# A CHARACTERIZATION OF THE WEAK CONVERGENCE OF MEASURES

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**0. Summary.** In this paper we shall investigate the so-called weak convergence of measures. Although the origin of the concept of the weak convergence of measures is a probabilistic one, the concept itself is purely measure-theoretical, and should be, therefore, treated by measure-theoretical methods. In Probability Theory the notion of the weak convergence of measures first appeared in Central Limit Problem. Its full importance, however, has been recognized only recently. It is now known as Donsker's Invariance Principle.

In this paper we shall follow Prohorov's approach, as presented in [1]. The list of all necessary definitions and results is given in the Introduction.

We shall give some conditions for the weak convergence of measures in separable and complete metric spaces, which are expressed in terms of convergence of measures generated in finite dimensional Euclidean spaces. The last convergence can be treated by standard mathematical tools, like the Theory of Fourier Transformations. It should be noted that our theorems concerning the convergence of measures in separable complete metric spaces remain valid if we omit the assumption of completeness. The proofs will remain essentially unchanged; only instead of dealing with compact sets, we should deal with totally bounded closed sets.

The theorems given in Section 4 are of interest for the Theory of Stochastic Processes, since they give the conditions for the weak convergence of measures in the functional spaces $D[0, 1]$ and $C[0, 1]$, and to a large class of stochastic processes there correspond measures generated in space $D[0, 1]$ or $C[0, 1]$, and these measures are usually given in terms of $\mu^{t_1, \cdots, t_m}$, i.e. in terms of finite dimensional distribution functions of the process.

**1. Introduction.** Let $R$ be a complete separable metric space with the metric $\rho$. Denote by $M(R)$ the space of all finite measures defined on the Borel $\sigma$-fielp of subsets of $R$. A sequence $\mu_n$ of elements of $M(R)$ will be called weakly convergent to $\mu \in M(R)$ if for every bounded and continuous function $f(x)$ on $R$

$$\lim_{n\to\infty} \int_R f(x)\mu_n\,(dx) = \int_R f(x)\mu\,(dx). \tag{1.1}$$

We shall denote weak convergence by $\Rightarrow$. The following Theorems A–F can be found in [1]:

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THEOREM A. *Let* $\mu_n \varepsilon M(R)$, $n = 0, 1, 2, \cdots$. *Then* $\mu_n \Rightarrow \mu_0$ *if, and only if,* $\lim_{n\to\infty} \mu_n(R) = \mu_0(R)$ *and* $\limsup_{n\to\infty} \mu_n(F) \leqq \mu_0(F)$ *for every closed set* $F \subset R$.

Let $\mu_1, \mu_2 \varepsilon M(R)$. Denote by $\epsilon_{1,2}$ (resp. $\epsilon_{2,1}$) the greatest lower bound of those $\epsilon$, that for every closed set $F \subset R$ we have $\mu_1(F) \leqq \mu_2(F^\epsilon) + \epsilon$ (resp. $\mu_2(F) \leqq \mu_1(F^\epsilon) + \epsilon$) where $F^\epsilon$ denotes the $\epsilon$-neighborhood of the closed set $F$. Let

$$L(\mu_1, \mu_2) = \max(\epsilon_{1,2}, \epsilon_{2,1}). \tag{1.2}$$

The following theorem holds:

THEOREM B. *The function L, defined by* (1.2), *is a metric in the space* $M(R)$, *and the conditions* $\mu_n \Rightarrow \mu_0$ *and* $L(\mu_n, \mu_0) \to 0$ *are equivalent. Moreover,* $M(R)$ *with the metric L is a complete separable space.*

A condition for compactness of subsets of $M(R)$ is given by the following theorem:

THEOREM C. *The set* $B \subset M(R)$ *is compact if and only if* $\sup_{\mu\epsilon B} \mu(R) < \infty$ *and for every* $\epsilon > 0$ *there exists a compact set* $K_\epsilon \subset R$ *such that* $\sup_{\mu\epsilon B} \mu(K_\epsilon^c) < \epsilon$.[3]

Let $R^*$ be a complete separable metric space and let $\mu \varepsilon M(R)$. If $f$ is a continuous function mapping $R$ into $R^*$, then, the condition $\mu^f(A) = \mu\{f^{-1}(A)\}$ for the $\mu$-measurable $f^{-1}(A)$ defines the measure $\mu^f \varepsilon M(R^*)$. The following theorem holds:

THEOREM D. *The condition* $\mu_n \Rightarrow \mu_0$ *holds if and only if for every real* $\mu$-*almost everywhere continuous function f on R we have* $\mu_n^f \Rightarrow \mu_0^f$.

REMARK 1. In the definition of metric $L$, it is sufficient to take the greatest lower bound with respect to compact sets only. In fact, let the inequality $\mu_1(K) \leqq \mu_2(K^\epsilon) + \epsilon$ hold for all compact sets $K \subset R$ and let $F \subset R$ be an arbitrary closed set. Take a sequence $\{K_n\}$ of compact sets, such that $K_n \subset K_{n+1}$, $n = 1, 2, \cdots$, and $\mu_1[(\bigcup_{n=1}^\infty K_n)^c] = 0$ (see, for example, [2]). Then, for every $n$ we can write

$$\mu_1([F \cap K_n]) \leqq \mu_2([F \cap K_n]^\epsilon) + \epsilon \leqq \mu_2(F^\epsilon) + \epsilon,$$

and on the left hand side we can pass to the limit with $n \to \infty$, obtaining

$$\mu_1(F) = \mu_1(F \cap \bigcup_{n=1}^\infty K_n) \leqq \mu_2(F^\epsilon) + \epsilon.$$

REMARK 2. An analogous distance of measures has been defined by Lévy when the space $R$ is one-dimensional Euclidean space. He defined the distance between measures $\mu_1$ and $\mu_2$ as

$$\begin{aligned} L^*&(\mu_1, \mu_2) \\ &= \inf\{\epsilon; \text{for every } x: F_1(x - \epsilon) - \epsilon \leqq F_2(x) \leqq F_1(x + \epsilon) + \epsilon\} \end{aligned} \tag{1.3}$$

where $F_1(x)$ and $F_2(x)$ are the distribution functions of the measures $\mu_1$ and $\mu_2$, respectively.

[3] In this paper $A^c$ will denote the complement of $A$.

