THE SAMPLE MEAN AMONG THE EXTREME NORMAL ORDER STATISTICS[1]

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0. Summary. This paper begins with a discussion of convex spherical polyhedra. This discussion touches on conditions for degeneracy, sub-polyhedra, lunes, and simplices, and terminates with Schläfli's fundamental differential relation for the measure of such a polyhedron. The paper then proceeds to the computation of bounds for the measure of an equilateral spherical simplex. The asymptotic measure of an equilateral spherical simplex then is computed by means of these bounds. (As was brought out in the course of the refereeing, this asymptotic measure has recently been computed elsewhere, but by a different method.) These results are applied to the computation of the asymptotic value and bounds of the probability that a normal sample mean falls between successive order statistics of fixed order. These computations constitute the asymptotic solution of Youden's "Demon Problem", and yield probabilities of order substantially lower than had previously been hypothesized.

1. Introduction. This is one of two papers dealing with the magnitude of the sample mean relative to the order statistics for normal populations. In this paper, emphasis will be placed on the extreme order statistics for normal populations; intermediate order statistics are considered in the other paper [8].

Section 2 marshals some geometry needed in the subsequent development. Though some of this material appears to be new (notably Theorem 1 of Subsection 2.1), the important formula is Schläfli's expression (formula 1 of Subsection 2.3) for the differential of the measure of a spherical simplex.

Based on Section 2, Section 3 derives the asymptotic value of the measure of an equilateral spherical simplex,[2] and also exact bounds for this measure. Let $P(n, k)$ be the probability that a normal sample mean falls between the k'th and the $(k + 1)'$st order statistics. Leaning on Section 3, Section 4 derives an asymptotic expression for $P(n, k)$, as well as exact bounds. It appears from these investigations that a normal sample mean will reach the extreme order statistics considerably less often than has been suspected [14]. Specifically, $P(n, 1)$ asymptotically equals $(e/2\pi n)^{n/2}n^2\pi^{\frac{1}{2}}/e$ (Sub-section 4.4). Additional

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[1] This paper is part of a dissertation submitted to the Department of Statistics, University of Chicago, in partial fulfillment of the requirements for the Ph.D. degree.

[2] In the course of the refereeing of this paper, H. S. M. Coxeter and a referee pointed out that derivations in [5] show an argument by H. Daniels in [24] to have in effect established the asymptotic value for equilateral spherical simplices of unit edge. Subsequently, H. S. M. Coxeter further pointed out that Rogers recently obtained [25] this asymptotic value for arbitrary edge length, i.e., the asymptotic result of Section 3, but by a different method.

computations give the exact values of $P(4, k)$ and $P(5, k)$. It should be noted that multivariate normal probabilities other than $P(n, k)$ can be evaluated asymptotically as spherical measures, if only those corresponding to spherical polyhedra expressible in terms of equilateral simplices. However, no such extensions will be considered in this paper.

J. Youden has called the evaluation of $P(n, k)$ the Demon Problem and has evaluated these probabilities for low n by Monte Carlo computations at the National Bureau of Standards. He has kindly made these computations available to me (Sub-section 4.5). Other work on $P(n, k)$ is that of Kendall [14] who has approximated $P(n, k)$ by the Edgeworth series. The computation of $P(n, k)$ in this paper is in the language of the geometry discussed in Sections 2 and 3.

The probability $P(n, k)$ clearly equals $\binom{n}{k}$ times the probability $R(n, k)$ that the mean is larger than the first k unordered observations, and smaller than the last $n - k$ unordered observations, which in turn is the probability, under the spherical normal distribution, of a polyhedral cone C with bounding planes of the form $x_i - \bar{x} = 0$. By the spherical symmetry of the spherical normal, this last probability is equal to the spherical measure, relative to the spherical measure of the unit sphere S, of the spherical polyhedron $C \cap S$. Finally, as is shown in Sub-section 4.3, this relative or normed polyhedral measure is expressible as a linear combination of the normed measures of a set of equilateral simplices of descending dimension. Alternatively, transforming to, say, $y_i = x_i - \bar{x}$, $R(n, k)$ also equals the probability of the first orthant, induced by the normal density with covariance structure that of the transforms y_i, and it might have been possible to reach the results of this paper by exploiting this interpretation of $R(n, k)$.

Evaluating arbitrary normal first-orthant probabilities as normed spherical measures has been discussed by van der Vaart [33], [34]. Fisher [11] and Ruben [26], [27] have used normed spherical measures for related statistical problems.

Spherical simplex measures have interested several mathematicians, among them, Böhm [1], Dehn [9], p. 572, Poincaré [23], p. 116, eqn. (9), Schläfli [29] and [28], p. 71, eqn. (1), and Sommerville [30], p. 110, eqn. (3.74). These writers arrive at a generalization of the angular excess formula of the 2-sphere, which gives the normed spherical measure of a spherical simplex imbedded in a hypersphere of even dimensionality as a linear combination of the relative contents of simplices in spheres of next lowest and lower dimensions. F. N. David [7] has pointed out the relation of their formula to a well-known theorem of probability theory that she attributes to Boole.

The numerically most fruitful approach to the spherical simplex problem seems to have been that of Schläfli ([28], pp. 57–68), who derived the fundamental relation stated in Sub-section 2.3. In addition, Schläfli has given several extensions of the generalized spherical excess formula explored by Dehn, Poincaré, Sommerville and himself. The Schläfli theory is the fundamental tool in the present investigation. Schläfli's fundamental differential relation recently has been restated by Ruben in [26] eqns. (71), (71′), (72), and restated and reproved by van der Vaart in [33] eqn. (3.18) and in [34], Theorem 4.

Neither the generalized spherical excess formula nor Schläfli's relation yield closed expressions in terms of elementary functions for the normed spherical measure of an arbitrary spherical simplex of dimensionality greater than two, or, equivalently, of the arbitrary normal orthant for more than three dimensions. As a matter of fact, only Schläfli's differential relation yields exact and closed expressions, and these are not elementary. However, the problem of closed evaluation in special cases has been considered by several authors. Coxeter [4] has given closed expressions for a class of simplices in the 3-sphere, and Schläfli [28] has considered various classes of right simplices on spheres of various low dimensionalities including 3.

As indicated above, the evaluation of $R(n, k)$ corresponds as well to computing the probability of the first orthant under a normal density with a certain covariance structure. Several authors have studied the "normal orthant" problem without reference to the language of spherical geometry. As mentioned earlier, the results of Section 4 might perhaps have been obtained as well by extending some of the work in this area.

