi_{N,t-\Gamma}^p(x).$$

We use (32) to get

$$\begin{aligned} E\left\{U_N^p\right\} &\le \sup_x E\left[\sum_{y\in\mathbb{Z}^d} \bar{\xi}_{N,0,t-\Gamma}(y)(x)\right]^p \\ &\le C^{(3)}(p)\sup_x \sum_{k=1}^p \sum_{n_1,\dots,n_k} \sum_{\substack{y_1,\dots,y_k \\ \text{distinct}}} E\left\{[\bar{\xi}_{N,0,t-\Gamma}(y_1)(x)]^{n_1}\right\} \times \cdots \\ &\qquad \times E\left\{[\bar{\xi}_{N,0,t-\Gamma}(y_k)(x)]^{n_k}\right\}, \end{aligned}$$

where $n_1,\dots,n_k$ run over the partitions of p into k nonzero integers. $\bar{\xi}_{N,0,t-\Gamma}(y)(x)$ is an indicator function, and each of the above expectations can be rewritten as

$$P\left\{\bar{\xi}_{N,0,t-\Gamma}(y_i)(x) = 1\right\} = P\left\{S_{t-\Gamma}^{y_i} = x\right\},$$

so that

$$E\left\{U_N^p\right\} \le C^{(3)}(p)\sup_x \sum_{k=1}^p \sum_{n_1,\dots,n_k} \sum_{\substack{y_1,\dots,y_k \\ \text{distinct}}} \prod_{i=1}^k P\left\{S_{t-\Gamma}^{y_i} = x\right\} \le C^{(4)}(p),$$

where $C^{(4)}(p)$ is independent of N.

By Jensen's inequality, this last inequality holds if we replace p by any positive real number of size at least one. In particular, this allows us to write

$$E\left\{|U_N - EU_N|^{2(p-2+\varepsilon)/\varepsilon}\right\} \le C^{(5)}(p,\varepsilon). \tag{37}$$

Applying (37) to (36) gives us

$$E\left\{U_N^p\right\} \le C^{(6)}(p,\varepsilon)\left[\mathrm{Var}(U_N)\right]^{(1-\varepsilon/2)} + C^{(2)}(p)[EU_N]^p.$$

This, together with (35), proves that

$$\begin{aligned} &E\left\{\left[\sum_{z\in\mathbb{Z}^d} \xi_{N,t-\Gamma}(z)\left[\sum_{j=1}^p \alpha_\Gamma(z-u_j)\right]\right]^p\right\} \\ &\qquad \le C^{(7)}\left[[EU_N]^p \vee (t-\Gamma)^{\frac{(n-2)(1-\varepsilon)}{n-1}}\Gamma^{-\frac{d(1-\varepsilon)}{2}}\right] \end{aligned} \tag{38}$$

for $n \ge 3$, where $C^{(7)} = C^{(7)}(n,d,\varepsilon,p)$ is independent of $N > 0$.

We establish (33) by taking the lim inf as $N \to \infty$ of both sides of (38) and applying Fatou's lemma to the left hand side and (34) to the right hand side. $\square$

The usefulness of this lemma will be enhanced through the following additional lemma:

LEMMA 14. *If we use the notation $\Lambda_t(x)_p$ (p a strictly positive integer) to denote the number of ordered p-tuples of distinct particles which we can select from the particles present at the site x at time t, then the following inequalities hold*:

$$P\{\xi_t(\mathbf{0}) \geq p\} \leq E\{\xi_t(\mathbf{0});\ \xi_t(\mathbf{0}) \geq p\} \leq E\Lambda_t(\mathbf{0})_p, \tag{39}$$

$$|E(t) - P(t)| = \sum_{k\geq 2} kP\{\xi_t(\mathbf{0}) = k\} \leq E\Lambda_t(\mathbf{0}, \mathbf{0}). \tag{40}$$

For $n \geq 2$ and $x \in \mathbb{Z}^d$,

$$[\xi_t(x)]^n \leq n^n\,[\Lambda_t(x)_n \vee \mathbb{1}[\xi_t(x) > 0]]. \tag{41}$$

Let $x_1, \dots, x_k$ be distinct points in $\mathbb{Z}^d$, and $\delta_1, \dots, \delta_k$ be strictly positive integers with $N := \max_i \delta_i$ and $D := \sum_i \delta_i$. Then, for $\Gamma < t$, $\varepsilon < 1$ and large t,

$$E\left\{\prod_{i=1}^k \xi_t^{\delta_i}(x_i)\right\} \leq 2N^D C_5\left[E^k(t-\Gamma) \vee (t-\Gamma)^{\frac{(n-2)(1-\varepsilon)}{n-1}}\Gamma^{-\frac{d(1-\varepsilon)}{2}}\right]. \tag{42}$$

PROOF. For (39), we have

$$P\{\xi_t(\mathbf{0}) \geq p\} \leq E\{\xi_t(\mathbf{0});\ \xi_t(\mathbf{0}) \geq p\} = \sum_{k\geq p} kP\{\xi_t(\mathbf{0}) = k\}$$
$$\leq \sum_{k\geq p} k!/(k-p)!P\{\xi_t(\mathbf{0}) = k\} = E\Lambda_t(\mathbf{0})_p.$$

We can see that (41) holds for $\xi_t(x) \leq n$. If $\xi_t(x) = k > n$, then, since $\Lambda_t(x)_n = k(k-1)\cdots(k-n+1)$, it suffices to show that

$$a_k := \frac{k^n}{k(k-1)\dots(k-n+1)} \quad \text{is decreasing in } k \geq n.$$

This is true, as $1/a_k = 1\,(1-1/k)\cdots(1-(n-1)/k)$ is increasing in k.

To prove (42), note that (41) gives us

$$[\xi_t(x_i)]^{\delta_i} \leq N^{\delta_i}\big[\Lambda_t(x_i)_{\delta_i} \vee \mathbb{1}[\xi_t(x_i) > 0]\big],$$

so that

$$\prod_{i=1}^k \xi_t^{\delta_i}(x_i) \leq N^D \prod_{i=1}^k \big[\Lambda_t(x_i)_{\delta_i} \vee \mathbb{1}[\xi_t(x_i) > 0]\big] \leq N^D \sum \Lambda_t(y_1, \dots, y_\ell).$$

The sum is over combinations $y_1, \dots, y_\ell$ such that, for each $j = 1 \dots k$, $y_i = x_j$ for at least one but at most δ_j different y_i's. We take the expectation of both sides and note that Lemma 13 shows that, for large t, the highest order bound on any term of the sum is that on the term with the fewest number of variables. Equation (42) follows. □

5. Solving an approximate differential equation for $E(t)$. The following lemma is an estimate for noninteracting random walks. It is, essentially, Lemma 12 in Kesten and van den Berg (2000). If $s \to S_s^{(u_1,k_1)}, \dots, S_s^{(u_n,k_n)}$ (k_i strictly positive integers) are random walk paths starting at $u_1, \dots, u_n \in \mathbb{Z}^d$, respectively, we will say that *the paths $S_s^{(u_1,k_1)}, \dots, S_s^{(u_n,k_n)}$ meet exactly m times during a time interval J* if there exist exactly m times in the interval J when one of these paths enters a site currently occupied by the other $n-1$ paths.