Generally, if $F_1(x_1, \cdots, x_m)$ and $F_2(x_1, \cdots, x_m)$ are the distribution functions of measures $\mu_1$ and $\mu_2$ in $m$-dimensional Euclidean space $R_m$, then we can define the Lévy distance $L^*(F_1, F_2)$ as

$$(1.4)\quad \begin{aligned} \inf \{\epsilon; \text{for every } x_1, \cdots, x_m : F_1(x_1 - \epsilon, \cdots, x_m - \epsilon) - \epsilon \\ \leqq F_2(x_1, \cdots, x_m) \leqq F_1(x_1 + \epsilon, \cdots, x_m + \epsilon) + \epsilon\}. \end{aligned}$$

The convergence $L^*(F_n, F_0) \to 0$ is equivalent to the condition that

$$\lim_{n\to\infty} F_n(x_1, \cdots, x_m) = F_0(x_1, \cdots, x_m)$$

at every continuity point of the function $F_0(x_1, \cdots, x_m)$, and therefore, it is also equivalent to the weak convergence $\mu_n \Rightarrow \mu_0$ of the corresponding measures (see, for example, [3]).

In this paper, whenever the $m$-dimensional Euclidean space $R_m$ is considered it will be assumed that the metric in this space is defined as

$$\rho(\{x_1, \cdots, x_m\}, \{y_1, \cdots, y_m\}) = \max_{1\leqq k\leqq m} |x_k - y_k|.$$

By $C[0, 1]$ we shall denote the space of all real continuous functions $f(t)$ on $[0, 1]$ with the uniform metric

$$c(g, h) = \sup_{0\leqq t\leqq 1} |g(t) - h(t)|.$$

Denote by $D[0, 1]$ the space of all real functions $f(t)$ on $[0, 1]$ satisfying the following conditions:

(a) the limits $f(t + 0)$ and $f(t - 0)$ exist at every point $t \in (0, 1)$ and the limits $f(0+)$ and $f(1 - 0)$ exist at the points $t = 0$ and $t = 1$, respectively.

(b) at every point $t \in [0, 1]$ one of the two equalities $f(t) = f(t + 0)$ and $f(t) = f(t - 0)$ holds.

We shall add to the definition of the space $D[0, 1]$ the usual convention that every two functions $f_1(t)$ and $f_2(t)$ for which the equalities $f_1(t + 0) = f_2(t + 0)$ and $f_1(t - 0) = f_2(t - 0)$ are satisfied for all $t \in [0, 1]$ will be considered as one element of $D[0, 1]$.

Let $f \in D[0, 1]$ and let $\Gamma_f$ be the graph of function $f$, that is, the set of points $(t, u)$ such that $t \in [0, 1]$ and $u$ satisfies one of the following inequalities:

$$f(t - 0) \leqq u \leqq f(t + 0) \quad \text{and} \quad f(t + 0) \leqq u \leqq f(t - 0).$$

Note that every graph is a bounded closed set on the plane. Let

$$w_f(\Delta) = \sup_{t_1, t_2 \in \Delta} |f(t_1) - f(t_2)|.$$

We shall consider two functions

$$(1.5)\qquad w_f(a) = \sup_{\Delta; |\Delta| \leqq a} w_f(\Delta)$$

and

$$(1.6) \qquad \tilde{w}_f(a) = \sup_{\Delta; |\Delta| \leqq a} \sup_{\tau_0 \varepsilon \Delta} \min \left( w_f\{[\tau_1, \tau_0)\}, w_f\{(\tau_0, \tau_2]\} \right),$$

where $\Delta$ denotes the interval $[\tau_1, \tau_2]$. The following propositions have been proved by Prohorov [1]:

I. *If the function f has no jump greater than c, then*

$$(1.7) \qquad w_f(a) \leqq 4(\tilde{w}_f(a) + c) \quad \textit{for all} \quad 0 \leqq a \leqq 1.$$

II. *$\tilde{w}_f(a)$ is a non-decreasing function, and $\tilde{w}_f(a) \downarrow 0$ as $a \downarrow 0$.*

Let

$$(1.8) \qquad r_f(z) = \begin{cases} \tilde{w}_f(e^z) & \text{for} \quad z \leqq 0 \\ \tilde{w}_f(1) & \text{for} \quad z > 0. \end{cases}$$

Let $f, g \varepsilon D[0, 1]$. Define

$$(1.9) \qquad d_1(f, g) = \max \{\sup_{p \varepsilon \Gamma_f} \inf_{q \varepsilon \Gamma_g} |p - q|, \sup_{p \varepsilon \Gamma_g} \inf_{q \varepsilon \Gamma_f} |p - g|\}$$

and

$$(1.10) \qquad d_2(f, g) = L^*(r_f(z), r_g(z)),$$

where $L^*$ is the Lévy distance defined by (1, 3). Then the following theorem holds:

THEOREM E. *The function*

$$(1.11) \qquad d(f, g) = d_1(f, g) + d_2(f, g)$$

*defines a metric in the space $D[0, 1]$. The space $D[0, 1]$ with the metric d is separable and complete, the subspace $C[0, 1]$ is a closed set in $D[0, 1]$ and for the subspace $C[0, 1]$ the d-convergence is equivalent to the uniform one. Moreover, if $d(f_n, f) \rightarrow 0$, then $f_n(t) \rightarrow f(t)$ at every point of continuity of $f(t)$.*

Conditions for compactness of the subsets of the space $D[0, 1]$ are given by the theorem:

THEOREM F. *The set $B \subset D[0, 1]$ is compact if and only if, there exists a constant $M > 0$ and a function $h(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$, such that for all $f \varepsilon B$*

$$\sup_{0 \leqq t \leqq 1} |f(t)| < M$$

$$\tilde{w}_f(\epsilon) \leqq h(\epsilon) \quad \textit{for} \quad 0 \leqq \epsilon \leqq 1.$$

Now we shall prove the following inequality

$$(1.12) \qquad d(f, g) \leqq 3 \sup_{0 \leqq t \leqq 1} |f(t) - g(t)|.$$

In fact, if $\sup_{0 \leqq t \leqq 1} |f(t) - g(t)| \leqq \epsilon$, then, of course, $d_1(f, g) \leqq \epsilon$. Since the functions $\tilde{w}_f(a)$ and $\tilde{w}_g(a)$ satisfy the inequality $|\tilde{w}_f(a) - \tilde{w}_g(a)| \leqq 2\epsilon$ for every $a$, then also $|r_f(z) - r_g(z)| \leqq 2\epsilon$ for every $z$, hence $d_2(f, g) \leqq 2\epsilon$, which was to be proved.

Let $t_1, \cdots, t_m$ be fixed points from $[0, 1]$. Denote by $\varphi = \varphi_{t_1, \cdots, t_m}$ the function mapping $D[0, 1]$ (or $C[0, 1]$) into $R_m$, defined as $\varphi(f) = \{f(t_1 - 0), \cdots, f(t_m - 0)\}$. For $\mu \varepsilon M(D[0, 1])$ (or $\mu \varepsilon M(C[0, 1])$), we shall denote by $\mu^{t_1, \cdots, t_m}$ the measure in $M(R_m)$ defined as $\mu^{t_1, \cdots, t_m}(A) = \mu\{\varphi^{-1}(A)\}$ for $\mu$-measurable $\varphi^{-1}(A)$.

