EQUALITIES FOR STATIONARY PROCESSES SIMILAR TO AN EQUALITY OF WALD

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I. Introduction. Let Ω be a non-empty set with elements ω, $\mathfrak{F}$ be a σ-algebra of subsets of Ω and P be a probability measure on $\mathfrak{F}$. Let T be a one to one map of Ω onto Ω which, together with its inverse T^{-1} are $\mathfrak{F}$-measurable and P measure preserving. For any random variable (real $\mathfrak{F}$-measurable function) X on Ω, let TX be the function on Ω defined by $TX(\omega) = X(T\omega)$ so that $[TX \varepsilon B] = T^{-1}[X \varepsilon B]$ for any Borel set B. Consider an $\mathfrak{F}$-measurable set E with $P(E) > 0$. For any $\omega \varepsilon E$ consider the images of ω under iterates of T: $T\omega$, $T^2\omega$, $\cdots$, $T^n\omega$, $\cdots$. If n_1 is the smallest positive integer for which $T^{n_1}\omega \varepsilon E$ we say that the first recurrence time ν_1 of E is equal to n_1. The Poincaré recurrence theorem ([2], p. 10) asserts that ν_1 is well defined and finite almost everywhere on E. In fact the stronger version of the Poincaré recurrence theorem asserts that, for almost all $\omega \varepsilon E$, there are infinitely many positive integers n such that $T^n\omega \varepsilon E$. Let us write down these integers according to their natural order, n_1, $n_1 + n_2$, $n_1 + n_2 + n_3$, $\cdots$. Then n_k is defined to be the value of the kth recurrence time ν_k of ω. Thus the successive recurrence times of E: ν_1, ν_2, $\cdots$ are well defined almost everywhere on E. If we introduce the conditional probability measure given E, P_E, on $\mathfrak{F}$ by

$$P_E(A) = P(E \cap A)/P(E), \tag{1}$$

then ν_1, ν_2, $\cdots$ are well defined and finite valued with P_E probability one on the whole space Ω. In [3] it was proved that $\{\nu_k\}$ is a stationary sequence under P_E measure. In this paper we shall introduce a P_E measure preserving transformation S which associates with $\{\nu_k\}$ in a very natural way. It is shown that $S^{k-1}\nu_1 = \nu_k$, $k = 1, 2, \cdots$, so that the stationarity of $\{\nu_k\}$ is actually due to the P_E measure preserving property of S. Let $X_n = T^nX$. It is then shown that sequences $\{X_{\nu_k}\}$ and $\{X_{\nu_1+\cdots+\nu_{k-1}+1} + \cdots + X_{\nu_1+\cdots+\nu_k}\}$ are stationary under P_E measure. This leads to equalities (13) and (15), which resemble an equality of Wald for an independent sequence of random variables [1]. In fact, the proofs of (13) and (15) are also rather similar to the proof given in [1].

II. The Transformation S. Let $\bar{E}$, $\underline{E}$ be subsets of Ω defined by

$$\bar{E} = E \cap \left(\bigcup_{n=1}^{\infty} T^{-n}E\right), \tag{2}$$

$$\underline{E} = E \cap \left(\bigcup_{n=1}^{\infty} T^{n}E\right). \tag{3}$$

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$\bar{E}$ may be decomposed into disjoint, countably many pieces, $D_1, D_2, \cdots$, where

$$D_n = E \cap T^{-1}E' \cap \cdots \cap T^{-(n-1)}E' \cap T^{-n}E \tag{4}$$

with $E' = \Omega - E$. Similarly, $\underline{E}$ may be decomposed into disjoint, countably many pieces, $F_1, F_2, \cdots$, where

$$F_n = T^nE \cap T^{n-1}E' \cap \cdots \cap TE' \cap E. \tag{5}$$

We shall define a one to one map S of $\bar{E}$ onto $\underline{E}$ as follows:

$$S\omega = T^n\omega \quad \text{if} \quad w \, \varepsilon \, D_n, \qquad n = 1, 2, 3, \cdots. \tag{6}$$