LEMMA 15. *Let $d \geq 3$ and let $S_t^{(u_1,k_1)}, \dots, S_t^{(u_n,k_n)}$, $(u_i, k_i) \in \mathbb{Z}^d \times \{1, 2, \dots\}$, be independent random walks such that $S_0^{(u_i,k_i)} = u_i$ for $i = 1, \dots, n$.*
Choose $\Delta > 1$ and define, for $y_1, \dots, y_n \in \mathbb{Z}^d$,

$$\begin{aligned}&\mathscr{E}(u_1, \dots, u_n, k_1, \dots, k_n, m, \Delta, y_1, \dots, y_n)\\ &\quad := \{S_\Delta^{(u_i,k_i)} = y_i \text{ for each } i = 1, \dots, n, \text{ and the paths } s \to S_s^{(u_i,k_i)}\\ &\qquad\quad \text{meet exactly } m \text{ times during } (0, \Delta]\},\end{aligned}$$

then there exists a $\delta = \delta(d, n)$ with $0 < \delta(d, n) \leq 1$ and a positive constant $C_6 = C_6(d, n)$ such that, uniformly in the y_i's and m,

$$\begin{aligned}\sum_{u_1,\dots,u_n} \Bigg| & P\{\mathscr{E}(u_1, \dots, u_n, k_1, \dots, k_n, m, \Delta, y_1, \dots, y_n)\}\\ & -P\big\{s \to S_s^{-y_i}, i = 1, \dots, n, \text{ meet exactly } m \text{ times during } (0, \infty)\big\}\\ & \qquad \times \prod_{i=1}^n \alpha_\Delta(u_i - y_i)\Bigg| \leq C_6 \Delta^{-\delta}.\end{aligned}$$

Although we omit the details, following the proof in Kesten and van den Berg (2000) shows that this lemma holds with

$$\delta = \left(\frac{2}{n}\right)\left(\frac{d-2}{3d^2 - 3d - 4}\right).$$

Define

$$\begin{aligned}\rho(m, y) := P\big\{&s \mapsto S_s^{\mathbf{0}} \text{ and } n-1 \text{ independent copies of } s \mapsto S_s^{-y} \text{ meet}\\ &\text{exactly } m \text{ times during } [0, \infty)\big\}.\end{aligned}$$

In addition, define

$$D(y) := \sum_{m=0}^{\infty} (1 - a_{n-1} - c_{n-1})^m \rho(m, y).$$

Finally, let $\Lambda_t^*(u_1, \dots, u_p)$ be the number of ordered $p-$tuples of distinct particles, the first particle being present at u_1 at time t, the second at u_2, etc. Note that

$$\Lambda_t^*(u_1, \dots, u_p) \le \Lambda_t(u_1, \dots, u_p).$$

Also,

$$\Lambda_t^*(u_1, \dots, u_p) = \prod_{i=1}^{p} \xi_t(u_i) \tag{43}$$

if all of the u_i's are distinct.

LEMMA 16. *Assume* (1), (2) *and* (3). *For* $y \neq \mathbf{0}$, $0 < \Delta < t/4$ *and* $n \ge 3$,

$$\begin{aligned} &\Bigg| E\{\xi_t(\mathbf{0})(2a_{\xi_t(y)} + c_{\xi_t(y)})\} \\ &\quad -(2a_{n-1} + c_{n-1}) D(y) \sum_{u_1,\dots,u_n \in \mathbb{Z}^d} E\{\Lambda_{t-\Delta}^*(u_1, \dots, u_n)\} \alpha_\Delta(u_1) \prod_{i=2}^{n} \alpha_\Delta(u_i - y) \Bigg| \\ &\qquad \le C_7 \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big] \\ &\qquad\quad + C_8 \Delta \Big[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big] \\ &\qquad\quad + C_9 \Delta^{-\delta(d,n)} \Big[E^{n}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big]. \end{aligned}$$

PROOF. We start by showing that

$$\begin{aligned} &\Big| E\big\{\xi_t(\mathbf{0})(2a_{\xi_t(y)} + c_{\xi_t(y)}) - (2a_{n-1} + c_{n-1}) P\big\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1\big\}\Big| \\ &\quad \le C^{(2)} \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big]. \end{aligned} \tag{44}$$

For $y \neq \mathbf{0}$,

$$\begin{aligned} &\Big| E\big\{\xi_t(\mathbf{0})(2a_{\xi_t(y)} + c_{\xi_t(y)})\big\} - (2a_{n-1} + c_{n-1}) E\big\{\xi_t(\mathbf{0}) \mathbb{1}[\xi_t(y) = n-1]\big\}\Big| \\ &\quad \le 2E\big\{\xi_t(\mathbf{0}) \mathbb{1}[\xi_t(y) \ge n]\big\} \\ &\quad \le 2E\big\{\Lambda_t(\mathbf{0}, y, \dots, y); \xi_t(y) \ge n\big\} \qquad (n \text{ copies of } y) \\ &\quad \le 2E\Lambda_t(\mathbf{0}, y, \dots, y) \\ &\quad \le 2C^{(3)} \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big], \end{aligned}$$

where the last inequality above comes from Lemma 13, taking $\Gamma := t/2$. In addition,

$$\begin{aligned}
&\left|E\{\xi_t(\mathbf{0})\mathbb{1}[\xi_t(y) = n-1]\} - P\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1\}\right| \\
&\quad = E\{\xi_t(\mathbf{0})\mathbb{1}[\xi_t(\mathbf{0}) \geq 2]\mathbb{1}[\xi_t(y) = n-1]\} \\
&\quad \leq E\{\Lambda_t(\mathbf{0}, \mathbf{0}, y, \dots, y); \xi_t(\mathbf{0}) \geq 2, \xi_t(y) = n-1\} \qquad (n-1 \text{ copies of } y) \\
&\quad \leq E\Lambda_t(\mathbf{0}, \mathbf{0}, y, \dots, y) \\
&\quad \leq C^{(3)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right].
\end{aligned}$$

This proves (44).