**2. Convergence of measures in metric spaces.** In the present section, $R$ will denote an arbitrary fixed complete separable metric space with the metric $\rho$. Let $f_1(x), f_2(x), \cdots$ be a fixed sequence of continuous mappings of $R$ into the real line $R_1$. Suppose that for every $x$ we have $\sup_n |f_n(x)| < \infty$.

Denote by $\varphi_k$ the mapping of $R$ into $k$-dimensional Euclidean space $R_k$ defined as $\varphi_k(x) = \{f_1(x), \cdots, f_k(x)\}$. If $\mu \varepsilon M(R)$, we shall write $\mu^k = \mu^{\varphi_k}$. Further we shall use the notation $\rho^*(x, y) = \sup_n |f_n(x) - f_n(y)|$.

The following theorems hold:

THEOREM 1. *If the functions $f_1(x), f_2(x), \cdots$ are equicontinuous at each point $x \varepsilon R$, that is, if the condition $\rho(x_m, x) \to 0$ implies $\rho^*(x_m, x) \to 0$ then a necessary condition for the convergence $\mu_n \Rightarrow \mu_0 (\mu_n \varepsilon M(R), n = 0, 1, \cdots)$ is*

$$\lim_{n \to \infty} \sup_k L(\mu_n^k, \mu_0^k) = 0. \tag{2.1}$$

THEOREM 2. *If the condition $\rho^*(x_m, x) \to 0$ implies $\rho(x_m, x) \to 0$, then* (2.1) *is a sufficient condition for the convergence $\mu_n \Rightarrow \mu_0 (\mu_n \varepsilon M(R), n = 0, 1, \cdots)$.*

To prove these theorems we need some lemmas giving the connections between $\epsilon$-neighborhoods in the spaces $R$ and $R_k$.

LEMMA 1. *If the conditions of Theorem 1 are satisfied, then for an arbitrary compact set $D \subset R$, any integer $k$ and any set $F \subset R_k$*

$$D \cap [\varphi_k^{-1}(F)]^\epsilon \subset \varphi_k^{-1}(F^{\pi_D(\epsilon)}),$$

*where $\pi_D(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$.*

PROOF. At first suppose that $F = \{x_1, \cdots, x_k\}$ and let $\xi \varepsilon D \cap [\varphi_k^{-1}(F)]^\epsilon$. It follows that there exists $\eta \varepsilon \varphi_k^{-1}(F)$ such that $\rho(\xi, \eta) < \epsilon$. Since $D$ is supposed to be compact it follows that $\rho^*(\xi, \eta) < \pi_D(\epsilon)$, where $\pi_D(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. Then also

$$\max_{1 \leq i \leq k} |f_i(\xi) - f_i(\eta)| < \pi_D(\epsilon)$$

and since $f_i(\eta) = x_i$, $(i = 1, \cdots, k)$, we have

$$\xi \varepsilon \varphi_k^{-1}(\{x_1, \cdots, x_k\}^{\pi_D(\epsilon)}).$$

To complete the proof it is sufficient to note that pre-images and $\epsilon$-neighborhoods are additive and $\pi_D(\epsilon)$ does not depend on $\{x_1, \cdots, x_k\}$.

LEMMA 2. *If $F \subset R$ is compact, then for arbitrary $\epsilon > 0$ and $\delta > 0$*

$$\bigcap_{k=1}^{\infty} \varphi_k^{-1}[\varphi_k(F)^\epsilon] \subset F_*^{\epsilon + \delta},$$

*where $F_*^a$ denotes the a-neigborhood of the set $F$ in the metric $\rho^*$.*

PROOF. Let $\epsilon > 0$ and $\delta > 0$ be arbitrary, and let $\xi \in \bigcap_{k=1}^{\infty} \varphi_k^{-1}[\varphi_k(F)^{\epsilon}]$. It follows, that for every $k$ we have $\varphi_k(\xi) \in \varphi_k(F)^{\epsilon}$. Then, there exists $\eta_k \in F$ such that $\max_{1 \leqq i \leqq k} |f_i(\xi) - f_i(\eta_k)| < \epsilon$. Since, by assumption, $F$ is compact, we can select a convergent subsequence from the sequence $\{\eta_k\}$. Suppose, without loss of generality, that $\eta_k \to \eta \in F$. Let $n$ be an arbitrary integer. Then

$$|f_n(\xi) - f_n(\eta)| \leqq |f_n(\xi) - f_n(\eta_k)| + | f_n(\eta_k) - f_n(\eta)|.$$

For a sufficiently large $k$, the first term on the right hand side of the last formula is less than $\epsilon$ and the second is less than $\delta$. Hence it follows that

$$\rho^*(\xi, \eta) = \sup_n |f_n(\xi) - f_n(\eta)| < \epsilon + \delta,$$

which was to be proved.

PROOF OF THEOREM 1. Suppose that $\mu_n \Rightarrow \mu_0$. Then $L(\mu_n, \mu_0) \to 0$ and for any $\delta > 0$ there exists a compact set $D_\delta$ such that $\sup_n \mu_n(D_\delta^c) < \delta$ (see [1]). Let $L(\mu_n, \mu_0) \leqq \alpha$; then it follows that for every closed set $A \subset R$ we have $\mu_n(A) \leqq \mu_0(A^{\alpha}) + \alpha$ and $\mu_0(A) \leqq \mu_n(A^{\alpha}) + \alpha$. Let $k$ be an arbitrary integer and $F \subset R_k$ be an arbitrary closed set. Then, by Lemma 1 and the fact that the set $\varphi_k^{-1}(F)$ is closed, we obtain

$$\begin{aligned} \mu_0^k(F) = \mu_0\{\varphi_k^{-1}(F)\} &\leqq \mu_n\{\varphi_k^{-1}(F)^{\alpha}\} + \alpha \\ &\leqq \mu_n\{D_\delta \cap [\varphi_k^{-1}(F)]^{\alpha}\} + \alpha + \delta \\ &\leqq \mu_n\{\varphi_k^{-1}(F^{\pi_\delta(\alpha)})\} + \alpha + \delta = \mu_n^k(F^{\pi_\delta(\alpha)}) + \alpha + \delta. \end{aligned}$$

Similarly we obtain $\mu_n^k(F) \leqq \mu_0^k(F^{\pi_\delta(\alpha)}) + \alpha + \delta$, which implies that

$$L(\mu_n^k, \mu_0^k) \leqq \max (\alpha + \delta, \pi_\delta(\alpha)),$$

and also

$$\sup_k L(\mu_n^k, \mu_0^k) \leqq \max (\alpha + \delta, \pi_\delta(\alpha)).$$

Let $\epsilon > 0$ be arbitrary. Choose a fixed $\delta < \epsilon/2$ and then find $\alpha$ such that $\alpha < \epsilon/2$ and $\pi_\delta(\alpha) < \epsilon$. Then choose $N = N_\alpha$, such that for $n > N$ we have $L(\mu_n, \mu_0) \leqq \alpha$. It follows that for $n > N$

$$\sup_k L(\mu_n^k, \mu_0^k) < \epsilon$$

which was to be proved.