In other words, S is identical to T^n on D_n. It is clear that D_n is mapped onto F_n under S. $\bar{E}$ consists of all points of E for which there is a positive integer n such that $T^n\omega \, \varepsilon \, E$, therefore, has P_E measure one according to the Poincaré recurrence theorem. Applying the same theorem to T^{-1} we conclude that $\underline{E}$ is also of P_E measure one. Hence S and its integral (positive or negative) powers are well defined with P_E probability one on Ω.

Lemma 1. *If $A \, \varepsilon \, \mathfrak{F}$ and $A \subset \bar{E}$ then $P_E(A) = P_E(SA)$.*

Proof. It is sufficient to prove that $P(A) = P(SA)$.

$$P(SA) = P\left[\bigcup_{n=1}^{\infty} S(A \cap D_n)\right] = \sum_{n=1}^{\infty} P[S(A \cap D_n)] = \sum_{n=1}^{\infty} P[T^n(A \cap D_n)]$$
$$= \sum_{n=1}^{\infty} P(A \cap D_n) = P\left[\bigcup_{n=1}^{\infty} (A \cap D_n)\right] = P(A).$$

For any $\mathfrak{F}$-measurable function Z which is well defined up to a set of P_E measure 0, SZ, $S^{-1}Z$ are defined by $SZ(\omega) = Z(S\omega)$, $S^{-1}Z(\omega) = Z(S^{-1}\omega)$. Again SZ, $S^{-1}Z$ are well defined up to sets of P_E measure 0 and $SS^{-1}Z = S^{-1}SZ = Z$ with P_E measure one. The following theorem follows immediately from Lemma 1.

Theorem 1. *Let Z be any random variable and $Z_k = S^kZ$, $k = 1, 2, 3, \cdots$. Then $\{Z_k\}$ is a stationary sequence under the conditional probability measure P_E.*

The natural connection between S and the successive recurrence times, $\nu_1, \nu_2, \cdots$, is revealed by the following theorem.

Theorem 2. *$S^{k-1}\nu_1 = \nu_k$ with P_E measure one for $k = 1, 2, 3, \cdots$. For any positive integer k and any k positive integers $n_1, n_2, \cdots, n_k$, $S^k = T^{n_1+n_2+\cdots+n_k}$ on the set $[\nu_1 = n_1, \nu_2 = n_2, \cdots \nu_k = n_k]$.*

Proof. For any k positive integers $n_1, n_2, \cdots, n_k$, let

$$D_{n_1,n_2,\cdots,n_k} = E \cap T^{-1}E' \cap \cdots \cap T^{-(n_1-1)}E' \cap T^{-n_1}E \cap T^{-(n_1+1)}E'$$
$$\cap \cdots \cap T^{-(n_1+n_2-1)}E' \cap T^{-(n_1+n_2)}E \cap T^{-(n_1+n_2+1)}E'$$
$$\cap \cdots \cap T^{-(n_1+n_2+\cdots+n_k-1)}E' \cap T^{-(n_1+n_2+\cdots+n_k)}E.$$

Then

$$\begin{aligned}
&D_{n_1,n_2,\cdots,n_k} \subset D_{n_1}, \\
&T^{n_1}D_{n_1,n_2,\cdots,n_k} \subset D_{n_2}, \\
&T^{n_1+n_2}D_{n_1,n_2,\cdots,n_k} \subset D_{n_3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&T^{n_1+n_2+\cdots+n_{k-1}}D_{n_1,n_2,\cdots,n_k} \subset D_{n_k}.
\end{aligned}$$