We next approximate

$$P\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1\}. \tag{45}$$

We will henceforth apply Lemma 13 with $\Gamma := t/2 - \Delta$. In particular, this gives the inequality

$$E\Lambda_{t-\Delta}(u_1, \dots, u_p) \leq C^{(1)}\left[\left(E(t/2)\right)^p \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right],$$

for some $C^{(1)} = C^{(1)}(n, d, \varepsilon, p)$.

At any time $s \geq 0$, we can arbitrarily order the particles located at any position $u \in \mathbb{Z}^d$. In this way, each particle will have a unique index. If $\xi_t(\mathbf{0}) = 1$ and $\xi_t(y) = n-1$, then there must be n distinct particles, $\pi_1, \dots, \pi_n$, in the system at time $t-\Delta$; one of these must move to $\mathbf{0}$ and the rest must move to y at time t without coalescing into or being annihilated by another particle in the mean time. Let $u_1, \dots, u_n$ be the positions and $k_1, \dots, k_n$ the indices of these n particles at time $t-\Delta$.

We define $\mathscr{G}_1$ as the event that:

(i) there exist distinct particles $\pi_1, \dots, \pi_{n+1}$ positioned at some (u_i, k_i) at time $t-\Delta$ such that π_1 moves to be at $\mathbf{0}$ at time t, $\pi_2, \dots, \pi_n$ move to be at y at time t, and

(ii) an $n-$particle collision involving $n-1$ of the $\pi_1, \dots, \pi_n$ along with the particle π_{n+1} occurs in the time interval $(t-\Delta, t]$.

$\mathscr{G}_1$ can be written as the union of several sub-events:

The first sub-event, which we will call $\mathscr{G}_{1,0}$, is the event that $u_1, \dots, u_{n+1}$, the positions of the particles $\pi_1, \dots, \pi_{n+1}$ at time $t-\Delta$, are not distinct. The conditional probability of $\mathscr{G}_{1,0}$ given $\mathscr{F}_{t-\Delta}$ is at most

$$\sum_{u_i \text{ not distinct}} \Lambda_{t-\Delta}(u_1, \dots, u_{n+1})\alpha_\Delta(u_1)\prod_{j=2}^{n}\alpha_\Delta(u_j - y).$$

Taking expectations and using Lemma 13, we see that

$$P\{\mathscr{G}_{1,0}\} \leq C^{(4)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right].$$

The next sub-event, $\mathscr{G}_{1,1}$, is that the $u_1, \dots, u_{n+1}$ are distinct and that the particle π_{n+1}, starting at some site u_{n+1} at time $t-\Delta$, jumps onto a site occupied by $\pi_1, \dots, \pi_{n-1}$ at some time during the interval $(t-\Delta, t]$. Decomposing with respect to the time of the jump and the positions z' and z just before and after the jump, we find that the conditional probability of $\mathscr{G}_{1,1}$ given $\mathscr{F}_{t-\Delta}$ is at most

$$\sum_{u_i \text{ distinct}} \Lambda_{t-\Delta}(u_1, \dots, u_{n+1}) \times \int_0^\Delta \Bigg[\sum_{z,z'} \alpha_s(u_1 - z) \cdots \alpha_s(u_{n-1} - z) \alpha_s(u_{n+1} - z') q(z - z') \times \alpha_{\Delta-s}(z) [\alpha_{\Delta-s}(z-y)]^{n-2} \alpha_\Delta(u_n - y) \Bigg] ds.$$

We take the expectation to get

$$\begin{aligned}
P\{\mathscr{G}_{1,1}\} &\le \sum_{u_i \text{ distinct}} E\Lambda_{t-\Delta}(u_1, \dots, u_{n+1}) \\
&\qquad \times \int_0^\Delta \Bigg[\sum_{z,z'} \alpha_s(u_1 - z) \cdots \alpha_s(u_{n-1} - z) \alpha_s(u_{n+1} - z') q(z - z') \\
&\qquad\qquad \times \alpha_{\Delta-s}(z) [\alpha_{\Delta-s}(z-y)]^{n-2} \alpha_\Delta(u_n - y) \Bigg] ds \\
&\le C^{(3)} \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big] \\
&\qquad \times \sum_{u_i \text{ distinct}} \int_0^\Delta \Bigg[\sum_{z,z'} \alpha_s(u_1 - z) \cdots \alpha_s(u_{n-1} - z) \alpha_s(u_{n+1} - z') q(z - z') \\
&\qquad\qquad \times \alpha_{\Delta-s}(z) [\alpha_{\Delta-s}(z-y)]^{n-2} \alpha_\Delta(u_n - y) \Bigg] ds \\
&\le C^{(5)} \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big] \int_0^\Delta [1 \wedge (\Delta - s)^{-d(n-2)/2}] \, ds \\
&\le C^{(6)} \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big].
\end{aligned}$$

We define the sub-event $\mathscr{G}_{1,2}$ identically to $\mathscr{G}_{1,1}$, except that the collision involves the particles $\pi_2, \dots, \pi_n, \pi_{n+1}$. The bound on the probability of this event is the same as that on $\mathscr{G}_{1,1}$.

We also have the sub-events that one of the particles $\pi_1, \dots, \pi_n$ jumps to a site which is already occupied by π_{n+1} and $n-2$ of the $\pi_1, \dots, \pi_n$. These sub-events are treated in the same way as $\mathscr{G}_{1,2}$, and the bound on their probabilities is the same. The result is that

$$P\{\mathscr{G}_1 \mid \mathscr{F}_{t-\Delta}\} \le C^{(7)} \Big[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}} \Big]. \tag{46}$$

We next define $\mathscr{G}_2$ as the event that:

(i) there exist distinct particles $\pi_1, \dots, \pi_{n+2}$ positioned at some (u_i, k_i) at time $t-\Delta$ such that π_1 moves to be at $\mathbf{0}$ at time t, $\pi_2, \dots, \pi_n$ move to be at y at time t, and

(ii) an n−particle collision involving $n-2$ of the $\pi_1, \dots, \pi_n$ along with the particles π_{n+1} and π_{n+2} occurs in the time interval $(t-\Delta, t]$.

