SAMPLING MOMENTS OF MEANS FROM FINITE MULTIVARIATE POPULATIONS[1]

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Summary. A method is described for deriving the sampling moments of means of random vectors obtained by sampling without replacement from a finite k-variate population of n vector members. A table of results is presented listing the moments of order less than or equal to six as a function of the population moments. These moments were originally derived, in a less general form, in the course of developing the Simplex-Sum Designs discussed in [1]. Their possible wider applicability to sampling problems, however, motivated the extension of the work to the general formulas given here.

Notation and Description of the Method. The n vectors comprising a finite k-variate population will be denoted by

$$\mathbf{x}_u' = (x_{1u}, x_{2u}, \cdots, x_{ku}), \qquad u = 1, 2, \cdots, n$$

and the population moments by

$$\langle 1^{\alpha_1} 2^{\alpha_2} \cdots k^{\alpha_k} \rangle' = \frac{1}{n} \sum_{u=1}^{n} x_{1u}^{\alpha_1} x_{2u}^{\alpha_2} \cdots x_{ku}^{\alpha_k}.$$

The order of a moment is given by $\alpha = \sum \alpha_i$.

We are concerned with the sampling moments of the mean vector $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$ of s vectors $\mathbf{x}_{u_1}', \mathbf{x}_{u_2}', \cdots, \mathbf{x}_{u_s}'$, randomly chosen from the population without replacement, viz., $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k) = s^{-1} \sum_{i=1}^{s} \mathbf{x}_{u_i}'$. The sampling moments are written as Ave $[\bar{x}_1^{\alpha_1} \bar{x}_2^{\alpha_2} \cdots \bar{x}_k^{\alpha_k}]$ where Ave denotes the expectation or average over all samples of s from the population. In deriving these results it has been convenient to use the bracket notation developed by Tukey in [2] and [3] and extended by Robson and Hooke in [4] and [5]. Univariate brackets are defined by Tukey as

$$\langle p \rangle = \frac{1}{s} \sum_{i=1}^{s} x_i^p,$$

and in general

$$\langle p_1 \, p_2 \cdots p_m \rangle = \frac{\sum_{i,j,\cdots,q}^{s \neq} x_i^{p_1} x_j^{p_2} \cdots x_q^{p_m}}{s(s-1) \cdots (s-m+1)}$$

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where the summation takes place over unequal indices and the denominator consists of the total number of terms in the numerator. These expressions are "inherited on the average" which is to say that their average over all samples of size s is equal to the same function of the n population elements. Using a prime to denote the average value then we have

$$\text{Ave}[\langle p_1 p_2 \cdots p_m\rangle] = \langle p_1 p_2 \cdots p_m\rangle' = \frac{\sum\limits_{i,j,\cdots,q}^{n\neq} x_i^{p_1} x_j^{p_2} \cdots x_q^{p_m}}{n(n-1)\cdots(n-m+1)}.$$

In extending this notation to a bivariate population (x_{1u}, x_{2u}), $u = 1, 2, \cdots, n$ the bracket $\langle 1\ 2\rangle$ can take on several meanings. For example we may have the symmetric means

$$\frac{\sum\limits^{s\neq} x_{1u}\, x_{2v}^2}{s(s-1)} \quad \text{and} \quad \frac{\sum\limits^{s\neq} x_{1u}\, x_{2u}\, x_{2v}}{s(s-1)}.$$

Departing slightly from Hooke and Robson we will adopt the notation $\langle 1^1, 2^2\rangle$ and $\langle 1^1\, 2^1, 2^1\rangle$ respectively for the above two expressions. An obvious extension of the principle yields multivariate brackets of any desired order. By using this notation we may omit from the bracket expression any vector elements whose exponents are zero. That is if we consider a k-variate population $(x_{1u}\, x_{2u} \cdots x_{ku})$ we may use

$$\langle 1^2, 2^2, 3^2\rangle = \frac{\sum\limits^{s\neq} x_{1u}^2\, x_{2v}^2\, x_{3w}^2}{s(s-1)(s-2)}$$

to represent a symmetric mean involving the first three population elements only, while $\langle i^2, j^2, l^2\rangle$ represents the same expression for a general set of three elements chosen from the population vector. It can be noted here that, by definition, primed brackets involving commas are not regarded as population moments.