PROOF OF THEOREM 2. Suppose that the condition (2.1) is satisfied. Let $\epsilon > 0$ be arbitrary and let for $n > N_\epsilon$

$$\sup_k L(\mu_n^k, \mu_0^k) < \epsilon. \tag{2.2}$$

By Lemma 2, for every $n$ and every compact set $F$ we can find $k = k_{n,F}$ such that

$$\mu_n\{\varphi_k^{-1}[\varphi_k(F)^{\epsilon}]\} \leqq \mu_n(F_*^{2\epsilon}) + \epsilon. \tag{2.3}$$

Then, for every compact set $F$ we can write the following chain of inequalities, using the conditions (2.2) and (2.3):

$$\mu_n(F) \leqq \mu_n\{\varphi_k^{-1}[\varphi_k(F)]\} = \mu_n^k\{\varphi_k(F)\}$$
$$\leqq \mu_0^k\{\varphi_k(F)^\epsilon\} + \epsilon = \mu_0\{\varphi_k^{-1}[\varphi_k(F)^\epsilon]\} + \epsilon \leqq \mu_0(F_*^{2\epsilon}) + 2\epsilon.$$

In a similar way we prove the inequality

$$\mu_0(F) \leqq \mu_n(F_*^{2\epsilon}) + 2\epsilon.$$

Hence, by Remark 1 (made at the beginning of this paper) we get

$$L_*(\mu_n, \mu_0) \leqq 2\epsilon \quad \text{for} \quad n > N_\epsilon,$$

where the distance $L_*$ is calculated according to the metric $\rho^*$. Thus $\mu_n \Rightarrow \mu_0$ in the topology generated by the metric $\rho^*$, and $\lim\sup_{n\to\infty} \mu_n(F) \leqq \mu_0(F)$ for all sets $F$ closed in the metric $\rho^*$, hence, for all sets $F$ closed in the original metric $\rho$, which was to be proved.

**3. Some lemmas.** Let $\mu \in M(D[0, 1])$. We shall say that $t_0$ is a continuity point of the measure $\mu$ if the set of those $f \in D[0, 1]$ which are discontinuous at the point $t_0$ is of $\mu$-measure zero.

The first lemma we are going to prove gives some regularity properties of the behavior of the functions $f$ in the neighborhood of the continuity points.

LEMMA 3. *If $t_1, \cdots, t_m$ are arbitrary continuity points of the measure $\mu$, then for every $\epsilon > 0$*

$$\liminf_{c\to 0}\left\{\inf_{x_1,\cdots,x_m}\left[\mu\left(\bigcap_{k=1}^m\left\{f; \sup_{t\in T_k^c} f(t) < x_k + \epsilon\right\}\right) - \mu\left(\bigcap_{k=1}^m\left\{f; f(t_k) < x_k\right\}\right)\right]\right\} \geqq 0, \tag{3.1}$$

*where $T_k^c$ denotes the interval $[t_k - c, t_k + c]$. If for some $\epsilon_0 > 0$, and for some particular points $t_1, \cdots, t_m$ (without the assumption that they are continuity points) the relation* (3.1) *is not satisfied, then there exists $\alpha_0 > 0$ and $t_k$ (among $t_1, \cdots, t_m$) such that*

$$\mu\{f; |f(t_k + 0) - f(t_k - 0)| > \alpha_0\} > \alpha_0. \tag{3.2}$$

PROOF. Suppose that the points $t_1, \cdots, t_m$ are continuity points of the measure $\mu$. Then for arbitrary $c > 0$, $\epsilon > 0$ and arbitrary $x_1, \cdots, x_m$ we have

$$\mu\left(\bigcap_{k=1}^m \{f; f(t_k) < x_k\}\right)$$
$$= \mu\left(\bigcap_{k=1}^m \{f; f(t_k) < x_k\} \cap \bigcap_{k=1}^m \{f; \sup_{t\in T_k^c} |f(t) - f(t_k)| < \epsilon\}\right)$$
$$+ \mu\left(\bigcap_{k=1}^m \{f; f(t_k) < x_k\} \cap \left[\bigcap_{k=1}^m \{f; \sup_{t\in T_k^c} |f(t) - f(t_k)| \geqq \epsilon\}\right]^c\right)$$
$$\leqq \mu\left(\bigcap_{k=1}^m \{f; \sup_{t\in T_k^c} f(t) < x_k + \epsilon\}\right)$$
$$+ \sum_{k=1}^m \mu\{f; \sup_{t\in T_k^c} |f(t) - f(t_k)| \geqq \epsilon\},$$

or

$$\mu\left(\bigcap_{k=1}^{m}\{f;\sup_{t\in T_k^c} f(t) < x_k+\epsilon\}\right) - \mu\left(\bigcap_{k=1}^{m}\{f; f(t_k) < x_k\}\right)$$
$$+\sum_{k=1}^{m}\mu\{f;\sup_{t\in T_k^c}|f(t)-f(t_k)|\geqq\epsilon\}\geqq 0.$$

Let $c_n$ be an arbitrary sequence of numbers decreasing to 0. Denote by $A_{n,k}$ the set of those $f$'s, such that

$$\sup_{t\in T_k^{c_n}}|f(t)-f(t_k)|\geqq\epsilon.$$

Then $A_{n,k}\supset A_{n,k+1}$ and

$$\inf_{x_1,\cdots,x_m}\left[\mu\left(\bigcap_{k=1}^{m}\{f;\sup_{t\in T_k^{c_n}} f(t) < x_k+\epsilon\}\right) - \mu\left(\bigcap_{k=1}^{m}\{f; f(t_k) < x_k\}\right)\right]$$
$$+\sum_{k=1}^{m}\mu(A_{n,k})\geqq 0.$$

On the other hand, for every $k$ we have

$$A_k=\bigcap_{n=1}^{\infty}A_{n,k}\subset\{f;|f(t_k+0)-f(t_k-0)|\geqq\epsilon\},$$

and $0=\mu(A_k)=\lim_{n\to\infty}\mu(A_{n,k})$, which proves the inequality (3.1). The second part of Lemma 3 follows immediately from the first part.

The succeeding lemmas which we shall prove will give us some connection between neighborhoods in the spaces $D[0, 1]$ and $R_m$.