One attempt to evaluate the "normal orthant" is that of Kendall [15] and Moran [21], who developed an infinite series involving successively higher powers of the correlation coefficients. The convergence of this series apparently is felt to be rather slow (F. N. David [7]). Another idea is to express the n-fold integral involved as a single integral of a tabulated function. Two special cases have been studied in this light. Moran [20], Ruben [26], and Dunnett [10] have given such a reduction for the case of equal correlation coefficients; in this case the tabulated functions turn out to be essentially powers of the error function. Also, Ruben has pointed out that, in the case of $P(n, k)$, i.e., the particular problem under discussion in this paper, the multiple integral may be reduced to a single integral of a tabulated function which is essentially a product of G-functions (cf., Godwin [12]). Again, along somewhat different lines, McFadden [17] has considered an approximation suggested by the form of analogous probabilities occurring in certain dependent sampling schemes. Finally, Lomnicki and Zaremba [16] have obtained certain characterizations of the orthant probability in the case of four dimensions, and Steck [32] and Das [6] have given workable numerical methods for three dimensions.

A comprehensive bibliography covering all areas mentioned above has recently been compiled by Gupta [13].

A final word on notation: a reference to, say (17) means a reference to equation (17) in "this" sub-section; a reference to, say, (3.17) means a reference to equation (17) of Sub-section 3 of "this" section, while a reference to, say, (2.3.17) directs the reader to section, sub-section, and equation, respectively.

2. Convex spherical polyhedra.

2.1. *The supports of a convex spherical polyhedron.*

Definition 1. Let $\mathfrak{F}$ be a Euclidean space containing a set W of vectors w. The point set $C\{W\} = \{x:|x| = 1; x \cdot w \geqq 0, w \varepsilon W\}$ in $\mathfrak{F}$ is a convex spherical polyhedron, and W is a set of supports for $C\{W\}$.

Lemma 1. *Let $\mathcal{E}$ be a linear subspace of $\mathfrak{F}$, and let $E^{\perp}$ be a set of vectors spanning the orthogonal complement $\mathcal{E}^{\perp}$ of $\mathcal{E}$ in $\mathfrak{F}$. Let $C_1 = C\{W \cup E^{\perp} \cup -E^{\perp}\} = C\{W\} \cap \mathcal{E}$ and let $C_2 = C\{W^* \cup E^{\perp} \cup -E^{\perp}\} = C\{W^*\} \cap \mathcal{E}$, where W^* is obtained from W by replacing any set of vectors of W by their orthogonal projections on $\mathcal{E}$. Then $C_1 = C_2$.*

Proof. For any x in $\mathcal{E}$, since orthogonal projection is self-adjoint, $x \cdot w \geqq 0$ holds if and only if $\mathcal{E}x \cdot w \geqq 0$, which in turn holds if and only if $x \cdot \mathcal{E}w \geqq 0$ (where the symbol $\mathcal{E}$, by a notational device to be used repeatedly below, denotes orthogonal projection onto $\mathcal{E}$). Hence $x \varepsilon [C\{W\} \cap \mathcal{E}]$ if and only if $x \varepsilon [C\{W^*\} \cap \mathcal{E}]$.

Definition 2. $C\{W\}$ is called degenerate relative to a subspace $\mathcal{G}$ of $\mathfrak{F}$ if $C\{W\} \cap \mathcal{G}$ lies entirely in a proper linear subspace of $\mathcal{G}$. In other words, $C\{W\}$ is degenerate relative to $\mathcal{G}$ if the spherical m-measure of $C\{W\} \cap \mathcal{G}$ is zero, where $m = \dim \mathcal{G} - 1$.

Context allowing, the phrase "relative to" sometimes will be omitted.

Definition 3. $\bar{X}$ is the set of non-zero elements of X.

Lemma 2. *A convex spherical polyhedron $C\{W\}$ is degenerate relative to $\mathcal{G}$ if and only if $x \cdot \overline{\mathcal{G}W} > 0$ for no $x \varepsilon \mathcal{G}$.*

Proof of Necessity. Suppose $x \cdot \overline{\mathcal{G}W} > 0$ for some $x \varepsilon \mathcal{G}$, say x_o. Then there exists a neighborhood $N(x_o)$ of x_o in $\mathcal{G}$ such that $[x \varepsilon N(x_o)]$ implies $[x \cdot \overline{\mathcal{G}W} > 0]$. Hence, by scaling every x in $N(x_o)$ by an arbitrary positive constant, there exists a cone D in $\mathcal{G}$ such that the intersection of D with the unit m-sphere in $\mathcal{G}$ yields a *set S on the unit m-sphere in $\mathcal{G}$, of non-zero spherical m-measure, such that $[x \varepsilon S]$ implies $[x \cdot \overline{\mathcal{G}W} > 0]$.*

Now denote by W_1 the set of vectors w of W such that $\mathcal{G}w \neq 0$; denote by W_2 the set of vectors of W such that $\mathcal{G}w = 0$. W_2 clearly is in the orthogonal complement of $\mathcal{G}$, so that $[x \varepsilon \mathcal{G}]$ implies $[x \cdot W_2 = 0]$. Using this fact, plus the self-adjointness of orthogonal projection as it was used in the proof of Lemma 1, one obtains the following chain of implications: $[x \varepsilon \mathcal{G}; x \cdot \overline{\mathcal{G}W} > 0] \Rightarrow [x \varepsilon \mathcal{G}; x \cdot \mathcal{G}W_1 > 0] \Rightarrow [x \varepsilon \mathcal{G}; \mathcal{G}x \cdot W_1 > 0] \Rightarrow [x \varepsilon \mathcal{G}; x \cdot W_1 > 0] \Rightarrow [x \varepsilon \mathcal{G}; x \cdot W_2 = 0; x \cdot W_1 > 0] \Rightarrow [x \varepsilon \mathcal{G}; x \cdot W \geqq 0]$, or $[x \varepsilon (C\{W\} \cap \mathcal{G})]$. Hence, *if $S \subset \mathcal{G}$ is such that $[x \varepsilon S]$ implies $[x \cdot \overline{\mathcal{G}W} > 0]$, then $S \subset (C\{W\} \cap \mathcal{G})$.*

The two italicized portions of the above argument imply that $C\{W\} \cap \mathcal{G}$ is not degenerate relative to $\mathcal{G}$.