As with $\mathscr{G}_1$, we analyze $P\{\mathscr{G}_2 \mid \mathscr{F}_{t-\Delta}\}$ in terms of its sub-events. The same analysis leads to the bound

$$P\{\mathscr{G}_2 \mid \mathscr{F}_{t-\Delta}\} \le C^{(8)}\left[E^{n+2}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right].$$

Define $\mathscr{G}_3, \dots, \mathscr{G}_{n-2}$ in a corresponding fashion, with the subscript denoting the number of extra particles (not in the set $\{\pi_1, \dots, \pi_n\}$) involved in a collision. The conditional probability of each of these has a bound of order at most $\left[E^{n+2}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]$. Thus,

$$P\left\{\bigcup_{i=1}^{n-2} \mathscr{G}_i \mid \mathscr{F}_{t-\Delta}\right\} \le C^{(9)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]. \tag{47}$$

Lastly, define $\mathscr{G}_{n-1}$ as the event that:

(i) there exist distinct particles $\pi_1, \dots, \pi_{2n-1}$ positioned at some (u_i, k_i) at time $t-\Delta$ such that π_1 moves to be at $\mathbf{0}$ at time t, $\pi_2, \dots, \pi_n$ move to be at y at time t, and

(ii) an n−particle collision involving one of the $\pi_1, \dots, \pi_n$ along with the particles $\pi_{n+1}, \dots, \pi_{2n-1}$ occurs in the time interval $(t-\Delta, t]$.

In this case, sub-events are of the form

$$\mathscr{G}_{(n-1),1} := \big\{\pi_1 \text{ jumps onto a site occupied by } \pi_{n+1}, \dots, \pi_{2n-1} \text{ at some time during the interval } (t-\Delta, t]\big\}.$$

Then

$$\begin{aligned}
&P\{\mathscr{G}_{(n-1),1} \mid \mathscr{F}_{t-\Delta}\} \\
&\quad\le \sum_{u_i \text{ distinct}} \Lambda_{t-\Delta}(u_1, \dots, u_{2n-1}) \\
&\qquad\times \int_0^\Delta \left[\sum_{z,z'} \alpha_s(u_1 - z') \prod_{i=n+1}^{2n-1} \alpha_s(u_i - z) q(z-z') \alpha_{\Delta-s}(z) \prod_{j=2}^{n} \alpha_\Delta(u_j - y)\right] ds.
\end{aligned}$$

Taking the expectation of both sides, we have

$$\begin{aligned}
P\{\mathscr{G}_{(n-1),1}\} \le{}& \sum_{u_i \text{ distinct}} E\Lambda_{t-\Delta}(u_1, \dots, u_{2n-1}) \\
&\times \int_0^\Delta \left[\sum_{z,z'} \alpha_s(u_1 - z') \prod_{i=n+1}^{2n-1} \alpha_s(u_i - z) q(z-z') \alpha_{\Delta-s}(z) \right. \\
&\qquad\left. \times \prod_{j=2}^{n} \alpha_\Delta(u_j - y)\right] ds. \\
\le{}& C^{(10)}\Delta\left[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right].
\end{aligned}$$

The result is that

$$P\{\mathscr{G}_{n-1} \mid \mathscr{F}_{t-\Delta}\} \le C_8\Delta\left[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \tag{48}$$

for some positive constant C_8.

We have finished with the analysis of $\mathscr{G}$ and will now concern ourselves with $\mathscr{G}^c$. On the complement of $\mathscr{G}$, $\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1\}$ occurs if and only if:

(i) there exist distinct particles $\pi_1, \dots, \pi_n$ positioned at some (u_i, k_i) at time $t-\Delta$ such that π_1 moves to be at $\mathbf{0}$ at time t, $\pi_2, \dots, \pi_n$ move to be at y at time t, and

(ii) at any jump-time $s \in (t-\Delta, t]$ when one of these n particles enters a site occupied by the other $n-1$ particles, the corresponding $U[0,1]$ random variable exceeds $a_{n-1} + c_{n-1}$.

Conditionally on $\mathscr{F}_{t-\Delta}$, the probability of these three events is

$$\sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \times P\left\{\bigcup_{\substack{u_1,\dots,u_n\\ 1\le k_i\le \xi_{t-\Delta}(u_i)\\ (u_i,k_i)\neq(u_j,k_j) \text{ for } i\neq j}} \mathscr{E}(u_1,\dots,u_n,k_1,\dots,k_n,m,\Delta,\mathbf{0},y,\dots,y)\right\}.$$

We are now able to write the inequality

$$\begin{aligned}
&P\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1 \mid \mathscr{F}_{t-\Delta}\}\\
&\quad\le P\{\mathscr{G} \mid \mathscr{F}_{t-\Delta}\}\\
&\qquad + \sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \sum_{u_1,\dots,u_n} \sum_{\substack{1\le k_i\le \xi_{t-\Delta}(u_i)\\ (u_i,k_i)\neq(u_j,k_j) \text{ for } i\neq j}}\\
&\qquad\qquad \times P\{\mathscr{E}(u_1,\dots,u_n,k_1,\dots,k_n,m,\Delta,\mathbf{0},y,\dots,y)\}\\
&\quad= P\{\mathscr{G} \mid \mathscr{F}_{t-\Delta}\} + \sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \sum_{u_1,\dots,u_n} \Lambda^*_{t-\Delta}(u_1,\dots,u_n)\\
&\qquad\qquad \times\Bigg[P\{\mathscr{E}(u_1,\dots,u_n,1,\dots,n,m,\Delta,\mathbf{0},y,\dots,y)\}\\
&\qquad\qquad\qquad -\rho(m,y)\alpha_\Delta(u_1)\prod_{i=2}^{n}\alpha_\Delta(u_i-y)\Bigg]\\
&\qquad + \sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m\rho(m,y)\\
&\qquad\qquad \times \sum_{u_1,\dots,u_n}\Lambda^*_{t-\Delta}(u_1,\dots,u_n)\alpha_\Delta(u_1)\prod_{i=2}^{n}\alpha_\Delta(u_i-y).
\end{aligned} \tag{49}$$