Multivariate brackets are also inherited on the average. By making use of this convenient property the derivation of moment formulas is considerably simplified. In finding the sampling moments of means of samples of s drawn from a univariate population of n we seek

$$\text{Ave}[(\bar{x}_1)^\alpha] = \text{Ave}\left[\left(\frac{x_{1u_1} + x_{1u_2} + \cdots + x_{1u_s}}{s}\right)^\alpha\right] = \text{Ave}[\langle 1^1\rangle^\alpha],$$

or in general for the k-variate case we seek

$$\text{Ave}\,[(\bar{x}_1)^{\alpha_1}(\bar{x}_2)^{\alpha_2} \cdots (\bar{x}_k)^{\alpha_k}] = \text{Ave}\,[\langle 1^1\rangle^{\alpha_1}\langle 2^1\rangle^{\alpha_2} \cdots \langle k^1\rangle^{\alpha_k}].$$

We may first expand the product $\langle 1^1\rangle^{\alpha_1}\langle 2^1\rangle^{\alpha_2} \cdots \langle k^1\rangle^{\alpha_k}$ as a linear function of multivariate brackets and then take the average by simply adding primes to each bracket. Each of these brackets can then be expanded in terms of single index summations, i.e., in terms of the population moments. This is most easily accomplished by using tables of symmetric functions [6]. Although these tables provide formulas for univariate summations only, they are helpful in writing the multivariate generalizations.

Illustrative Example. The method is illustrated below for the trivial case Ave $[\bar{x}_i\bar{x}_j]$.

$$\begin{aligned}\text{Ave}\,[\bar{x}_i\bar{x}_j] &= s^{-1}\,\text{Ave}\,[\langle i^1 j^1\rangle + (s-1)\langle i^1, j^1\rangle],\\ &= s^{-1}[\langle i^1 j^1\rangle' + (s-1)\langle i^1, j^1\rangle'].\end{aligned}$$

The bracket $\langle i^1, j^1\rangle'$ is not a population moment and is therefore expanded in the form

$$\langle i, j\rangle' = \frac{n}{n-1}\,\langle i^1\rangle'\langle j^1\rangle' - \frac{1}{n-1}\,\langle i^1 j^1\rangle'.$$

The desired moment can then be written as

$$\text{Ave}[\bar{x}_i\,\bar{x}_j] = \frac{n}{s^2}\left[\left(\frac{s^{(1)}}{n^{(1)}} - \frac{s^{(2)}}{n^{(2)}}\right)\langle i^1 j^1\rangle' + \frac{ns^{(2)}}{n^{(2)}}\,\langle i^1\rangle'\langle j^1\rangle'\right]$$

where we use the factorial notation, $n^{(p)} = n(n-1)(n-2)\cdots(n-p+1)$. While for higher moments the initial expansion of the expectation in brackets involves many terms, they may readily be written down in systematic fashion by simply forming all possible comma partitions within the bracket, utilizing in turn from 1 to $k-1$ commas.

The second moment of the ith element of the vector is easily obtained by letting $i = j$ and hence obtaining the familiar univariate expression

$$\begin{aligned}\text{Ave}[\bar{x}_i^2] &= \frac{n}{s^2}\left[\left(\frac{s^{(1)}}{n^{(1)}} - \frac{s^{(2)}}{n^{(2)}}\right)\langle i^2\rangle' + \frac{ns^{(2)}}{n^{(2)}}\,(\langle i^1\rangle')^2\right]\\ &= \frac{1}{s}\left[\frac{n-s}{(n-1)}\,\langle i^2\rangle' + \frac{n(s-1)}{(n-1)}\,(\langle i^1\rangle')^2\right].\end{aligned}$$

The coefficient of $\langle i^2\rangle'$ in the above expression for the moment could be and was simplified in this case. We will in general however leave the coefficients in the unsimplified form which is convenient for higher order moments where such simplification is not always possible or desirable. As can be seen in the table of results which follows, moments of order α will involve factorial coefficient terms up to and including $s^{(\alpha)}/n^{(\alpha)}$. These formulas only apply strictly however when $n \geqq \alpha$ since for $n < \alpha$ only terms up to $s^{(n)}/n^{(n)}$ can be obtained. It is convenient however to list the formulas for $n \geqq \alpha$ and to leave it to the user to delete the meaningless coefficient terms should the order of the moment exceed the population size. Thus to retain generality these factorial elements of the coefficients are left in identifiable form. Actually, of course, these terms in the coefficients only extend to $s^{(s)}/n^{(s)}$ when $s < \alpha$ but as long as either $n \geqq \alpha$, or terms of higher order than $s^{(n)}/n^{(n)}$ are deleted when $n < \alpha$, the unwanted terms will automatically vanish since $s^{(s+1)} = 0$.