Proof of Sufficiency. Suppose $x \cdot \overline{\mathcal{G}W} > 0$ for no $x \varepsilon \mathcal{G}$. Then, unless $C\{W\}$ is empty, there is for every x of $C\{W\} \cap \mathcal{G}$ a subset $\overline{\mathcal{G}W}(x)$ of $\overline{\mathcal{G}W}$ such that $x \cdot \mathcal{G}w = 0$ for $\mathcal{G}w \varepsilon \overline{\mathcal{G}W}(x)$, $x \cdot \mathcal{G}w > 0$ for $\mathcal{G}w \not\varepsilon \overline{\mathcal{G}W}(x)$. We now show that $C\{W\}$ is degenerate relative to $\mathcal{G}$ by showing that $\bigcap_{x \varepsilon (C(W) \cap \mathcal{G})} \overline{\mathcal{G}W}(x)$ is not empty, i.e., that $C\{W\} \cap \mathcal{G}$ lies entirely in some hyperplane $x \cdot \mathcal{G}w_o = 0$ in $\mathcal{G}$, $\mathcal{G}w_o$ some vector of $\overline{\mathcal{G}W}$.

Suppose that $\bigcap_{x \varepsilon (C(W) \cap \mathcal{G})} \overline{\mathcal{G}W}(x)$ is empty. Partition the points x of $C\{W\} \cap \mathcal{G}$ into, say, K partitions, according to the composition of $\overline{\mathcal{G}W}(x)$. (Note that

$K \leqq 2^n - 1$; n the number of vectors in $\overline{\mathcal{G}W}$; where the subtraction of 1 is due to the original premise excluding $x \cdot \overline{\mathcal{G}W} > 0$.) Now, from each of the K partitions, pick an x, say x_k, $1 \leqq k \leqq K$. Then if $\bigcap_{x \varepsilon C\{W\} \cap \mathcal{G}} \overline{\mathcal{G}W}(x)$ is empty then $\bigcap_k \overline{\mathcal{G}W}(x_k)$ is empty. This last in turn implies that $\bar{x} \cdot \overline{\mathcal{G}W} > 0$, where $\bar{x} = \sum x_k/k \; \varepsilon \; \mathcal{G}$ since $x_k \; \varepsilon \; \mathcal{G}$.

In summary, if $\bigcap_{x \varepsilon C\{W\} \cap \mathcal{G}} \overline{\mathcal{G}W}(x)$ were empty, then there would exist an $\bar{x} \; \varepsilon \; \mathcal{G}$ with $\bar{x} \cdot \overline{\mathcal{G}W} > 0$, contradicting the original premise that $x \cdot \overline{\mathcal{G}W} > 0$ for no $x \; \varepsilon \; \mathcal{G}$.

LEMMA 3. *$x \cdot \overline{\mathcal{G}W} > 0$ for no $x \; \varepsilon \; \mathcal{G}$ if and only if $0 \; \varepsilon \; cc\overline{\mathcal{G}W}$, where $cc\overline{\mathcal{G}W}$ is the convex closure of $\overline{\mathcal{G}W}$.*

PROOF OF NECESSITY. If $0 \not\varepsilon \; cc\overline{\mathcal{G}W}$, then the normal ξ to the supporting hyperplane separating 0 and $cc\overline{\mathcal{G}W}$ satisfies $\xi \cdot \overline{\mathcal{G}W} > 0$.

PROOF OF SUFFICIENCY. Let $\{g_i\}$ denote the elements of $\overline{\mathcal{G}W}$. If there exists an x, say x_o, such that $x_o \cdot \overline{\mathcal{G}W} > 0$, then $x_o \cdot \sum \alpha_i g_i = \sum (\alpha_i)(x_o \cdot g_i) > 0$ for any set $\{\alpha_i\}$ of convex weights, i.e., $\sum \alpha_i g_i = 0$ for no set $\{\alpha_i\}$ of convex weights, i.e., $0 \not\varepsilon \; cc\overline{\mathcal{G}W}$.

THEOREM 1. *$C\{W\}$ is degenerate relative to $\mathcal{G}$ if and only if $0 \; \varepsilon \; cc\overline{\mathcal{G}W}$.*

PROOF. By Lemmas 2 and 3.

A result analogous to Theorem 1 for convex Euclidean polyhedra is given on page 16 of [2].

DEFINITION 4.

(i) $C\{W\}$ is a proper lune (orange section peel) relative to $\mathcal{G}$ if the subspace $\mathcal{V}$ spanned by $\overline{\mathcal{G}W}$ has dimension less than $(m + 1)$, $(m + 1)$ being the dimension of $\mathcal{G}$.

(ii) the type of the proper lune $C\{W\}$ relative to $\mathcal{G}$ is $(m + 1) - \dim(\mathcal{V})$.

(iii) for $\mathcal{S}$ any subspace of $\mathcal{G}$ such that $\mathcal{V} \subset \mathcal{S} \subset \mathcal{G}$, $\mathcal{S}^\perp$ the orthogonal complement of $\mathcal{S}$ in $\mathcal{G}$, and $S^\perp$ any vector set spanning $\mathcal{S}^\perp$, $C\{W\} \cap \mathcal{S} = C\{W \cup S^\perp \cup -S^\perp\}$ is the base of $C\{W\}$ corresponding to $\mathcal{S}$, of order $(m + 1) - \dim(\mathcal{S})$ relative to $\mathcal{G}$.

(iv) $C\{W\} \cap \mathcal{V}$ is the minimal base of $C\{W\}$ relative to $\mathcal{G}$.

The type of a proper lune clearly is the order of its minimal base.

LEMMA 4. *If $C\{W\}$ is a proper lune relative to $\mathcal{G}$, then $C\{W\}$ may be degenerate relative to $\mathcal{G}$.*

PROOF. By Definition 4 $C\{W\}$ is a proper lune relative to $\mathcal{G}$ if $\dim(\mathcal{V}) \leqq m$. By Theorem 1, $C\{W\}$ is degenerate relative to $\mathcal{G}$ if and only if $0 \; \varepsilon \; cc\overline{\mathcal{G}W}$. These two conditions clearly are not inconsistent.

DEFINITION 5. $C\{W\}$ is a simplex relative to $\mathcal{G}$ if $\overline{\mathcal{G}W}$ is a maximal independent set of $\mathcal{G}$, i.e., a set of $(m + 1)$ independent vectors.