Hence if $\omega \varepsilon D_{n_1,n_2,\cdots,n_k}$

$$\begin{aligned}
&S\omega = T^{n_1}\omega \varepsilon D_{n_2}, \\
&S^2\omega = SS\omega = T^{n_2}S\omega = T^{n_1+n_2}\omega \varepsilon D_{n_3}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&S^k\omega = SS^{k-1}\omega_1 = T^{n_k}S^{k-1}\omega = T^{n_1+n_2+\cdots+n_k}\omega.
\end{aligned}$$

We observe that $[\nu_1 = n] = D_n$ and $[\nu_1 = n_1, \nu_2 = n_2, \cdots, \nu_k = n_k] = D_{n_1,n_2,\cdots,n_k}$. Therefore the second half of the theorem is proved. The first half of the theorem will be proved only for the case $k = 2$. The general case can be proved similarly.

In the following, two sets are equal if they differ at most by a set of P_E measure 0. From the definition of S,

$$S^{-1}\omega = T^{-n}\omega \quad \text{if} \quad \omega \varepsilon F_n, \qquad n = 1, 2, 3, \cdots. \tag{7}$$

Hence for any positive integer j,

$$\begin{aligned}
[S\nu_1 = j] &= S^{-1}[\nu_1 = j] = \bigcup_{k=1}^{\infty} T^{-k}[F_k \cap D_j] \\
&= \bigcup_{k=1}^{\infty} T^{-k}[T^kE \cap T^{k-1}E' \cap \cdots \cap TE' \cap E \cap T^{-1}E' \cap \cdots \cap T^{-(j-1)}E' \cap T^{-j}E] \\
&= \bigcup_{k=1}^{\infty} [E \cap T^{-1}E \cap \cdots \cap T^{-(k-1)}E' \cap T^{-k}E \cap T^{-(k+1)}E' \\
&\qquad\qquad \cap \cdots \cap T^{-(k+j-1)}E' \cap T^{-(k+j)}E] \\
&= \bigcup_{k=1}^{\infty} [\nu_1 = k, \nu_2 = j] = [\nu_2 = j].
\end{aligned}$$

Hence $S\nu_1 = \nu_2$ with P_E measure one.

III. Two Equalities for a Stationary Sequence. Let X be any random variable and let

$$X_n = T^nX, \qquad n = 1, 2, \cdots. \tag{8}$$

Then $\{X_n\}$ is a stationary sequence under the measure P. For any positive integer k define $X_{\nu_1+\cdots+\nu_k}$ by (9).

$$X_{\nu_1+\cdots+\nu_k} = X_{n_1+\cdots+n_k} = T^{n_1+\cdots+n_k}X \tag{9}$$

on the set $[\nu_1 = n_1, \cdots, \nu_k = n_k]$. Then $X_{\nu_1+\cdots+\nu_k}$ are well defined with P_E measure one. By Theorem 2, $S^k X = T^{n_1+\cdots+n_k} X$ on the set $[\nu_1 = n_1, \cdots, \nu_k = n_k]$. Hence

$$S^k X = X_{\nu_1+\cdots+\nu_k}. \tag{10}$$

More generally, for any positive integer n, let $f_n(x_1, \cdots, x_n)$ be a Borel measurable function defined on the n-dimensional Euclidean space. Define

$$Z_k = f_{\nu_k}(X_{\nu_1+\cdots+\nu_{k-1}+1}, X_{\nu_1+\cdots+\nu_{k-1}+2}, \cdots, X_{\nu_1+\cdots+\nu_k})$$

as follows:

$$Z_k = f_{n_k}(x_{n_1+\cdots+n_{k-1}+1}, X_{n_1+\cdots+n_{k-1}+2}, \cdots, X_{n_1+\cdots+n_k}) \tag{11}$$

on the set $[\nu_1 = n_1, \cdots \nu_k = n_k]$. Then $Z_1, Z_2, \cdots$ are well defined with P_E measure one. We shall show that