In order to consolidate these results only the most general moment of the bulky fifth and sixth order expressions is given since any other moment of the same order is easily obtained by equating subscripts. That is, for example, Ave $[\bar{x}_i^2\bar{x}_j^2\bar{x}_k^2]$ can be obtained by letting $i = l$, $j = m$ and $k = n$ in the expression for Ave $[\bar{x}_i\bar{x}_j\bar{x}_k\bar{x}_l\bar{x}_m\bar{x}_n]$.

The table which follows summarizes results obtained by use of this procedure.

Sampling Moments of Means of Samples of Size s from a Multivariate Population of n Vector Elements

A. Moment Formulas

$$\text{Ave }[\bar{x}_i] = \langle i^1\rangle'.$$

$$\text{Ave }[\bar{x}_i\bar{x}_j] = A_{21}\langle i^1j^1\rangle' + A_{22}\langle i^1\rangle'\langle j^1\rangle.$$

$$\text{Ave }[\bar{x}_i^2] = A_{21}\langle i^2\rangle' + A_{22}(\langle i^1\rangle')^2.$$

$$\begin{aligned}\text{Ave }[\bar{x}_i\bar{x}_j\bar{x}_k] = {} & A_{31}\langle i^1j^1k^1\rangle' + A_{32}(\langle i^1\rangle'\langle j^1k^1\rangle' + \langle j^1\rangle'\langle i^1k^1\rangle' \\ & + \langle k^1\rangle'\langle i^1j^1\rangle') + A_{33}\langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'.\end{aligned}$$

$$\begin{aligned}\text{Ave }[\bar{x}_i^2\bar{x}_j] = {} & A_{31}\langle i^2j^1\rangle' + A_{32}(2\langle i^1\rangle'\langle i^1j^1\rangle' + \langle j^1\rangle'\langle i^2\rangle') \\ & + A_{33}(\langle i^1\rangle')^2\langle j^1\rangle'.\end{aligned}$$

$$\text{Ave }[\bar{x}_i^3] = A_{31}\langle i^3\rangle' + 3A_{32}\langle i^1\rangle'\langle i^2\rangle' + A_{33}(\langle i^1\rangle')^3.$$

$$\begin{aligned}\text{Ave }[\bar{x}_i\bar{x}_j\bar{x}_k\bar{x}_l] = {} & A_{41}\langle i^1j^1k^1l^1\rangle' + A_{42}(\langle i^1\rangle'\langle j^1k^1l^1\rangle' + \langle j^1\rangle'\langle i^1k^1l^1\rangle' \\ & + \langle k^1\rangle'\langle i^1j^1l^1\rangle' + \langle l^1\rangle'\langle i^1j^1k^1\rangle') \\ & + A_{43}(\langle i^1j^1\rangle'\langle k^1l^1\rangle' + \langle i^1k^1\rangle'\langle j^1l^1\rangle' \\ & + \langle i^1l^1\rangle'\langle j^1k^1\rangle') \\ & + A_{44}(\langle i^1\rangle'\langle j^1\rangle'\langle k^1l^1\rangle' + \langle i^1\rangle'\langle k^1\rangle'\langle j^1l^1\rangle' \\ & + \langle i^1\rangle'\langle l^1\rangle'\langle j^1k^1\rangle' + \langle j^1\rangle'\langle k^1\rangle'\langle i^1l^1\rangle' \\ & + \langle j^1\rangle'\langle l^1\rangle'\langle i^1k^1\rangle' + \langle k^1\rangle'\langle l^1\rangle'\langle i^1j^1\rangle') \\ & + A_{45}\langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'\langle l^1\rangle'.\end{aligned}$$