Let $t_1, \cdots, t_m$ be fixed points of the interval $[0, 1]$ and let $\varphi$ denote the function mapping $D[0, 1]$ into $R_m$, defined as $\varphi(f)=\{f(t_1-0),\cdots,f(t_m-0)\}$. Let $B$ be an arbitrary compact set in $D[0, 1]$ and let $C=C[0, 1]$ denote, as usual, the space of all real continuous functions on $[0, 1]$.

LEMMA 4. *For an arbitrary set* $K\subset R_m$

$$[\varphi^{-1}(K)\cap B]^{\epsilon}\cap C\subset\varphi^{-1}(K^{\psi_B(\epsilon)}) \tag{3.3}$$

*and*

$$[\varphi^{-1}(K)\cap B\cap C]^{\epsilon}\subset\varphi^{-1}(K^{\theta_B(\epsilon)}), \tag{3.4}$$

*where* $\psi_B(\epsilon)\downarrow 0$ *and* $\theta_B(\epsilon)\downarrow 0$ *as* $\epsilon\downarrow 0$.

PROOF. Note first, that since pre-images and $\epsilon$-neighborhoods are additive, it is sufficient to prove inclusions (3.3) and (3.4) for sets $K=\{x_1,\cdots,x_m\}$.

To prove (3.3) suppose that $f\in[\varphi^{-1}(\{x_1,\cdots,x_m\})\cap B]^{\epsilon}\cap C$. It means that there exists a function $g\in D[0, 1]$, such that $g\in\varphi^{-1}(\{x_1,\cdots,x_m\})\cap B$ and $d(f, g)<\epsilon$. In other words

(1) $g(t_k)=x_k$, $k=1, 2, \cdots, m$,

(2) $g \varepsilon B$,
(3) $d(f, g) < \epsilon$,
(4) $f$ is continuous.

Condition (2) means that $\tilde{w}_g(a) \leqq h_B(a)$ for $0 \leqq a \leqq 1$, where function $h_B(a)$ is a function appearing in the condition for compactness of the subsets of $D[0, 1]$ given by Theorem E of the Introduction. From condition (3) it follows that $d_1(f, g) < \epsilon$ and $d_2(f, g) < \epsilon$, where $d_1$ and $d_2$ are defined by (1.9) and (1.10). Then, there exist points $t_1', \cdots, t_m'$, with

$$\max_{1 \leq k \leq m} |t_1' - t_k|$$

and

$$\max_{1 \leq k \leq m} |f(t_k') - g(t_k)| < \epsilon.$$

Using (1) we obtain

$$\max_{1 \leq k \leq m} |f(t_k') - x_k| < \epsilon. \tag{3.5}$$

Condition $d_2(f, g) < \epsilon$ implies that for every $z$

$$\tilde{w}_g(e^{z-\epsilon}) - \epsilon \leqq \tilde{w}_f(e^z) \leqq \tilde{w}_g(e^{z+\epsilon}) + \epsilon,$$

and hence by (2)

$$\tilde{w}_f(e^z) \leqq \epsilon + h_B(e^{z+\epsilon}).$$

Putting $e^z = \epsilon$ we obtain for sufficiently small $\epsilon$

$$\tilde{w}_f(\epsilon) \leqq \epsilon + h_B(e^\epsilon \cdot \epsilon) \leqq \epsilon + h_B(2\epsilon).$$

Since $f$ is supposed to be continuous, we may use (1.7) obtaining

$$w_f(\epsilon) \leqq 4(\epsilon + h_B(2\epsilon)),$$

which means that

$$\sup_{\tau_1, \tau_2; |\tau_1 - \tau_2| < \epsilon} |f(\tau_1) - f(\tau_2)| \leqq 4(\epsilon + h_B(2\epsilon)). \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\max_{1 \leq k \leq m} |f(t_k) - x_k| \leqq \epsilon + 4(\epsilon + h_B(2\epsilon)) = \psi_B(\epsilon)$$

which implies that $f \varepsilon \varphi^{-1}(\{x_1, \cdots, x_m\}^{\psi_B(\epsilon)})$, and according to Theorem E, we have $h_B(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$, which proves the first part of Lemma 4. To prove inclusion (3.4), suppose that $f \varepsilon [\varphi^{-1}(\{x_1, \cdots, x_m\}) \cap B \cap C]^\epsilon$. This means that there exists function $g \varepsilon D[0, 1]$ such that

(1) $g(t_k) = x_k$, $k = 1, 2, \cdots, m$,
(2) $g \varepsilon B$,
(3) $g$ is continuous,
(4) $d(f, g) < \epsilon$.

First we shall prove that function $f$ cannot have jumps greater than $8h_B(\epsilon) + 2\epsilon$, where $h_B(\epsilon)$ denotes, as before, the function appearing in the condition for compactness of the set $B$. In fact, suppose that for some $t_0$ we have

$$|f(t_0 + 0) - f(t_0 - 0)| > 8h_B(\epsilon) + 2\epsilon.$$

It follows, that $\max\{|g(t_0) - f(t_0 + 0)|, |g(t_0) - f(t_0 - 0)|\} > 4h_B(\epsilon) + \epsilon$. Suppose, for instance, that $|f(t_0 + 0) - g(t_0)| > 4h_B(\epsilon) + \epsilon$. Since $g \varepsilon B$ and $g$ is continuous, it follows from (1.7) that

$$w_g(\epsilon) \leqq 4\, \tilde{w}_g(\epsilon) \leqq 4\, h_B(\epsilon).$$

This condition implies that

$$\sup_{t;|t-t_0|<\epsilon} |g(t) - g(t_0)| \leqq 4\, h_B(\epsilon)$$

and we obtain for some $t'$ with $|t_0 - t'| < \epsilon$:

$$|g(t') - f(t_0 + 0)| > \epsilon + 4\, h_B(\epsilon) - 4\, h_B(\epsilon) = \epsilon$$

hence $d_1(f, g) \geqq \epsilon$ in contradiction with (4).

Now, from (1) and (4) it follows that there exist points $t_1', \cdots, t_m'$ with

$$\max_{1\leqq k\leqq m} |t_k' - t_k| < \epsilon$$

and

$$\max_{1\leqq k\leqq m} |f(t_k') - x_k| < \epsilon. \tag{3.7}$$

From (4) it follows that for every $z$

$$\tilde{w}_g(e^{z-\epsilon}) - \epsilon \leqq \tilde{w}_f(e^z) \leqq \tilde{w}_g(e^{z+\epsilon}) + \epsilon,$$

which implies that

$$\tilde{w}_f(e^z) \leqq \epsilon + h_B(e^{z+\epsilon}).$$

Putting $e^z = \epsilon$ we get for sufficiently small $\epsilon$

$$\tilde{w}_f(\epsilon) \leqq \epsilon + h_B(2\epsilon)$$

and applying (1.7) with $c = 2\epsilon + 8\, h_B(\epsilon)$ we get

$$w_f(\epsilon) \leqq 4[\epsilon + h_B(2\epsilon) + (2\epsilon + 8\, h_B(\epsilon))]. \tag{3.8}$$

Combining (3.7) and (3.8) we obtain

$$\max_{1\leqq k\leqq m} |f(t_k) - x_k| \leqq \epsilon + 4[\epsilon + h_B(2\epsilon) + (2\epsilon + 8h_B(\epsilon))] = \theta_B(\epsilon)$$

which means that $f \varepsilon \varphi^{-1}(\{x_1, \cdots, x_m\}^{\theta_B(\epsilon)})$. Since $\theta_B(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$, Lemma 4 is proved.