$$Z_{k+1} = SZ_k, \qquad k = 1, 2, \cdots \tag{12}$$

with P_E measure one. First,

$$SZ_k(\omega) = Z_k(S\omega) = f_{n_{k+1}}[X_{n_2+\cdots+n_k+1}(S\omega), \cdots, X_{n_2+\cdots+n_{k+1}}(S\omega)]$$

if $S\omega \,\varepsilon\, [\nu_1 = n_2, \nu_2 = n_3, \cdots, \nu_k = n_{k+1}]$ or, equivalently, if

$$\omega \,\varepsilon\, [\nu_2 = n_2, \nu_3 = n_3, \cdots \nu_{k+1} = n_{k+1}].$$

However, $S\omega = T^{n_1}\omega$ if $\omega \,\varepsilon\, [\nu_1 = n_1]$. Hence, for

$$\omega \,\varepsilon\, [\nu_1 = n_1, \nu_2 = n_2, \cdots, \nu_{k+1} = n_{k+1}],$$

$$\begin{aligned} SZ_k(\omega) &= f_{n_{k+1}}(X_{n_2+\cdots+n_k+1}(T^{n_1}\omega), \cdots, X_{n_2+\cdots+n_{k+1}}(T^{n_1}\omega)) \\ &= f_{n_{k+1}}(X_{n_1+n_2+\cdots+n_k+1}(\omega), \cdots, X_{n_1+n_2+\cdots+n_{k+1}}(\omega)). \\ &= Z_{k+1}(\omega). \end{aligned}$$

Hence (12) is proved and $Z_1, Z_2, \cdots$ form a stationary sequence under P_E measure.

For special cases of $\{Z_k\}$, we have (a), (b), (c).

(a) Let $f_n(x_1, \cdots, x_n) \equiv n$, then $Z_1 = \nu_1, Z_2 = \nu_2, \cdots$.

(b) Let $f_n(x_1, \cdots, x_n) = x_n$, then $Z_1 = X_{\nu_1}, Z_2 = X_{\nu_1+\nu_2}, \ldots$.

(c) Let $f_n(x_1, \cdots, x_n) = x_1 + \cdots + x_n$, then $Z_1 = X_1 + \cdots + X_{\nu_1}$, $Z_2 = X_{\nu_1+1} + \cdots + X_{\nu_1+\nu_2}, \cdots$.

THEOREM 3. *Let X be a P integrable random variable and let $X_n = T^n X$, $n = 1, 2, 3, \cdots$. If T is ergodic then $X_1 + \cdots + X_{\nu_1}$ is P_E integrable and*

$$\begin{aligned} \int (X_1 + \cdots + X_{\nu_1})\, dP_E &= \left(\int X_1\, dP\right)\left(\int \nu_1\, dP_E\right) \\ &= [1/P(E)]\left(\int X_1\, dP\right). \end{aligned} \tag{13}$$

PROOF. It has been proved in [3] that, if T is ergodic, then $\nu_1, \nu_2, \cdots$ are well defined with P measure one, and $\lim_{k\to\infty} (\nu_1 + \cdots + \nu_k)k^{-1} = [1/P(E)]$ with P measure one, and also $\int \nu_k \, dP_E = [1/P(E)]$. Let Ω' be the set of all ω for which we have simultaneously

$$\lim_{k\to\infty} [\nu_1(\omega) + \cdots + \nu_k(\omega)]k^{-1} = [1/P(E)]$$

and

$$\lim_{n\to\infty} [X_1(\omega) + \cdots + X_n(\omega)]n^{-1} = \int X_1 \, dP.$$

Then $P(\Omega') = 1$. Hence the following equalities are true on Ω'.