$$\begin{aligned}\text{Ave }[\bar{x}_i^2\bar{x}_j\bar{x}_k] = {} & A_{41}\langle i^2j^1k^1\rangle' + A_{42}(2\langle i^1\rangle'\langle i^1j^1k^1\rangle' + \langle j^1\rangle'\langle i^2k^1\rangle' \\ & + \langle k^1\rangle'\langle i^2j^1\rangle') \\ & + A_{43}(2\langle i^1j^1\rangle'\langle i^1k^1\rangle' + \langle i^2\rangle'\langle j^1k^1\rangle') \\ & + A_{44}(2\langle i^1\rangle'\langle j^1\rangle'\langle i^1k^1\rangle' + 2\langle i^1\rangle'\langle k^1\rangle'\langle i^1j^1\rangle' \\ & + (\langle i^1\rangle')^2\langle j^1k^1\rangle' + \langle j^1\rangle'\langle k^1\rangle'\langle i^2\rangle') \\ & + A_{45}[(\langle i^1\rangle')^2\langle j^1\rangle'\langle k^1\rangle'].\end{aligned}$$

$$\begin{aligned}
\text{Ave}\,[\bar{x}_i^3\bar{x}_j] = {} & A_{41}\langle i^3 j^1\rangle' + A_{42}(3\langle i^1\rangle'\langle i^2 j^1\rangle' + \langle j^1\rangle'\langle i^3\rangle') \\
& + 3A_{43}\langle i^1 j^1\rangle'\langle i^2\rangle' + 3A_{44}[\langle i^1\rangle'\langle j^1\rangle'\langle i^2\rangle' \\
& + (\langle i^1\rangle')^2\langle i^1 j^1\rangle'] \\
& + A_{45}(\langle i^1\rangle')^3\langle j^1\rangle'.
\end{aligned}$$

$$\begin{aligned}
\text{Ave}\,[\bar{x}_i^2\bar{x}_j^2] = {} & A_{41}\langle i^2 j^2\rangle' + 2A_{42}(\langle i^1\rangle'\langle i^1 j^2\rangle' + \langle j^1\rangle'\langle i^2 j^1\rangle') \\
& + A_{43}[2(\langle i^1 j^1\rangle')^2 + \langle i^2\rangle'\langle j^2\rangle'] \\
& + A_{44}[4\langle i^1\rangle'\langle j^1\rangle'\langle i^1 j^1\rangle' \\
& + (\langle i^1\rangle')^2\langle j^2\rangle' + (\langle j^1\rangle')^2\langle i^2\rangle'] \\
& + A_{45}(\langle i^1\rangle')^2(\langle j^1\rangle')^2.
\end{aligned}$$

$$\begin{aligned}
\text{Ave}\,[\bar{x}_i^4] = {} & A_{41}\langle i^4\rangle' + 4A_{42}\langle i^1\rangle'\langle i^3\rangle' + 3A_{43}(\langle i^2\rangle')^2 \\
& + 6A_{44}(\langle i^1\rangle')^2\langle i^2\rangle' + A_{45}(\langle i^1\rangle')^4.
\end{aligned}$$

$$\begin{aligned}
\text{Ave}\,[\bar{x}_i\bar{x}_j\bar{x}_k\bar{x}_l\bar{x}_m] = {} & A_{51}\langle i^1 j^1 k^1 l^1 m^1\rangle' + A_{52}\begin{Bmatrix}\text{5 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1 k^1 l^1 m^1\rangle'\end{Bmatrix} \\
& + A_{53}\begin{Bmatrix}\text{10 distinct arrangements} \\ \text{of } \langle i^1 j^1\rangle'\langle k^1 l^1 m^1\rangle'\end{Bmatrix} \\
& + A_{54}\begin{Bmatrix}\text{10 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1\rangle'\langle k^1 l^1 m^1\rangle'\end{Bmatrix} \\
& + A_{55}\begin{Bmatrix}\text{15 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1 k^1\rangle'\langle l^1 m^1\rangle'\end{Bmatrix} \\
& + A_{56}\begin{Bmatrix}\text{10 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'\langle l^1 m^1\rangle'\end{Bmatrix} \\
& + A_{57}\langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'\langle l^1\rangle'\langle m^1\rangle'.
\end{aligned}$$