Taking the expectation and applying (47) and (48) to the first term and Lemmas 13 and 15 to the other terms, we see that

$$
\begin{aligned}
&P\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1\} \\
&\quad \le C^{(10)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right] + C_8\Delta\left[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + \sum_{m=0}^{\infty}(1 - a_{n-1} - c_{n-1})^m C_5 \left[E^n(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad\qquad \times \sum_{u_1,\dots,u_n} \Bigg| P\{\mathscr{E}(u_1, \dots, u_n, 1, \dots, n, m, \Delta, \mathbf{0}, y, \dots, y)\} \\
&\qquad\qquad\qquad - \rho(m, y)\alpha_\Delta(u_1)\prod_{i=2}^{n}\alpha_\Delta(u_i - y)\Bigg| \\
&\qquad + \sum_{m=0}^{\infty}(1 - a_{n-1} - c_{n-1})^m \rho(m, y) \\
&\qquad\qquad \times \sum_{u_1,\dots,u_n} E\{\Lambda^*_{t-\Delta}(u_1, \dots, u_n)\}\alpha_\Delta(u_1)\prod_{i=2}^{n}\alpha_\Delta(u_i - y) \\
&\quad \le C^{(10)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + C_8\Delta\left[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + C_9\Delta^{-\delta(d,n)}\left[E^{n}(t/2) \vee t^{\frac{n-2}{n-1} - \frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + D(y)\sum_{u_1,\dots,u_n} E\{\Lambda^*_{t-\Delta}(u_1, \dots, u_n)\}\alpha_\Delta(u_1)\prod_{i=2}^{n}\alpha_\Delta(u_i - y).
\end{aligned}
$$

To get an inequality in the other direction,

$$
\begin{aligned}
&P\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1 \mid \mathscr{F}_{t-\Delta}\} \\
&\quad \ge \sum_{m=0}^{\infty}(1 - a_{n-1} - c_{n-1})^m \\
&\qquad \times P\Bigg\{ \bigcup_{\substack{u_1,\dots,u_n \\ 1 \le k_i \le \xi_{t-\Delta}(u_i) \\ (u_i,k_i) \ne (u_j,k_j)}} S_\Delta^{(u_1,k_1)} = \mathbf{0}, S_\Delta^{(u_i,k_i)} = y, i = 2, \dots, n \\
&\qquad\qquad \text{and the paths } s \to S_s^{(u_i,k_i)} \text{ meet exactly } m \text{ times during } (0, \Delta]\Bigg\} \\
&\qquad - P\{\mathscr{G} \mid \mathscr{F}_{t-\Delta}\}.
\end{aligned}
$$

Apply inclusion-exclusion to see that this is at least

$$
\begin{aligned}
&-P\{\mathscr{G} \mid \mathscr{F}_{t-\Delta}\} + \sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \\
&\quad\times \sum_{\substack{u_1,\dots,u_n\\ 1\le k_i\le \xi_{t-\Delta}(u_i)\\ (u_i,k_i)\neq(u_j,k_j)}} P\big\{\mathscr{E}(u_1,\dots,u_n,k_1,\dots,k_n,m,\Delta,\mathbf{0},y,\dots,y)\big\} \qquad (50)\\
&-\sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \\
&\quad\times \sum_{\substack{u_1,\dots,u_{2n}\\ 1\le k_i\le \xi_{t-\Delta}(u_i)}} P\Big\{\mathscr{E}(u_1,\dots,u_n,k_1,\dots,k_n,m,\Delta,\mathbf{0},y,\dots,y) \qquad (51)\\
&\qquad\qquad \cap\, \mathscr{E}(u_{n+1},\dots,u_{2n},k_{n+1},\dots,k_{2n},m,\Delta,\mathbf{0},y,\dots,y)\Big\}.
\end{aligned}
$$

The last sum in (51) is over (u_i, k_i) such that $(u_i, k_i) \neq (u_j, k_j)$ for $i \neq j$ and $i, j \in [1, \dots, n]$, $(u_\ell, k_\ell) \neq (u_p, k_p)$ for $\ell \neq p$ and $\ell, p \in [(n+1), \dots, 2n]$ and, finally, $(u_1, \dots, u_n, k_1, \dots, k_n) \neq (u_{n+1}, \dots, u_{2n}, k_{n+1}, \dots, k_{2n})$.

We separate this last sum into n sub-sums grouped according to the number of distinct (u_i, k_i) present in each term (ranging from $n+1$ to $2n$). From Lemma 13, we see that the highest order bound on the expectation of any of these sub-sums is the bound on the sum over $n+1$ distinct particles. Thus, for large t, the expectation of each of the n sub-sums in (50) is bounded by

$$\sum_{u_1,\dots,u_{n+1}} C^{(3)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]\alpha_\Delta(u_1)\alpha_\Delta(u_{n+1})\prod_{i=2}^{n}\alpha_\Delta(u_i-y).$$

We sum over the u_i's and m and see that the expected value of (51) is at most

$$C^{(11)}(a_{n-1}+c_{n-1})^{-1}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right].$$

Estimating (50) as we did (49), we see that

$$
\begin{aligned}
&P\{\xi_t(\mathbf{0}) = 1, \xi_t(y) = n-1\}\\
&\quad \ge D(y)\sum_{u_1,\dots,u_n}\left[E\{\Lambda^*_{t-\Delta}(u_1,\dots,u_n)\}\alpha_\Delta(u_1)\prod_{i=2}^{n}\alpha_\Delta(u_i-y)\right]\\
&\qquad - C^{(10)}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]\\
&\qquad - C_8\Delta\left[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]\\
&\qquad - C_9\Delta^{-\delta(d,n)}\left[E^{n}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]\\
&\qquad - C^{(11)}(a_{n-1}+c_{n-1})^{-1}\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]. \qquad \square
\end{aligned}
$$

PROOF OF THEOREM 1. We assume throughout that $n \geq 3$ (for Lemma 11), $t \geq 1$ and $d \geq 2n+4$. With $\delta = \delta(d, n)$ taken from Lemma 15, we choose $\zeta > 0$ such that

$$\zeta < \left[\frac{1}{(n-1)} \wedge \frac{2}{(d+2)} \wedge \frac{\delta}{(\delta+1)}\right]$$

and set

$$\Delta := t^{1-\zeta}.$$

These three bounds on ζ assure us that

$$E(t/2) \leq 2^{\frac{1}{n-1}} C_2 t^{-\zeta}, \tag{52}$$

$$t^{\frac{n-2}{n-1}} \Delta^{-\frac{d}{2}} \leq t^{-n-\zeta} \tag{53}$$

and

$$\Delta^{-\delta} \leq t^{-\zeta}, \tag{54}$$

respectively. Here we have applied Proposition 1 to get (52). Note that (53), along with Lemma 11, gives us the inequality

$$\Delta^{-d/2} \leq C^{(1)} E(t/2) t^{-\zeta} \tag{55}$$

for some positive constant $C^{(1)} = C^{(1)}(d)$.