Let $t_1, \cdots, t_m, \cdots$ be a fixed sequence of points dense in [0, 1]. Denote by $\varphi_k$ the mapping of $D[0, 1]$ into $R_k$, defined as $\varphi_k(f) = \{f(t_1 - 0), \cdots, f(t_k - 0)\}$.

We shall prove

LEMMA 5. *If* $F \subset D\,[0, 1]$ *is compact, then*

$$\bigcap_{k=1}^{\infty} \varphi_k^{-1}[\varphi_k(F)^{\epsilon}] \subset F^{4\epsilon}. \tag{3.9}$$

PROOF. Suppose that $f \,\varepsilon\, \bigcap_{k=1}^{\infty} \varphi_k^{-1}[\varphi_k(F)^{\epsilon}]$. It follows that for every $k$ there exists $g_k \,\varepsilon\, F$, such that

$$\max_{1 \leqq j \leqq k} |f(t_j) - g_k(t_j)| < \epsilon. \tag{3.10}$$

Since $F$ is supposed to be compact, we can select a convergent subsequence from the sequence $\{g_k\}$. Without the loss of generality, we may assume that $g_k \to g \,\varepsilon\, F$. Let $\tau$ be a continuity point of functions $f$ and $g$. For an arbitrary $\delta > 0$ let us choose a point $t_m$ from the sequence $\{t_k\}$ which is a continuity point of the functions $f$ and $g$ and such that

$$|f(\tau) - f(t_m)| < \delta, \qquad |g(\tau) - g(t_m)| < \delta.$$

For all sufficiently large $k$ we have

$$|g(t_m) - g_k(t_m)| < \delta,$$

hence by (3.10)

$$\begin{aligned} |f(\tau) - g(\tau)| \leqq |f(\tau) - f(t_m)| &+ |f(t_m) - g_k(t_m)| \\ &+ |g_k(t_m) - g(t_m)| + |g(t_m) - g(\tau)| < \epsilon + 3\delta. \end{aligned}$$

Since $\delta$ is arbitrary we get

$$\sup_{\tau \varepsilon A} |f(\tau) - g(\tau)| \leqq \epsilon,$$

where $A$ denotes the set of points in $[0, 1]$ at which both functions $f$ and $g$ are continuous. Hence

$$\sup_{0 \leqq t \leqq 1} |f(t - 0) - g(t - 0)| \leqq \epsilon,$$

$$\sup_{0 \leqq t \leqq 1} |f(t + 0) - g(t + 0)| \leqq \epsilon,$$

and by the convention concerning the identification of elements of $D[0, 1]$ and the relation (1.12), we get $d(f, g) \leqq 3\epsilon < 4\epsilon$, which was to be proved.

**4. Criteria for convergence of measures in functional spaces.** Let

$$\mu_n \,\varepsilon\, M(D[0, 1]), \qquad n = 0, 1, 2, \cdots.$$

We shall give some conditions for the weak convergence $\mu_n \Rightarrow \mu_0$ expressed in terms of distances $L(\mu_n^{t_1, \cdots, t_m}, \mu_0^{t_1, \cdots, t_m})$ of corresponding measures generated in an $m$-dimensional space $R_m$. All these conditions consist of requirements of some kind of uniformity in $t_1, \cdots, t_m$.

We shall repeat that point $t_0$ is a continuity point of measure $\mu$ if the set of those functions $f \varepsilon D[0, 1]$ which are discontinuous at the point $t_0$ is of $\mu$-measure zero. If every point $t \varepsilon [0, 1]$ is a continuity point of measure $\mu$, then we shall say that the measure $\mu$ has no fixed points of discontinuity (note that this does not imply that $\mu\{(C[0, 1])^C\} = 0$).

THEOREM 3. *Let $\mu_n \varepsilon M(D[0, 1])$, $n = 0, 1, 2, \cdots$ and let measure $\mu_0$ have no fixed points of discontinuity. Then for the convergence $\mu_n \Rightarrow \mu_0$ it is necessary that*

$$\lim_{n\to\infty} \sup_{t_1,\cdots,t_m} L(\mu_n^{t_1,\ldots,t_m}, \mu_0^{t_1,\ldots,t_m}) = 0 \tag{4.1}$$

*for every $m$.*

PROOF. Note first that the convergence

$$\lim_{n\to\infty} L(\mu_n^{t_1,\ldots,t_m}, \mu_0^{t_1,\ldots,t_m}) = 0$$

for every fixed $t_1, \cdots, t_m \varepsilon [0, 1]$ follows from Theorem D and from the fact that the function $\varphi_{t_1,\cdots,t_m}(f) = \{f(t_1 - 0), \cdots, f(t_m - 0)\}$ is a $\mu_0$-almost everywhere continuous mapping of $D[0, 1]$ into $R_m$.

To prove Theorem 3 suppose that condition (4.1) is not satisfied. Then, there exists a number $\epsilon_0 > 0$, a sequence $n_i \to \infty$, a number $m_0$ and a sequence

$$\{t_1^i, \cdots, t_{m_0}^i\}$$

such that

$$L(\mu_{n_i}^{t_1^i,\ldots,t_{m_0}^i}, \mu_0^{t_1^i,\ldots,t_{m_0}^i}) > \epsilon_0, \qquad i = 1, 2, \cdots. \tag{4.2}$$

Notice that for fixed $m$ the set of measures $\{\mu_0^{t_1,\ldots,t_m}\}$, $t_1, \cdots, t_m \varepsilon [0, 1]$ is compact. In fact, consider an arbitrary sequence $\{\mu_0^{t_1^k,\ldots,t_m^k}\}$, $k = 1, 2, \cdots$. Let us select from the sequence $\{t_1^k, \cdots, t_m^k\}$ a subsequence $\{t_1^{k_j}, \cdots, t_m^{k_j}\}$ convergent to $(t_1^0, \cdots, t_m^0)$ Since the measure $\mu_0$ has no fixed points of discontinuity, we have

$$\mu_0^{t_1^{k_j},\ldots,t_m^{k_j}} \Rightarrow \mu_0^{t_1^0,\ldots,t_m^0} \quad \text{as} \quad j \to \infty.$$