$$\begin{aligned} &\lim_{k\to\infty} [(X_1 + \cdots + X_{\nu_1}) + \cdots + (X_{\nu_1+\cdots+\nu_{k-1}+1} + \cdots + X_{\nu_1+\cdots+\nu_k})]k^{-1} \\ &\qquad = \lim_{k\to\infty} \frac{X_1 + \cdots + X_{\nu_1+\cdots+\nu_k}}{\nu_1 + \cdots + \nu_k} \cdot \frac{\nu_1 + \cdots + \nu_k}{k} \\ &\qquad = \left[\int X_1 \, dP\right][1/P(E)]. \end{aligned} \tag{14}$$

if X is non-negative with P_E measure one, then $X_{\nu_1+\cdots+\nu_{k-1}+1} + \cdots + X_{\nu_1+\cdots+\nu_k}$, $k = 1, 2, \cdots$, are non-negative with P_E measure one. The conclusion of Theorem 3 follows easily from the fact that $X_1 + \cdots + X_{\nu_1}$, $X_{\nu_1+1} + \cdots + X_{\nu_1+\nu_2}$, $\cdots$, form a stationary sequence under P_E measure and the following statement. If non-negative functions, $g_1, g_2, \cdots$ form a stationary sequence and $\lim_{k\to\infty} (g_1 + \cdots + g_k)k^{-1} = g$ with probability one with g integrable, then g_1 is integrable and the integral of g_1 is equal to the integral of g. This statement can be easily proved by the ergodic theorem. If X is not non-negative apply Theorem 3 to $|X|$. Thus we have that $|X_1| + \cdots + |X_{\nu_1}|$ is P_E integrable and, therefore, $X_1 + \cdots + X_{\nu_1}$ is also P_E integrable. The ergodic theorem again implies (13).

THEOREM 4. *Let random variable X be P_E integrable and let $X_n = T^nX$, $n = 1, 2, \cdots$. If T is ergodic then X_{ν_1} is P_E integrable and*

$$\int X_{\nu_1} \, dP_E = \int X \, dP_E. \tag{15}$$

PROOF. For any subset A of Ω, let I_A be the real valued function define on Ω by

$$\begin{aligned} I_A(\omega) &= 1 \quad \text{if} \quad \omega \in A \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then $T^nI_E = I_{T^{-n}E}$. Let $X' = XI_E$ and $X_n' = T^nX'$, $n = 1, 2, \cdots$. Then $\int X' \, dP = \int_E X \, dP = P(E) \int X \, dP_E$, so that X' is P integrable. Applying Theorem 3 to the sequence $X_1', X_2' \cdots$, we have

$$\int (X_1' + \cdots + X_{\nu_1}') \, dP_E = [1/P(E)]\left[\int X' \, dP\right] = \int X \, dP_E.$$

However

$$X_1' + \cdots + X_{\nu_1}' = X_1 I_{T^{-1}E} + \cdots + X_{\nu_1} I_{T^{-\nu_1}E} = X_{\nu_1}.$$

Hence X_{ν_1} is P_E integrable and $\int X_{\nu_1} \, dP_E = \int X \, dP_E$.

COROLLARY 1. *If T is ergodic, so is S.*

PROOF. Let X, X_n, X_n' be as in Theorem 4. Applying (14) to $\{X_n'\}$, we have

$$\lim_{k\to\infty} [(X_1' + \cdots + X_{\nu_1}') + \cdots + (X'_{\nu_1+\cdots+\nu_{k-1}+1} + \cdots + X'_{\nu_1+\cdots+\nu_k})]k^{-1}$$

$$= \left[\int X_1' \, dP\right] [1/P(E)] = \int X \, dP_E$$

with P_E measure one. However, by (10),

$$X'_{\nu_1+\cdots+\nu_{k-1}+1} + \cdots + X'_{\nu_1+\cdots+\nu_k}) = X_{\nu_1+\cdots+\nu_k} = S^k X.$$

Hence

$$\lim_{k\to\infty} (SX + \cdots + S^k X)k^{-1} = \int X \, dP_E \tag{16}$$

with P_E measure one. Since (16) is true for any P_E integrable random variable X, the conclusion that S is ergodic is thus proved.

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