We choose $\varepsilon < 1$ small enough so that, for $d \geq 2n+4$,

$$t^{1+\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}} = o(t^{-n}) = o\big(E^n(t/2)\big) \quad \text{(by Lemma 11)}. \tag{56}$$

Lemmas 7 and 16, along with (54) and (56), show that

$$\begin{aligned}
&\Bigg|\frac{d}{dt}E(t) + \sum_y q(y)(2a_{n-1}+c_{n-1})D(y) \sum_{\substack{u_i \in \mathbb{Z}^d \\ i=1,\ldots,n}} E\left\{\Lambda^*_{t-\Delta}(u_1,\ldots,u_n)\right\} \\
&\qquad\qquad \times \alpha_\Delta(u_1)\prod_{i=2}^n \alpha_\Delta(u_i-y)\Bigg| \qquad (57)\\
&\quad \leq C_7\left[E^{n+1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + C_8\Delta\left[E^{2n-1}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + C_9\Delta^{-\delta}\left[E^{n}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \\
&\quad \leq C^{(2)}E(t/2)\left[E^{n}(t/2) \vee t^{1+\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \qquad (58)\\
&\qquad + C_8 t^{-\zeta}\left[E^{n}(t/2)[tE^{n-1}(t/2)] \vee t^{1+\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad + C_9\Delta^{-\delta}\left[E^{n}(t/2) \vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right].
\end{aligned}$$

Here we have applied Lemma 11 in rewriting the first term. By Proposition 1, we have, for some positive constant $C^{(3)}$, the bound

$$tE^{n-1}(t/2) \le C^{(3)}. \tag{59}$$

Applying (56) to the three terms in (58), as well as (52) to the first term, (59) to the second term, and (54) to the third term, we see that (57) can be bounded by

$$C^{(4)}E^n(t/2)t^{-\zeta},$$

for some constant $C^{(4)}$.

Recall that the local central limit theorem gives the bound

$$\sup_x \alpha_\Delta(x) \le C^{(5)}\Delta^{-d/2}, \tag{60}$$

so that we can write

$$\begin{aligned}
&\left| \sum_{\substack{u_1,\dots,u_n\in\mathbb{Z}^d \\ u_i=u_j \text{ for at} \\ \text{least one pair } i,j}} E\Lambda^*_{t-\Delta}(u_1,\dots,u_n)\alpha_\Delta(u_1)\prod_{i=2}^n \alpha_\Delta(u_i-y)\right| \\
&\qquad \le C^{(6)}\Delta^{-d/2}\sup_{u_1,\dots,u_n} E\Lambda^*_{t-\Delta}(u_1,\dots,u_n) \\
&\qquad \le C^{(7)}t^{-\zeta}\left[E^{n+1}(t/2)\vee t^{1+\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] && (61)\\
&\qquad \le C^{(8)}E^n(t/2)t^{-\zeta}, && (62)
\end{aligned}$$

by (55) and (56). Similarly,

$$\begin{aligned}
&\left| \sum_{\substack{u_1,\dots,u_n\in\mathbb{Z}^d \\ u_i=u_j \text{ for at} \\ \text{least one pair } i,j}} E\left\{\prod_{i=1}^n \xi_{t-\Delta}(u_i)\right\}\alpha_\Delta(u_1)\prod_{i=2}^n \alpha_\Delta(u_i-y)\right| \\
&\qquad \le C^{(9)}\sup_{k=1,\dots,n-1}(n-k+1)^n\Delta^{-d(n-k)/2}\left[E^k(t/2)\vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right] \\
&\qquad \le C^{(10)}E^n(t/2)t^{-\zeta}. && (63)
\end{aligned}$$

Here we have applied (42) to arrive at the first inequality. The variable k indicates the number of distinct sites in the summation. Equations (62) and (63) allow us, with an error of order $E^n(t/2)t^{-\zeta}$, to apply equation (43) and replace (57) with

$$S\left|\frac{d}{dt}E(t)+\sum_y q(y)(2a_{n-1}+c_{n-1})D(y)\right.$$

$$\left.\times\sum_{\substack{u_i\in\mathbb{Z}^d\\ i=1,\dots,n}} E\left\{\prod_{i=1}^n \xi_{t-\Delta}(u_i)\right\}\alpha_\Delta(u_1)\prod_{i=2}^n \alpha_\Delta(u_i-y)\right| \tag{64}$$

$$\leq C^{(11)}E^n(t/2)t^{-\zeta}.$$

We continue by writing

$$\sum_{\substack{u_i\in\mathbb{Z}^d\\ i=1,\dots,n}} \alpha_\Delta(u_1)\xi_{t-\Delta}(u_1)\prod_{i=2}^n \alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)$$

$$=\left[\sum_{u_1}\alpha_\Delta(u_1)\xi_{t-\Delta}(u_1)\right]\prod_{i=2}^n\left[\sum_{u_i}\alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)\right].$$

We can then use the variance estimate (Lemma 12) to bound the error involved in replacing the expectation of this product of sums with the product of the expectations of the sums. As a first step, we have

$$\left|\sum_{\substack{u_i\in\mathbb{Z}^d\\ i=1,\dots,n}} E\left\{\prod_{i=1}^n \xi_{t-\Delta}(u_i)\right\}\alpha_\Delta(u_1)\prod_{i=2}^n \alpha_\Delta(u_i-y)\right.$$

$$\left.-E\left\{\sum_{u_1}\alpha_\Delta(u_1)\xi_{t-\Delta}(u_1)\right\}E\left\{\prod_{i=2}^n\left[\sum_{u_i}\alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)\right]\right\}\right| \tag{65}$$

$$\leq\left[\operatorname{Var}\left(\sum_{u_1}\alpha_\Delta(u_1)\xi_{t-\Delta}(u_1)\right)\right]^{\frac12}\left[\operatorname{Var}\left(\prod_{i=2}^n\left[\sum_{u_i}\alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)\right]\right)\right]^{\frac12}.$$

By Lemma 12 and (60), the first variance is bounded by $C_4C^{(5)}t^{\frac{n-2}{n-1}}\Delta^{-\frac{d}{2}}$.