$$\begin{aligned}
\text{Ave}\,[\bar{x}_i\bar{x}_j\bar{x}_k\bar{x}_l\bar{x}_m\bar{x}_n] = {} & A_{61}\langle i^1 j^1 k^1 l^1 m^1 n^1\rangle' + A_{62}\begin{Bmatrix}\text{6 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1 k^1 l^1 m^1 n^1\rangle'\end{Bmatrix} \\
& + A_{63}\begin{Bmatrix}\text{15 distinct arrangements} \\ \text{of } \langle i^1 j^1\rangle'\langle k^1 l^1 m^1 n^1\rangle'\end{Bmatrix} \\
& + A_{64}\begin{Bmatrix}\text{10 distinct arrangements} \\ \text{of } \langle i^1 j^1 k^1\rangle'\langle l^1 m^1 n^1\rangle'\end{Bmatrix} \\
& + A_{65}\begin{Bmatrix}\text{15 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1\rangle'\langle k^1 l^1 m^1 n^1\rangle'\end{Bmatrix} \\
& + A_{66}\begin{Bmatrix}\text{60 distinct arrangements} \\ \text{of } \langle i^1\rangle'\langle j^1 k^1\rangle'\langle l^1 m^1 n^1\rangle'\end{Bmatrix}
\end{aligned}$$

$$+ A_{67}\begin{Bmatrix}\text{15 distinct arrangements}\\ \text{of } \langle i^1 j^1\rangle'\langle k^1 l^1\rangle'\langle m^1 n^1\rangle'\end{Bmatrix}$$

$$+ A_{68}\begin{Bmatrix}\text{20 distinct arrangements}\\ \text{of } \langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'\langle l^1 m^1 n^1\rangle'\end{Bmatrix}$$

$$+ A_{69}\begin{Bmatrix}\text{45 distinct arrangements}\\ \text{of } \langle i^1\rangle'\langle j^1\rangle'\langle k^1 l^1\rangle'\langle m^1 n^1\rangle'\end{Bmatrix}$$

$$+ A_{6\,10}\begin{Bmatrix}\text{15 distinct arrangements}\\ \text{of } \langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'\langle l^1\rangle'\langle m^1 n^1\rangle'\end{Bmatrix}$$

$$+ A_{6\,11}\,\langle i^1\rangle'\langle j^1\rangle'\langle k^1\rangle'\langle l^1\rangle'\langle m^1\rangle'\langle n^1\rangle'.$$

B. Coefficients[2]

$$A_{21} = \frac{n}{s^2}\left(\frac{s}{n} - \frac{s^{(2)}}{n^{(2)}}\right), \qquad A_{22} = \frac{n^2}{s^2}\cdot\frac{s^{(2)}}{n^{(2)}}.$$

$$A_{31} = \frac{n}{s^3}\left(\frac{s}{n} - 3\frac{s^{(2)}}{n^{(2)}} + 2\frac{s^{(3)}}{n^{(3)}}\right), \qquad A_{32} = \frac{n^2}{s^3}\left(\frac{s^{(2)}}{n^{(2)}} - \frac{s^{(3)}}{n^{(3)}}\right), \qquad A_{33} = \frac{n^3}{s^3}\cdot\frac{s^{(3)}}{n^{(3)}}.$$

$$A_{41} = \frac{n}{s^4}\left(\frac{s}{n} - 7\frac{s^{(2)}}{n^{(2)}} + 12\frac{s^{(3)}}{n^{(3)}} - 6\frac{s^{(4)}}{n^{(4)}}\right),$$

$$A_{42} = \frac{n^2}{s^4}\left(\frac{s^{(2)}}{n^{(2)}} - 3\frac{s^{(3)}}{n^{(3)}} + 2\frac{s^{(4)}}{n^{(4)}}\right),$$

$$A_{43} = \frac{n^2}{s^4}\left(\frac{s^{(2)}}{n^{(2)}} - 2\frac{s^{(3)}}{n^{(3)}} + \frac{s^{(4)}}{n^{(4)}}\right),$$

$$A_{44} = \frac{n^3}{s^4}\left(\frac{s^{(3)}}{n^{(3)}} - \frac{s^{(4)}}{n^{(4)}}\right), \qquad A_{45} = \frac{n^4}{s^4}\cdot\frac{s^{(4)}}{n^{(4)}}.$$

$$A_{51} = \frac{n}{s^5}\left(\frac{s}{n} - 15\frac{s^{(2)}}{n^{(2)}} + 50\frac{s^{(3)}}{n^{(3)}} - 60\frac{s^{(4)}}{n^{(4)}} + 24\frac{s^{(5)}}{n^{(5)}}\right),$$

$$A_{52} = \frac{n^2}{s^5}\left(\frac{s^{(2)}}{n^{(2)}} - 7\frac{s^{(3)}}{n^{(3)}} + 12\frac{s^{(4)}}{n^{(4)}} - 6\frac{s^{(5)}}{n^{(5)}}\right),$$

$$A_{53} = \frac{n^2}{s^5}\left(\frac{s^{(2)}}{n^{(2)}} - 4\frac{s^{(3)}}{n^{(3)}} + 5\frac{s^{(4)}}{n^{(4)}} - 2\frac{s^{(5)}}{n^{(5)}}\right),$$