LEMMA 5. *If $C\{W\}$ is a simplex relative to $\mathcal{G}$, then $C\{W\}$ is not degenerate relative to $\mathcal{G}$.*

PROOF. If $C\{W\}$ were degenerate relative to $\mathcal{G}$, then some convex combination of the elements of $\overline{\mathcal{G}W}$ would equal zero, contrary to the independence required in Definition 5.

LEMMA 6. *Let $C\{W\}$ be a proper lune relative to $\mathcal{G}$, W spanning $\mathcal{V} \subset \mathcal{G}$;*

let $C\{W\} \cap \mathcal{S}$ be a base of $C\{W\}$, where $\mathcal{V} \subset \mathcal{S} \subset \mathcal{G}$. Then $C\{W\}$ is degenerate relative to $\mathcal{G}$ if and only if $C\{W\} \cap \mathcal{S}$ is degenerate relative to $\mathcal{S}$.

PROOF. $C\{W\} \cap \mathcal{S} = C\{W \cup S^{\perp} \cup -S^{\perp}\}$. Hence, by Theorem 1, $\overline{C\{W\}} \cap \mathcal{S}$ is degenerate relative to $\mathcal{S}$ if and only if $cc(\mathcal{S}W \cup \mathcal{S}S^{\perp} \cup \mathcal{S} - S^{\perp}) = cc\overline{\mathcal{S}W}$ contains zero. Again, by Theorem 1, $C\{W\}$ is degenerate relative to $\mathcal{G}$ if and only if $cc\overline{\mathcal{G}W}$ contains zero. But W spans $\mathcal{V} \subset \mathcal{S} \subset \mathcal{G}$, so that $\mathcal{S}W = W = \mathcal{G}W$, or $cc\overline{\mathcal{S}W} = cc\overline{\mathcal{G}W}$.

2.2. *Subpolyhedra and dihedral angles.* To simplify the exposition, it will be assumed in this section that $W \varepsilon \mathcal{G}$, so that $\overline{\mathcal{G}W} = \bar{W}$. In addition, statements regarding $C\{W\}$, concerning degeneracy, and simplicial or lunar nature, often will omit the phrase "relative to $\mathcal{G}$." Statements regarding subpolyhedra of $C\{W\}$ will always contain the corresponding phrases, specifying the pertinent subspaces of $\mathcal{G}$.

DEFINITION 1. Let $C\{W\}$ be non-degenerate, and let U be an arbitrary subset of $\bar{W}$, spanning a subspace $\mathcal{U}$ of $\mathcal{G}$ of dimension k, $0 \leqq k \leqq m$. Then $C\{W \cup -U\}$ is the subpolyhedron of order k of $C\{W\}$ corresponding to U.

LEMMA 1. *Let $\mathcal{U}^{\perp}$ be the orthogonal complement of $\mathcal{U}$ in $\mathcal{G}$. Let $\tilde{U} = \bar{W} - U$. Then the order-k subpolyhedron $C\{W \cup -U\}$ of $C\{W\}$ equals $C\{\mathcal{U}^{\perp}\tilde{U}\}$.*

PROOF. Applying Lemma 1.1 with $\mathcal{E} = \mathcal{U}^{\perp}$,

$$C\{W \cup -U\} = C\{\tilde{U} \cup U \cup -U\} = C\{\mathcal{U}^{\perp}\tilde{U} \cup \mathcal{U}^{\perp}U \cup \mathcal{U}^{\perp} - U\} = C\{\mathcal{U}^{\perp}\tilde{U}\}.$$

LEMMA 2. *A simplex $C\{W\}$ has subpolyhedra of all orders, i.e., of order k, $0 \leqq k \leqq m$. All these subpolyhedra are themselves simplices; specifically, $C\{W \cup -U\} = C\{\mathcal{U}^{\perp}\tilde{U}\}$ is a simplex relative to $\mathcal{U}^{\perp}$.*

PROOF. Let U be a set of k vectors of $\bar{W}$. We must show that $\mathcal{U}^{\perp}\tilde{U}$ is a maximal independent set in $\mathcal{U}^{\perp}$. But, since the elements of U are independent, $\dim(\mathcal{U}^{\perp}) = m + 1 - k$; hence we must show that $\mathcal{U}^{\perp}\tilde{U}$ is a set of $(m + 1 - k)$ independent vectors.

Let $\tilde{U} = \bar{W} - U$ as in Lemma 1, and denote the elements of $\mathcal{U}^{\perp}\tilde{U}$ by $\mathcal{U}^{\perp}\tilde{u}_i$. Now suppose that $\sum a_i\mathcal{U}^{\perp}\tilde{u}_i = 0$; then $\mathcal{U}^{\perp}(\sum a_i\tilde{u}_i) = 0$, or $\sum a_i\tilde{u}_i = u^* \varepsilon \mathcal{U}$. But the elements u_i of U span $\mathcal{U}$, so that $u^* = \sum b_i u_i$, and we conclude that $\sum a_i\tilde{u}_i = \sum b_i u_i$, which implies that $a_i = b_i = 0$ by the independence of the elements of $\bar{W} = \{u_i, \cdots, u_k, \tilde{u}_1, \cdots, \tilde{u}_{m+1-k}\}$. In particular then, $a_i = 0$.

LEMMA 3. *If $C\{W\}$ is a proper lune of type t relative to $\mathcal{G}$, then $C\{W\}$ has no subpolyhedra of order higher than $(m + 1 - t)$.*

PROOF. Since the subspace $\mathcal{V}$ of $\mathcal{G}$ spanned by W has dimension $(m + 1 - t)$, no subset of W spans a subspace of dimension greater than $(m + 1 - t)$.

Note, incidentally, that the subpolyhedron of order $(m + 1 - t)$ of a proper lune of type t is the full $(t - 1)$-sphere.

LEMMA 4. *Consider a non-degenerate proper lune $C\{W\}$ of type t, with W spanning $\mathcal{V} \subset \mathcal{G}$. Consider a subspace $\mathcal{S}$ of $\mathcal{G}$, $\mathcal{V} \subset \mathcal{S} \subset \mathcal{G}$, determining the order-$s$ base $C\{W\} \cap \mathcal{S}$ corresponding to $\mathcal{S}$. Consider also a subset U of $\bar{W}$ determining the order-k subpolyhedron $C\{W \cup -U\} = C\{\mathcal{U}^{\perp}\tilde{U}\}$ of $C\{W\}$ corresponding to U, $C\{\mathcal{U}^{\perp}\tilde{U}\}$ assumed non-degenerate relative to $\mathcal{U}^{\perp}$. Then*

(1) *$C\{\mathcal{U}^\perp U\}$ is a proper lune of type t relative to $\mathcal{U}^\perp$; $C\{\mathcal{U}^\perp \tilde{U}\}$ possesses a base $B = C\{\mathcal{U}^\perp \tilde{U}\} \cap (\mathcal{S} \cap \mathcal{U}^\perp)$ corresponding to $\mathcal{S} \cap \mathcal{U}^\perp$, of order* $\dim (\mathcal{U}^\perp) - \dim (\mathcal{S} \cap \mathcal{U}^\perp) = (m + 1 - k) - (m + 1 - k - s) = s$ *relative to $\mathcal{U}^\perp$, and B is non-degenerate relative to* $(\mathcal{S} \cap \mathcal{U}^\perp)$.