For the second variance, we use the bound $\operatorname{Var}(X)\leq E(X^2)$, where

$$X^2=\prod_{i=1}^{2n-2}\left[\sum_{u_i\in\mathbb{Z}^d}\alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)\right].$$

As seen previously in this proof, we use (42), Lemma 13 and (60) to show that the expectation over the sum of the terms with $k\leq 2n-2$ distinct u_i's is bounded by $C^{(12)}[E^k(t/2)\vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}]\Delta^{-d(2n-2-k)/2}$, so that, again by (55),

$$\operatorname{Var}(X^2)\leq C^{(13)}\left[E^{2n-2}(t/2)\vee t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right].$$

From (53) and Lemma 11, we see that

$$\left[t^{\frac{n-2}{n-1}}\Delta^{-\frac{d}{2}}E^{2n-2}(t/2)\right]^{1/2} \le C^{(14)}E^n(t/2)t^{-\zeta}.$$

In addition, (53), (56) and Lemma 11 show that

$$\left[t^{\frac{n-2}{n-1}}\Delta^{-\frac{d}{2}}t^{\frac{n-2}{n-1}-\frac{d(1-\varepsilon)}{2}}\right]^{1/2} \le C^{(15)}E^n(t/2)t^{-\zeta}.$$

The result is that (66) is bounded above by $C^{(16)}E^n(t/2)t^{-\zeta}$.

Continuing to split up the product of expectations in this manner, we arrive at

$$\begin{aligned}&\Bigg|\sum_{\substack{u_i\in\mathbb{Z}^d\\ i=1,\dots,n}} E\left\{\prod_{i=1}^n \xi_{t-\Delta}(u_i)\right\}\alpha_\Delta(u_1)\prod_{i=2}^n\alpha_\Delta(u_i-y)\\ &\qquad - E\left\{\sum_{u_1\in\mathbb{Z}^d}\alpha_\Delta(u_1)\xi_{t-\Delta}(u_1)\right\}\prod_{i=2}^n E\left\{\sum_{u_i\in\mathbb{Z}^d}\alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)\right\}\Bigg|\\ &\qquad \le C^{(17)}E^n(t/2)t^{-\zeta}.\end{aligned}$$

The approximation in (64) now yields

$$\begin{aligned}&\Bigg|\frac{d}{dt}E(t)+\sum_y q(y)(2a_{n-1}+c_{n-1})D(y)\sum_{u_1}E\{\alpha_\Delta(u_1)\xi_{t-\Delta}(\mathbf{0})\}\\ &\qquad\qquad \times\prod_{i=2}^n E\left\{\sum_{u_i}\alpha_\Delta(u_i)\xi_{t-\Delta}(\mathbf{0})\right\}\Bigg|\\ &\quad=\Bigg|\frac{d}{dt}E(t)+\sum_y q(y)(2a_{n-1}+c_{n-1})D(y)\sum_{u_1}E\{\alpha_\Delta(u_1)\xi_{t-\Delta}(u_1)\}\\ &\qquad\qquad \times\prod_{i=2}^n E\left\{\sum_{u_i}\alpha_\Delta(u_i-y)\xi_{t-\Delta}(u_i)\right\}\Bigg|\\ &\quad\le C^{(18)}E^n(t/2)t^{-\zeta}.\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_y q(y)(2a_{n-1}+c_{n-1})D(y)\\ &\quad=\sum_y q(y)(2a_{n-1}+c_{n-1})\\ &\qquad\times\sum_{m=0}^\infty(1-a_{n-1}-c_{n-1})^m P\{s\mapsto S_s^{\mathbf{0}}\text{ and } n-1 \text{ copies of } s\mapsto S_s^{-y}\\ &\qquad\qquad \text{meet exactly } m \text{ times during } [0,\infty)\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_y q(y)(2a_{n-1}+c_{n-1}) \\
&\qquad \times \sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m P\{s\mapsto S_s^y \text{ and } n-1 \text{ copies of } s\mapsto S_s^{\mathbf{0}} \\
&\qquad\qquad\qquad \text{meet exactly } m \text{ times during } [0,\infty)\} \\
&= (2a_{n-1}+c_{n-1})\sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \\
&\qquad\qquad \times \sum_y q(y) P\{s\mapsto S_s^y \text{ and } n-1 \text{ copies of } s\mapsto S_s^{\mathbf{0}} \\
&\qquad\qquad\qquad \text{meet exactly } m \text{ times during } [0,\infty)\} \\
&= (2a_{n-1}+c_{n-1})\sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \\
&\qquad\qquad \times \sum_y q(y) P\{n \text{ copies of } s\mapsto S_s^{\mathbf{0}} \text{ meet exactly } m \text{ times during} \\
&\qquad\qquad\qquad [0,\infty) \mid \text{the first of the } n \text{ particles to move jumps to } y\} \\
&= (2a_{n-1}+c_{n-1})\sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m \\
&\qquad\qquad \times P\{n \text{ independent copies of } S_\cdot \text{ collide exactly } m \\
&\qquad\qquad\qquad \text{times after the first walk leaves } \mathbf{0}\} \\
&= (2a_{n-1}+c_{n-1})\sum_{m=0}^{\infty}(1-a_{n-1}-c_{n-1})^m(1-\gamma_{n,d})^m\gamma_{n,d} \\
&= [C_1(d,n)]^{-(n-1)}/(n-1),
\end{aligned}
$$

where

$$
\gamma_{n,d} := P\big\{n \text{ independent copies of } S_\cdot \text{ never coincide after the first walk leaves } \mathbf{0}\big\}.
$$

Thus, for $t \ge 1$,

$$
\left|\frac{d}{dt}E(t) + \frac{[C_1(d,n)]^{-(n-1)}}{(n-1)}\left[\sum_u \alpha_\Delta(u)E\xi_{t-\Delta}(\mathbf{0})\right]^n\right| \le C^{(18)}E^n(t/2)t^{-\zeta}. \tag{66}
$$

As in the proof of Lemma 16,

$$
\begin{aligned}
&E\{\xi_t(\mathbf{0}) \mid \mathscr{F}_{t-\Delta}\} \\
&\quad \ge \sum_{v\in\mathbb{Z}^d}\sum_{\ell\le\xi_{t-\Delta}(v)} P\big\{S_\Delta^{(v,\ell)} = \mathbf{0} \text{ and the path } s\mapsto S_s^{(v,\ell)} \text{ does not simultaneously} \\
&\qquad\qquad \text{coincide with } n-1 \text{ other paths } s\mapsto S_s^{(u_i,k_i)} \\
&\qquad\qquad \text{for any } s\le\Delta, u_1,\dots,u_{n-1}\in\mathbb{Z}^d, k_i\le\xi_{t-\Delta}(u_i)\big\} \\
&\quad \ge \sum_{v\in\mathbb{Z}^d}\sum_{\ell\le\xi_{t-\Delta}(v)} \alpha_\Delta(v)
\end{aligned}
$$

$$-n\sum_{u_1,\dots,u_n}\int_0^{\Delta}\Lambda_{t-\Delta}(u_1,\dots,u_n)\sum_{z,z'}\alpha_s(u_1-z')\alpha_{\Delta-s}(z')q(z-z')$$
$$\times\prod_{i=2}^{n}\alpha_s(u_i-z)\,ds,$$

since $P(A\cap B)\geq P(A)-P(A\cap B^c)$. Taking expectations and applying Lemma 13 and equation (56), we see that

$$E\xi_t(\mathbf{0})\geq E\left\{\sum_v \alpha_\Delta(v)\xi_{t-\Delta}(v)\right\}-C^{(19)}\Delta E^n(t/2).$$