By Theorem C, for arbitrary $\delta > 0$, there exists a compact set $B_\delta \subset R_m$ such that

$$\sup_{t_1,\cdots,t_m} \mu_0^{t_1,\ldots,t_m}(B_\delta^c) < \delta.$$

It follows that the distance $L$ in the formula (4.2) can be replaced by the Lévy distance $L^*$ defined by the formula (1.4). For simplicity of notation we may assume, without loss of generality, that $m_0 = 1$. Then formula (5.2) takes the form

$$L^*(\mu_{n_i}^{t_i}, \mu_0^{t_i}) > \epsilon_0, \qquad i = 1, 2, \cdots. \tag{4.3}$$

Furthermore, we may assume without loss of generality that $t_i \to t_0$. Since measure $\mu_0$ was supposed to be without fixed points of discontinuity, we have

$$\lim_{i\to\infty} L^*(\mu_0^{t_i}, \mu_0^{t_0}) = 0; \tag{4.4}$$

hence for all sufficiently large indices $i$ we have

$$L^*(\mu_{n_i}^{t_i}, \mu_0^{t_0}) > \epsilon_0/2. \tag{4.5}$$

From (4.5) it follows that there exists a sequence $\{x_i\}$ such that

$$\mu_{n_i}\{f; f(t_i) < x_i + \tfrac{1}{2}\epsilon_0\} + \tfrac{1}{2}\epsilon_0 < \mu_0\{f; f(t_0) < x_i\} \tag{4.6a}$$

or

$$\mu_{n_i}\{f; f(t_i) < x_i - \tfrac{1}{2}\epsilon_0\} - \tfrac{1}{2}\epsilon_0 > \mu_0\{f; f(t_0) < x_i\}. \tag{4.6b}$$

Suppose now, that the first inequality (4.6a) holds for infinitely many $i$'s. Then for any $c > 0$ and for every sufficiently large $i$ for which (4.6a) holds, we have

$$\mu_{n_i}\{f; f(t_i) < x_i + \tfrac{1}{2}\epsilon_0\} \geqq \mu_{n_i}\{f; \sup_{t\varepsilon T_0^c} f(t) < x_i + \tfrac{1}{2}\epsilon_0\} \tag{4.7}$$

where $T_0^c = [t_0 - c, t_0 + c]$.

On the other hand, according to Lemma 3, for $\epsilon = \frac{1}{4}\epsilon_0$, we have

$$\mu_0\{f; f(t_i) < x_i\} < \mu_0\{f; \sup_{t\varepsilon T_0^c} f(t) < x_i + \tfrac{1}{4}\epsilon_0\} + \tfrac{1}{4}\epsilon_0. \tag{4.8}$$

From (4.6a), (4.7) and (4.8) we obtain

$$\mu_0\{f; \sup_{t\varepsilon T_0^c} f(t) < x_i + \tfrac{1}{4}\epsilon_0\} > \mu_{n_i}\{f; \sup_{t\varepsilon T_0^c} f(t) < x_i + \tfrac{1}{4}\epsilon_0\} + \tfrac{1}{4}\epsilon_0$$

hence $L^*(\mu_{n_i}^{\varphi}, \mu_0^{\varphi}) > \frac{1}{4}\epsilon_0$, where

$$\varphi(f(t)) = \sup_{t\varepsilon T_0^c} f(t).$$

Since $\varphi$ is a $\mu_0$-almost everywhere continuous function on $D[0, 1]$, it follows that $\mu_{n_i} \not\Rightarrow \mu_0$.

Suppose now, that the inequality (4.6a) holds only for a finite number of indices $i$; hence, beginning from some $i_0$, the inequality (4.6b) holds. If the points $t_i$ are continuity points of infinitely many of the measures $\mu_{n_i}$, then the inequalities (4.7) and (4.8) are true with $\mu_{n_i}$ and $\mu_0$ interchanged, and the proof remains unchanged. If for all sufficiently large $i$ the points $t_i$ are the discontinuity points of the measures $\mu_{n_i}$, then, according to Lemma 1, there exists a sequence $\beta_i$ such that

$$\mu_{n_i}\{f; |f(t_i + 0) - f(t_i - 0)| > \beta_i\} > \beta_i. \tag{4.9}$$

Let $\beta_i'$ be the upper bound of the numbers $\beta_i$ for which the inequality (4.9) holds. If $\beta_i' \to 0$, then, again, we can make the above estimations. Suppose, then, that $\beta_{i_k}' > \beta_0 > 0$, $k = 1, 2, \cdots$. Take a $\delta > 0$ such that for the interval $A_\delta = [t_0 - \delta, t_0 + \delta]$ we have $\mu_0\{f; w_f(A_\delta) > \beta_0\} < \frac{1}{2}\beta_0$, where $w_f(A_\delta)$ is defined as $\sup_{t_1, t_2 \varepsilon A_\delta} |f(t_1) - f(t_2)|$. On the other hand, for all sufficiently large $i_k$, we have by (4.9)

$$\beta_0 < \beta'_{i_k} < \mu_{n_{i_k}}\{f; |f(t_{i_k}+0) - f(t_{i_k}-0)| > \beta'_{i_k}\} \leqq \mu_{n_{i_k}}\{f; w_f(A_\delta) > \beta_0\}$$

and hence $\mu^{\varphi}_{n_{i_k}} \not\Rightarrow \mu^{\varphi}_0$, where $\varphi(f(t)) = w_f(A_\delta)$. Since $\varphi$ is a $\mu_0$-almost everywhere continuous function on $D[0, 1]$ it follows that $\mu_{n_{i_k}} \not\Rightarrow \mu_0$, as asserted.

THEOREM 4. *Let $\mu_n \varepsilon M(D[0, 1])$, $n = 0, 1, 2, \cdots$ and let the measure $\mu_0$ satisfy the condition $\mu_0\{(C[0, 1])^C\} = 0$. Then a necessary condition for the convergence $\mu_n \Rightarrow \mu_0$ is*

$$\lim_{n\to\infty} \sup_m \sup_{t_1,\cdots,t_m} L(\mu_n^{t_1,\cdots,t_m}, \mu_0^{t_1,\cdots,t_m}) = 0. \tag{4.10}$$

PROOF. Suppose that $\mu_n \Rightarrow \mu_0$. According to Theorem B, it follows that $L(\mu_n, \mu_0) \to 0$ and according to Theorem C, it follows that for any given $\delta > 0$ there exists a compact set $B_\delta \subset D[0, 1]$ such that $\sup_n \mu_n(B_\delta^c) < \delta$. Let $t_1, \cdots, t_m$ be arbitrary points in $[0, 1]$ and let $\varphi$ be the function appearing in Lemma 4.