(2) *the base $C\{W\} \cap \mathcal{S} = C(W \cup S^\perp \cup -S^\perp\}$ of $C\{W\}$ corresponding to $\mathcal{S}$ is non-degenerate relative to $\mathcal{S}$, and has an order-k subpolyhedron $P = C\{W \cup S^\perp \cup -S^\perp \cup -U\}$. (For the definition of a subpolyhedron of the base $C\{W\} \cap \mathcal{S}$ read Definition* 2.2.1 *with $\mathcal{S}$ substituted for $\mathcal{G}$; $\mathcal{S}W$ need not be substituted for W, since $W \varepsilon \mathcal{S}$ by assumption.)*

(3) *$B = P$, from which it follows that P is non-degenerate relative to $\mathcal{S} \cap \mathcal{U}^\perp$, since B is non-degenerate relative to $\mathcal{S} \cap \mathcal{U}^\perp$.*

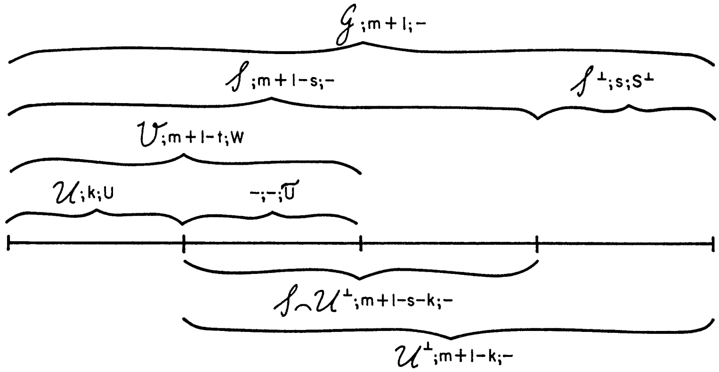

FIG. 1.

PROOF. Except for the non-degeneracy of $C\{W\} \cap \mathcal{S}$ and of B, which follows from Lemma 1.6, verification of (1) and (2) follows straightforwardly from the definitions; it may be facilitated by the schematic in figure 1, indicating the various subspaces of $\mathcal{G}$, their dimensions, and their spanning vector sets. (3) follows as readily, since

$$\begin{aligned} P = C\{W \cup S^\perp \cup -S^\perp \cup -U\} &= C\{W \cup -U \cup S^\perp \cup -S^\perp\} \\ &= C\{W \cup -U\} \cap \mathcal{S} = C\{W \cup -U\} \cap (\mathcal{S} \cap \mathcal{U}^\perp) \\ &= C\{\mathcal{U}^\perp \bar{U}\} \cap (\mathcal{S} \cap \mathcal{U}^\perp) = B. \end{aligned}$$

The content of Lemma 4 is summarized by the statement that a subpolyhedron of a base is the base of a subpolyhedron.

As will be shown below, subpolyhedra of order 2 corresponding to independent pairs (w_k, w_l) of supports of $C\{W\}$ are important in computing the spherical m-measure of $C\{W\}$. Equally important are the dihedral angles of $C\{W\}$ corresponding to (w_k, w_l).

DEFINITION 2. Given two independent vectors w_k and w_l of W, the dihedral angle of $C\{W\}$ corresponding to (w_k, w_l) is defined by

$$\theta_{k,l} = \arccos (-w_k \cdot w_l / |w_k| \cdot |w_l|) ; 0 \leqq \theta_{k,l} \leqq \pi.$$

Definition 3. Consider the order-k subpolyhedron $C\{W \cup -U\}$ of $C\{W\}$ corresponding to U: $w_i, w_j, \cdots, w_x$. Given two elements w_k and w_l of W such that $\mathcal{U}^\perp w_k$ and $\mathcal{U}^\perp w_l$ are independent, the dihedral angle of $C\{W \cup -U\}$ corresponding to (w_k, w_l) is defined by[3]

$$\theta_{k,l}^{i,j,\cdots,x} = \arccos\,(-\mathcal{U}^\perp w_k \cdot \mathcal{U}^\perp w_l / |\mathcal{U}^\perp w_k| \cdot |\mathcal{U}^\perp w_l|); \quad 0 \leqq \theta_{k,l}^{i,j,\cdots,x} \leqq \pi.$$

2.3. *Spherical simplices and their measures.* Since simplices with identical dihedral angles are congruent, differing at most in their position and orientation on the sphere, the $(m + 1)(m)/2$ dihedral angles of a simplex determine its measure; the dihedral angles determine as well existence:

Lemma 1. (Schläfli [29] and van der Vaart [33]). *Given $(m + 1)(m)/2$ numbers θ_{ij}, $0 \leqq \theta_{ij} \leqq \pi$, there exists a simplex with dihedral angles θ_{ij} if and only if the matrix $M = \|-\cos\theta_{ij}\|$ (with 1's on the diagonal) is positive definite.*

Proof. The Lemma follows from Definition 1.5, since the positive-definiteness of M is equivalent to the independence of the elements of W.

Lemma 2. *Given $(m + 1)(m)/2$ numbers θ_{ij}, $0 \leqq \theta_{ij} \leqq \pi$, there exists a proper lune (which may or may not be degenerate) with $(m + 1)$ supports and $(m + 1)(m)/2$ dihedral angles θ_{ij}, if and only if M (defined as in Lemma 1) is singular and positive semi-definite.*

Proof. The lemma follows from Definition 1.4, since M is singular if and only if the subspace spanned by W is of dimension less than $(m + 1)$.

Definition 1. The measure $|C\{W\}|$ (relative to $\mathcal{G}$) of the spherical polyhedron $C\{W\}$ is the spherical m-measure of $C\{W\} \cap \mathcal{G}$; in other words, $|C\{W\}|$ is the fraction of the unit m-sphere[4] in $\mathcal{G}$ occupied by $C\{W\} \cap \mathcal{G}$, multiplied by the spherical m-measure of the unit m-sphere.