[2] When the order of the moment exceeds the population size, i.e., $\alpha > n$, terms in the coefficients of the form $s^{(p)}/n^{(p)}$ where $p > n$ are to be deleted from the expression.

$$A_{54} = \frac{n^3}{s^5}\left(\frac{s^{(3)}}{n^{(3)}} - 3\,\frac{s^{(4)}}{n^{(4)}} + 2\,\frac{s^{(5)}}{n^{(5)}}\right),$$

$$A_{55} = \frac{n^3}{s^5}\left(\frac{s^{(3)}}{n^{(3)}} - 2\,\frac{s^{(4)}}{n^{(4)}} + \frac{s^{(5)}}{n^{(5)}}\right),$$

$$A_{56} = \frac{n^4}{s^5}\left(\frac{s^{(4)}}{n^{(4)}} - \frac{s^{(5)}}{n^{(5)}}\right),$$

$$A_{57} = \frac{n^5}{s^5}\cdot\frac{s^{(5)}}{n^{(5)}}.$$

$$A_{61} = \frac{n}{s^6}\left(\frac{s}{n} - 31\,\frac{s^{(2)}}{n^{(2)}} + 180\,\frac{s^{(3)}}{n^{(3)}} - 390\,\frac{s^{(4)}}{n^{(4)}} + 360\,\frac{s^{(5)}}{n^{(5)}} - 120\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{62} = \frac{n^2}{s^6}\left(\frac{s^{(2)}}{n^{(2)}} - 15\,\frac{s^{(3)}}{n^{(3)}} + 50\,\frac{s^{(4)}}{n^{(4)}} - 60\,\frac{s^{(5)}}{n^{(5)}} + 24\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{63} = \frac{n^2}{s^6}\left(\frac{s^{(2)}}{n^{(2)}} - 8\,\frac{s^{(3)}}{n^{(3)}} + 19\,\frac{s^{(4)}}{n^{(4)}} - 18\,\frac{s^{(5)}}{n^{(5)}} + 6\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{64} = \frac{n^2}{s^6}\left(\frac{s^{(2)}}{n^{(2)}} - 6\,\frac{s^{(3)}}{n^{(3)}} + 13\,\frac{s^{(4)}}{n^{(4)}} - 12\,\frac{s^{(5)}}{n^{(5)}} + 4\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{65} = \frac{n^3}{s^6}\left(\frac{s^{(3)}}{n^{(3)}} - 7\,\frac{s^{(4)}}{n^{(4)}} + 12\,\frac{s^{(5)}}{n^{(5)}} - 6\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{66} = \frac{n^3}{s^6}\left(\frac{s^{(3)}}{n^{(3)}} - 4\,\frac{s^{(4)}}{n^{(4)}} + 5\,\frac{s^{(5)}}{n^{(5)}} - 2\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{67} = \frac{n^3}{s^6}\left(\frac{s^{(3)}}{n^{(3)}} - 3\,\frac{s^{(4)}}{n^{(4)}} + 3\,\frac{s^{(5)}}{n^{(5)}} - \frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{68} = \frac{n^4}{s^6}\left(\frac{s^{(4)}}{n^{(4)}} - 3\,\frac{s^{(5)}}{n^{(5)}} + 2\,\frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{69} = \frac{n^4}{s^6}\left(\frac{s^{(4)}}{n^{(4)}} - 2\,\frac{s^{(5)}}{n^{(5)}} + \frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{6\,10} = \frac{n^5}{s^6}\left(\frac{s^{(5)}}{n^{(5)}} - \frac{s^{(6)}}{n^{(6)}}\right),$$

$$A_{6\,11} = \frac{n^6}{s^6}\cdot\frac{s^{(6)}}{n^{(6)}}.$$

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