Raising both sides to the nth power, expanding the right hand side and using Proposition 1 to bound the terms in the expansion, we have

$$\begin{aligned}[E\xi_t(\mathbf{0})]^n &\geq [E\xi_{t-\Delta}(\mathbf{0})]^n+\sum_{k=1}^{n}\binom{n}{k}(-\Delta E^{n-1}(t/2)C^{(19)})^k\,[E\xi_{t-\Delta}(\mathbf{0})]^{n-k}\,E^k(t/2)\\ &\geq [E\xi_{t-\Delta}(\mathbf{0})]^n-C^{(20)}E^n(t/2)t^{-\zeta}.\end{aligned}$$

Here we have again used the fact that $E(t)$ is decreasing in t and that $tE^{n-1}(t)$ is bounded above by a constant. In addition, we know that

$$E\xi_t(\mathbf{0})\leq E\left\{\sum_v \alpha_\Delta(v)\xi_{t-\Delta}(v)\right\}=E\xi_{t-\Delta}(\mathbf{0}).$$

Using these two inequalities, we get the bound

$$\left|\big[E\xi_t(\mathbf{0})\big]^n-\big[E\xi_{t-\Delta}(\mathbf{0})\big]^n\right|\leq C^{(20)}E^n(t/2)t^{-\zeta}.$$

Equation (66) then becomes

$$\left|\frac{d}{dt}E(t)+\frac{[C_1(d,n)]^{-(n-1)}}{(n-1)}E^n(t)\right|\leq C^{(21)}E^n(t/2)t^{-\zeta},\quad (t\geq 1). \tag{67}$$

We would like to rewrite this bound in terms of $E^n(t)$. Lemma 11 guarantees that there exists some $C^{(22)}>0$ such

$$\liminf_{t\to\infty}\frac{E(t)}{E(4t)}\leq C^{(22)}.$$

To see this, assume that for all $C>0$ there exists an $M(C)$ such that $E(4t)<E(t)/C$ for all $t\geq M(C)$. For any $k>0$ and $t=4^kM(16)$, we could then write $E(t)\leq E(M(16))/16^k=O(t^{-2})$, contradicting Lemma 11.

For ease of notation, we set

$$C^{(23)}:=\frac{[C_1(d,n)]^{-(n-1)}}{(n-1)}.$$

Without loss of generality, we take $C^{(22)} \geq 1$ large enough so that

$$\left(1 + C^{(22)}\right)^{n-1} \geq 1 + (n-1)C^{(3)}\left(1 - C^{(23)}\right), \tag{68}$$

where $C^{(3)}$ is as in (59). Choose T_0 large enough so that

$$T_0^{-\zeta} C^{(21)} \left(4C^{(22)}\right)^n \leq 1 \tag{69}$$

and

$$\frac{E(T_0)}{E(4T_0)} \leq 2C^{(22)}.$$

Define

$$T := \inf\left\{t \geq T_0 : \frac{E(t)}{E(2t)} \geq 4C^{(22)}\right\}.$$

Note that $T \geq 2T_0$, since, for $T_0 \leq s \leq 2T_0$, we have

$$\frac{E(s)}{E(2s)} \leq \frac{E(T_0)}{E(4T_0)} \leq 2C^{(22)}.$$

We want to prove that $T = \infty$. Assume the contrary, that T is finite, in which case we can write

$$\begin{aligned} E^{-(n-1)}(2T) - E^{-(n-1)}(T) &= (n-1)\int_T^{2T} E^{-n}(s)\frac{dE(s)}{ds}\, ds \\ &\leq (n-1)\int_T^{2T}\left[C^{(21)}s^{-\zeta}\frac{E^n(s/2)}{E^n(s)} - C^{(23)}\right] ds \\ &\leq (n-1)\int_T^{2T}\left[1 - C^{(23)}\right] ds, \end{aligned}$$

where the first inequality is from (67) and the second inequality follows from (69), since $T_0 \leq s/2 \leq T$. At this point, we have

$$E^{-(n-1)}(2T) - E^{-(n-1)}(T) \leq (n-1)T\left[1 - C^{(23)}\right],$$

so that

$$\begin{aligned} \left[\frac{E(T)}{E(2T)}\right]^{n-1} &\leq 1 + (n-1)(1 - C^{(23)})TE^{n-1}(T) \\ &\leq 1 + (n-1)(1 - C^{(23)})C^{(3)} \quad \text{(by (59))} \\ &\leq (1 + C^{(22)})^{n-1} \quad \text{(by (68))}. \end{aligned}$$

This provides the desired contradiction, since T is defined so that $\frac{E(T)}{E(2T)} \geq 4C^{(22)} > (1 + C^{(22)})$.

We have now proven that there exists a positive constant $C^{(24)}$ such that

$$\left[\frac{E(t)}{E(2t)}\right]^n \leq C^{(24)} \quad \text{for all } t \geq 0.$$

We express equation (66) as

$$\left|\frac{d}{dt}E(t) + \frac{[C_1(d,n)]^{-(n-1)}}{(n-1)}E^n(t)\right| \le C^{(25)}E^n(t)t^{-\zeta}, \quad (t \ge 1).$$

Integration gives

$$\begin{aligned} E^{-(n-1)}(t) - E^{-(n-1)}(0) &= -(n-1)\int_0^t E^{-n}(s)\frac{dE(s)}{ds}\,ds \\ &= \frac{[C_1(d,n)]^{-(n-1)}}{(n-1)}(n-1)t + O\left(t^{1-\zeta}\right). \end{aligned}$$

Then

$$\begin{aligned} E(t) &= C_1(d,n)t^{-\frac{1}{n-1}}\left[1 + O(t^{-\zeta})\right]^{-\frac{1}{n-1}} \\ &= C_1(d,n)t^{-\frac{1}{n-1}} + O\left(t^{-\frac{1}{n-1}-\zeta}\right), \end{aligned}$$

using a binomial series expansion to estimate $[1+O(t^{-\zeta})]^{-\frac{1}{n-1}}$. This gives (5). Results (4) and (6) in Theorem 1 then follow from (40) and (39), respectively. □

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