Definition 2. The measure $|C\{W \cup -U\}|$ (relative to $\mathcal{U}^\perp$) of the subpolyhedron $C\{W \cup -U)$ of $C\{W\}$ is the spherical $(m - k)$-measure of $C\{W \cup -U\} \cap \mathcal{U}^\perp$; in other words, $|C\{W \cup -U\}|$ is the fraction of the unit $(m - k)$-sphere in $\mathcal{U}^\perp$ occupied by $C\{W \cup -U\}$, multiplied by the spherical $(m - k)$-measure of the unit $(m - k)$-sphere.

Schläfli [28] first showed that the dihedral angles of a spherical simplex constitute natural vehicles for the study of its measure. Adapting our notation to this fact, we shall often designate the measure of a simplex $C\{W\}$ by $|C[\Theta]|$ rather than by $|C\{W\}|$, emphasizing thereby the dependence of the measure on the set Θ of dihedral angles $\theta_{k,l}$ of $C\{W\}$. In addition, we shall usually write, say, $|C_m[\Theta]|$ to remind the reader that, say, spherical m-measure is being computed. Similarly, the symbol $|C_{m-k}[\Theta^{i,j,\cdots,x}]|$ will often be used in place of $|C\{W \cup -U\}|$ to denote the measure of a subpolyhedron.

The fundamental formula, first given by Schläfli, and also given by Plackett [22], Ruben [26], and van der Vaart [34], is

(1a) $\partial|C_m[\Theta]|/\partial\theta_{ij} = |C_{m-2}[\Theta^{i,j}]|/(m - 1);\ m \geqq 2$

[3] As pointed out by Ruben in a forthcoming monograph, $-\cos\theta_{k,l}^{i,j,\cdots,x}$ is the same function of the quantities $\{-\cos\theta_{m,n}\}$ that the partial correlation coefficient $\rho_{k,l\cdot i,\cdots,x}$ is of the marginal correlation coefficients $\{\rho_{m,n}\}$.

[4] The unit m-sphere is taken here to be the set of points $\{x: \sum_{i=1}^{m+1} x_i^2 = 1\}$.

(1b) $|C_1[\Theta]| = \theta$; (Note that, for $m = 1$, the set Θ consists of the single angle θ.)

(1c) $|C_0| = 1$. (Note that, for $m = 0$, there exists only one simplex C_0, namely the one-point "hemisphere").

Explicit expressions for the $(m-1)(m-2)/2$ elements $\theta_{k,l}^{i,j}$ of the set $\Theta^{i,j}$ in terms of the $(m)(m+1)/2$ elements $\theta_{k,l}$ of the set Θ, given by Schläfli on page 62 of [28], are obtained by a process more recently called *Grammian projection*:[5]

$$-\cos \theta_{k,l}^{i,j} = \Delta_{k,l}^{i,j}(\Delta_{k,k}^{i,j}\Delta_{l,l}^{i,j})^{-\frac{1}{2}} \tag{2}$$

where

$$\Delta_{k,l}^{i,j} = \begin{vmatrix} -\cos \theta_{k,l} & -\cos \theta_{k,i} & -\cos \theta_{k,j} \\ -\cos \theta_{i,l} & 1 & -\cos \theta_{i,j} \\ -\cos \theta_{j,l} & -\cos \theta_{j,i} & 1 \end{vmatrix}$$

and

$$-\cos \theta_{ii} \equiv 1.$$

Relation (1a) can be integrated as follows.[6] Suppose that it is desired to find $|C_m[\theta_1]|$ while $|C_m[\theta_0]|$ is known. Since both spherical polyhedra are assumed to be simplices, their respective dihedral angle cosine matrices, call them $\|-\cos \theta_{1;ij}\|$ and $\|-\cos \theta_{0;ij}\|$, will be positive definite by Lemma 1. Hence, for every t, $0 \leqq t \leqq 1$, the matrix $\|-\cos \theta_{ij}(t)\| \equiv \|(\cos \theta_{0;ij})(t-1) - (\cos \theta_{1;ij})(t)\|$ will be positive definite, so that, by Lemma 1, there will exist a family of simplices $C[\Theta(t)]$, $0 \leqq t \leqq 1$, with the elements $\theta_{i,j}(t)$ of $\Theta(t)$ given by

$$\theta_{ij}(t) = \arccos((\cos \theta_{0;ij})(1-t) + (\cos \theta_{1;ij})(t)). \tag{3}$$

Hence (1a) yields

$$|C_m[\Theta_1]| = |C_m[\Theta_0]| + (m-1)^{-1} \int_0^1 \sum_{(i,j)} (|C_{m-2}[\Theta^{i,j}(t)]|)(\dot{\theta}_{ij}(t))\, dt, \tag{4}$$

where the summation extends over the $(m+1)(m)/2$ pairs (i, j), the set $\Theta^{i,j}(t)$ is derived from the set $\Theta(t)$ by (2), and where the derivative $\dot{\theta}_{ij}(t)$ of $\theta_{ij}(t)$ with respect to t is obtained from (3).

2.4. *The measure of an equilateral spherical simplex.* For equilateral simplices (writing the single dihedral angle θ in place of the set Θ), (3.1a) and (3.2), as well as (3.1b) and (3.1c), become (Ruben [26])

(1a) $\partial|C_m[\theta]|/\partial\theta = ((m+1)(m)/2(m-1))(|C_{m-2}[\operatorname{arcsec}(\sec\theta - 2)]|)$; $m \geqq 2$,

(1b) $|C_1[\theta]| = \theta$,

(1c) $|C_0| = 1$.

The integration could be performed here, as in (3.4), by selecting a straight-

[5] See footnote 3.

[6] See also Plackett [22] and Ruben [26].

line path in cosine space; this choice was made for (3.4) because, although leading to an unwieldy integration formula, it simplified the necessary verification of existence. However, in the equilateral case, choosing a straight-line path p_c in cosine space amounts to choosing a straight-line path p_a in angle space. Hence a path p_c may be chosen to simplify the integration, while the corresponding path p_c still establishes the required existence. Lemmas 1 and 2 effect the integration of (1a) along a suitable straight-line path p_a in angle space.