Suppose that for some fixed $n$ we have $L(\mu_n, \mu_0) \leqq a$. According to (1.2), it follows that

$$\mu_n(F) \leqq \mu_0(F^a) + a \quad \text{and} \quad \mu_0(F) \leqq \mu_n(F^a) + a \tag{4.11}$$

for all closed sets $F \subset D[0, 1]$.

Let $K \subset R_m$ be an arbitrary closed set. Then the sets

$$\varphi^{-1}(K) \cap B_\delta \quad \text{and} \quad \varphi^{-1}(K) \cap B \cap C[0, 1]$$

are also closed, and we can write the following two chains of inequalities, using Lemma 4 and (4.11):

$$\begin{aligned}\mu_n^{t_1,\cdots,t_m}(K) &= \mu_n\{\varphi^{-1}(K)\} \leqq \mu_n\{\varphi^{-1}(K) \cap B_\delta\} + \delta \\ &\leqq \mu_0\{[\varphi^{-1}(K) \cap B_\delta]^a\} + \delta + a = \mu_0\{[\varphi^{-1}(K) \cap B_\delta]^a \cap C[0, 1]\} \\ &\quad + \delta + a \leqq \mu_0\{\varphi^{-1}(K^{\psi_\delta(a)})\} + \delta + a = \mu_0^{t_1,\cdots,t_m}(K^{\psi_\delta(a)}) + \delta + a;\end{aligned} \tag{4.12}$$

$$\begin{aligned}\mu_0^{t_1,\cdots,t_m}(K) &= \mu_0\{\varphi^{-1}(K) \cap C[0, 1]\} \\ &\leqq \mu_0\{\varphi^{-1}(K) \cap C[0, 1] \cap B_\delta\} + \delta \leqq \mu_n\{[\varphi^{-1}(K) \cap C[0, 1] \cap B_\delta]^a\} \\ &\quad + \delta + a \leqq \mu_n\{\varphi^{-1}(K^{\theta_\delta(a)})\} + \delta + a = \mu_n^{t_1,\cdots,t_m}(K^{\theta_\delta(a)}) + \delta + a.\end{aligned} \tag{4.13}$$

From (4.12) and (4.13) it follows, according to the definition of metric $L$, that

$$L(\mu_n^{t_1,\cdots,t_m}, \mu_0^{t_1,\cdots,t_m}) \leqq \max(\delta + a, \psi_\delta(a), \theta_\delta(a))$$

and hence

$$\sup_m \sup_{t_1,\cdots,t_m} L(\mu_n^{t_1,\cdots,t_m}, \mu_0^{t_1,\cdots,t_m}) \leqq \max(a + \delta, \psi_\delta(a), \theta_\delta(a))$$

where $\delta > 0$ is arbitrary, $\psi_\delta(a) \downarrow 0$ and $\theta_\delta(a) \downarrow 0$ as $a \downarrow 0$.

Let $\epsilon > 0$ be arbitrary. Take $\delta < \frac{1}{2}\epsilon$ and then take $a < \delta/2$ such that

$$\max(\psi_\delta(a), \theta_\delta(a)) < \epsilon.$$

For this value of $a$ take $N = N_a$ such that for $n > N$ we have $L(\mu_n, \mu_0) < a$. Hence

$$\sup_m \sup_{t_1, \cdots, t_m} L(\mu_n^{t_1, \cdots, t_m}, \mu_0^{t_1, \cdots, t_m}) < \epsilon$$

for all $n > N$, which was to be proved.

THEOREM 5. *Let $\mu_n \varepsilon M(D[0, 1])$, $n = 0, 1, 2, \cdots$. Then the condition*

$$\lim_{n\to\infty} \sup_m \sup_{t_1, \cdots, t_m} L(\mu_n^{t_1, \cdots, t_m}, \mu_0^{t_1, \cdots, t_m}) = 0 \tag{4.14}$$

*is sufficient for the convergence $\mu_n \Rightarrow \mu_0$.*

PROOF. Suppose that condition (4.14) is satisfied. Let $\epsilon > 0$ be arbitrary and let for $n > N_\epsilon$

$$\sup_m \sup_{t_1, \cdots, t_m} L(\mu_n^{t_1, \cdots, t_m}, \mu_0^{t_1, \cdots, t_m}) < \tfrac{1}{4}\epsilon. \tag{4.15}$$

Let $t_1, \cdots, t_m, \cdots$ be a sequence of points dense in $[0, 1]$. Denote by $\varphi_k$ the function appearing in Lemma 5. By Lemma 5, for every compact set $F \subset D[0, 1]$ and for arbitrary $n = 0, 1, \cdots$ there exists $k = k(n, F)$ such that

$$\mu_n\{\varphi_k^{-1}[\varphi_k(F)^{\frac{1}{4}\epsilon}]\} \leqq \mu_n(F^\epsilon) + \tfrac{3}{4}\epsilon. \tag{4.16}$$

By (4.15) and (4.16), we can write for $n > N_\epsilon$ the following chain of inequalities:

$$\mu_n(F) \leqq \mu_n\{\varphi_k^{-1}[\varphi_k(F)]\} = \mu_n^{t_1, \cdots, t_k}\{\varphi_k(F)\} \leqq \mu_0^{t_1, \cdots, t_k}\{\varphi_k(F)^{\frac{1}{4}\epsilon}\} + \tfrac{1}{4}\epsilon$$

$$= \mu_0\{\varphi_k^{-1}[\varphi_k(F)^{\frac{1}{4}\epsilon}]\} + \tfrac{1}{4}\epsilon \leqq \mu_0(F^\epsilon) + \tfrac{1}{4}\epsilon + \tfrac{3}{4}\epsilon = \mu_0(F^\epsilon) + \epsilon.$$

In a similar way we prove that for all compact sets $F$

$$\mu_0(F) \leqq \mu_n(F^\epsilon) + \epsilon;$$

hence, by Remark 1 (made in the Introduction) $L(\mu_n, \mu_0) \leqq \epsilon$ for $n > N_\epsilon$, which was to be proved.

As an immediate consequence of Theorems 4 and 5 (and also Theorems 1 and 2), we obtain

THEOREM 6. *If $\mu_n \varepsilon M(C[0, 1])$, $n = 0, 1, 2 \cdots$ then for the convergence*

$$\mu_n \Rightarrow \mu_0$$

*it is necessary and sufficient that*

$$\lim_{n\to\infty} \sup_m \sup_{t_1, \cdots, t_m} L(\mu_n^{t_1, \cdots, t_m}, \mu_0^{t_1, \cdots, t_m}) = 0. \tag{4.17}$$

PROOF. Necessity of this condition has already been proved; sufficiency follows from the fact that for the space $C[0, 1] \subset D[0, 1]$ the $d$-convergence is equivalent to the uniform one.

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