LEMMA 1. $|C_m\,[\operatorname{arcsec} m]| = 0.$[7]

PROOF. Consider the $(m + 1)$ supports w_i of an equilateral simplex, each assumed normed to unit length. These supports satisfy the two sets of relations $w_i \cdot w_i = 1$ and $w_i \cdot w_j = \lambda$. Now suppose that $\lambda = -m^{-1}$. Then

$$\left(\sum w_i\right) \cdot \left(\sum w_i\right) = (m + 1) + (m)(m + 1)(\lambda) = 0,$$

so that $\sum w_i = 0$ and $|C_m\{W\}| = 0$ by Definition 2.2 and by Theorem 1.1.

LEMMA 2. $|C_m[\theta]| = \{[(m + 1)(m)]/2(m - 1)\} \int_{\operatorname{arcsec} m}^{\theta} |C_{m-2}\,[\operatorname{arcsec}\,(\sec t - 2)]|\,dt.$[8]

PROOF. The matrix with diagonal and off diagonal elements equal, respectively, to 1 and to $-(m + \epsilon)^{-1}$ is positive definite. Hence, by an argument analogous to that given in the paragraph preceding (3.3), the points of the half-open line segment p_c in cosine space from $(m^{-1}, \cdots, m^{-1})$ to $(\cos\theta, \cdots, \cos\theta)$ all determine simplices. Hence (1a) may be integrated along the half-open line segment p_a in angle space from $(\operatorname{arcsec} m, \cdots, \operatorname{arcsec} m)$ to $(\theta, \cdots, \theta)$, and the result follows from Lemma 1.

2.5. *The edge of an equilateral spherical simplex.*

DEFINITION 1. Let $C\{W\}$ be an equilateral simplex relative to an m-dimensional subspace of $\mathfrak{F}$. A subpolyhedron of order $(m - 1)$ of $C\{W\}$ is defined to be an edge of $C\{W\}$. The edge length of $C\{W\}$ is the measure of any of its subpolyhedra of order $(m - 1)$, or, by (4.1b), the dihedral angle ϕ of any of these subpolyhedra.

LEMMA 1. $\cos\phi = (\cos\theta)(1 - (m - 1)\cos\theta)^{-1}; \phi = \operatorname{arcsec}\,(\sec\theta - m + 1)$.

PROOF. By Definition 2.3, the lemma follows from a straightforward computation using Grammian projection (see (3.2), the Δ's now being $(m \times m)$ determinants with all non-unity elements equal to $-\cos\theta$).

Let $|C_m(x)|$ denote the measure of the equilateral simplex with edge $\operatorname{arcsec}\,(x + 1)$. The curved brackets of $|C_m(x)|$ indicate that measure is being considered as a function of $(\sec\phi - 1) \equiv x$. Introduction of the new argument

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[7] As pointed out by the referee, this result corresponds to the well-known fact that a $(m + 1)$-variate distribution with correlations all equal to $-1/m$ is singular, and therefore in particular the $(m + 1)$-variate normal distribution.

[8] In the present "acute" case, the integration limit arcsec m is more natural than the limit $\pi/2$ used by Ruben [26] for the "obtuse" case; the present limit further does away with the additive constant $(\frac{1}{2})^{m+1}$, a feature which is useful for the asymptotic computations below.

x will facilitate proofs in later sections. By Lemma 1, square brackets and curved are related by

$$(1)\qquad \begin{aligned} |C_m(x)| &= |C_m\,[\operatorname{arcsec}\,(x + m)]| \\ |C_m[\theta]| &= |C_m\,(\sec\theta - m)| \end{aligned}$$

and the relations (4.1) become in terms of x:

(2a) $\partial|C_m(x)|/\partial x = \dfrac{(m+1)(m)|C_{m-2}(x)|}{2(m-1)(x+m)((x+m)^2-1)}$; $\qquad m \geqq 2$,

(2b) $|C_1(x)| = \operatorname{arcsec}\,(x + 1)$,

(2c) $|C_0| = 1$.

2.6. *Arbitrary convex spherical polyhedra.* By decomposing arbitrary polyhedra into simplices, Schäfli [28] has shown that (3.1a) holds for the former as well as for the latter:

$$(1)\qquad d\,|C_m\{W\}| = (m-1)^{-1}\sum_{(i,j)} |C_{m-2}\{W \cup -u_i \cup -u_j\}|\, d\theta_{ij},$$

where the summation extends over all independent pairs (u_i, u_j) of elements of W. Observe that the required independence of u_i and u_j insures that $\mathfrak{U}^{\perp}$ has dimension $(m - 2)$. Hence, by Definition 3.2, the measures on the right of (1) are, as indicated by the subscript, spherical $(m - 2)$-measures. Observe as well that some of the subpolyhedra of order 2 entering (1) often will be degenerate, in which case their contribution to the summation is of course zero. Such a situation is illustrated by the specialization of (1) to the 2-sphere, where, unless the polyhedron in question is a proper lune, the second-order subpolyhedra corresponding to any two independent supports either are empty or equal C_0 (see (3.1c)). If the two supports correspond to adjacent sides the corresponding subpolyhedron equals C_0, and it is empty otherwise. Hence, by (3.1c), (1) becomes for $m = 2$:

$$(2)\qquad d\,|C_2\{W\}| = \sum{}^{*}\, d\theta_{ij}$$

where $\sum^*$ indicates summation over all generator pairs (u_i, u_j) corresponding to adjacent sides.

Equation (2), plus the fact that the interior angles of a Euclidean λ-gon add to $(\lambda - 2)(\pi)$, and the fact that near-degenerate convex spherical polyhedra are near-Euclidean, yields the familiar spherical excess formula for the spherical λ-gon:

$$(3)\qquad |C_2\{W\}| = \sum{}^{*}\, \theta_{i,j} - (\lambda - 2)\pi.$$

2.7. *The normed measure of a proper lune.*

Definition 1. Let Σ_m be the spherical m-measure of the unit m-sphere. The normed measure (relative to $\mathcal{G}$) of $C\{W\}$ equals $|C_m\{W\}|/\Sigma_m$. An analogous definition holds for subpolyhedra of $C\{W\}$.

Lemma 1. *The normed measure of any base of a proper lune equals the normed measure of the lune.*

PROOF. Let the base be of order s; we must show that

(1) $$|C_{m-s}\{W \cup S^{\perp} \cup -S^{\perp}\}|/\Sigma_{m-s} = |C_m\{W\}|/\Sigma_m .^9$$

This is proved straight forwardly by induction on m, using (6.1) and the fact that the base of a subpolyhedron of a proper lune is a subpolyhedron of a base of the lune (see Lemma 2.4).

3. The measure of an equilateral spherical simplex.

3.1. *Bounds*. The Euclidean $(m + 1)$-measure M of a Euclidean pyramid with altitude h and base of Euclidean m-measure b is ([31], p. 123)

(1) $$M = hb/(m + 1).$$

It follows from (1) that the Euclidean m-measure $|S_m(L)|$ of the m-dimensional equilateral Euclidean simplex $S_m(L)$ of side L is ([31], p. 125)

(2) $$|S_m(L)| = (m + 1)^{\frac{1}{2}}/(m!2^{m/2}) \cdot L^m.$$

Approximating the curved base by a Euclidean grid, (1) also implies that the Euclidean $(m + 1)$-measure $\mathfrak{M}$ of a spherical pyramid with altitude h and base of spherical m-measure b is given, as in (1), by

(3) $$\mathfrak{M} = hb/(m + 1).$$

Consider the equilateral spherical m-simplex $C_m(x)$ of edge arcsec $(x + 1)$, the notation being that introduced in (2.5.1). Let $|A|$ be the Euclidean m-measure of its exscribed equilateral Euclidean simplex A, and let $|B|$ be the Euclidean m-measure of its inscribed equilateral Euclidean simplex B.

The vertices of A are the intersections, with the hyperplane tangent to the unit m-sphere at the center of $C_m(x)$, of the vertices of $C_m(x)$, so that A has edge length

(4) $$(2x(m + 1))^{\frac{1}{2}}(x + m + 1)^{-\frac{1}{2}}$$

The edges of B are the line segments connecting the vertices of $C_m(x)$, so that B has edge length

(5) $$(2x)^{\frac{1}{2}}(x + 1)^{-\frac{1}{2}}$$

Let z_i be a vector oriented in the direction of the i'th vertex of the simplex A (or B). Formulas (4) and (5) are proved by noting that, for $|z_i| = r$, $z_i \cdot z_j = r^2/(x + 1)$, so that $|z_i - z_j| = (2r^2(1 - (x + 1)^{-1}))^{\frac{1}{2}}$, which yields (5) setting $r = 1$; on the other hand, $\bar{z} \cdot \bar{z} = r^2(m + x + 1)/(m + 1)(x + 1)$, or $r = |\bar{z}|(m + 1)(x + 1)/(m + x + 1)$, so that, setting $|\bar{z}| = 1$, $|z_i - z_j|$ is as given in (4).

Let $\mathfrak{A}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}$, and $\mathfrak{C}$ be, respectively, the Euclidean pyramid with base A and vertex at the origin, the Euclidean pyramid with base B and vertex at

[9] As pointed out by the referee, the probabilistic basis of this result is the joint normality of homogeneous linear functions of jointly normal random variables.

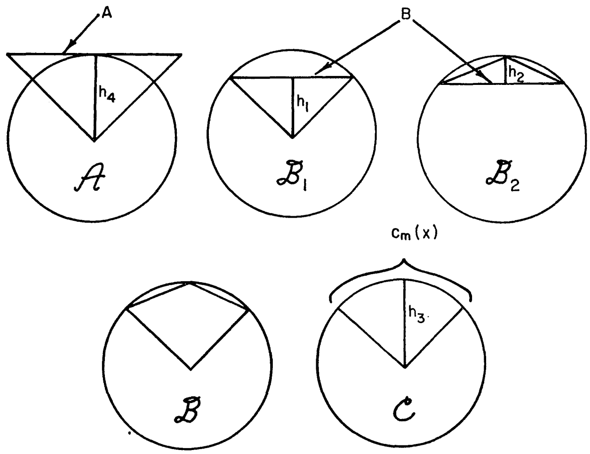

FIG. 2.

the origin, the Euclidean pyramid with base B and vertex at the center of $C_m(x)$, the $(2m + 2)$-sided Euclidean polyhedron $\mathcal{B}_1 \cup \mathcal{B}_2$, and the spherical Euclidean pyramid with base $C_m(x)$ and vertex at the origin.

It is clear that $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ so that $|\mathcal{B}_1| + |\mathcal{B}_2| = |\mathcal{B}| \leqq |\mathcal{C}| \leqq |\mathcal{A}|$ where the vertical bars indicate Euclidean $(m + 1)$-measure. Hence, by (1) and (3),

$$|B|\cdot h_1 + |B|\cdot h_2 \leqq |C_m(x)|\cdot h_3 \leqq |A|\cdot h_4, \tag{6}$$

where $|B|$ and $|A|$ are as defined in the paragraph following (3), and where h_1, h_2, h_3 and h_4 are the respective altitudes of $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$ and $\mathcal{A}$. But $h_1 + h_2 = h_3 = h_4 = 1$; hence (6) implies $|B| \leqq |C_m(x)| \leqq |A|$, or, by (2), (4) and (5),

$$\begin{aligned} \left(\frac{(m+1)^{\frac{1}{2}}}{m!}\right)\left(\frac{x}{(x+1)}\right)^{m/2} &\leqq |C_m(x)| \leqq \left(\frac{(m+1)^{\frac{1}{2}}}{m!}\right) \\ &\cdot \left(\frac{x(m+1)}{x+m+1}\right)^{m/2} = \left(\frac{(m+1)^{\frac{1}{2}}x^{m/2}e^{-x/2}}{m!}\right)\left(\frac{e^x}{\left(1+\dfrac{x}{m+1}\right)^m}\right)^{\frac{1}{2}} \end{aligned} \tag{7}$$

The factor $(e^x/\{1 + [x/(m + 1)]\}^m)^{\frac{1}{2}}$ clearly is approximately unity for m large or x small.

3.2. *A heuristic derivation.* In view of (1.7), it seems worthwhile to explore the possibility that

$$|C_m(x)| = \left(\frac{(m+1)^{\frac{1}{2}}}{m!}\right)(f(x))^{m/2}(g(x))(\eta(m, x)), \tag{1}$$

where $\eta(m, x)$ tends to unity for x fixed and m large. The program of this section is to produce tentative solutions for f and g; these tentative solutions are vindicated in the next section.

It follows from (1.7) that

$$\left(\frac{1}{x+1}\right)^{m/2} \leqq \frac{|C_m(x)|}{(x^{m/2})((m+1)^{\frac{1}{2}}/m!)} \leqq \left(\frac{m+1}{x+m+1}\right)^{m/2}$$

so that

$$\lim_{\substack{x\to 0\\ m \text{ fixed}}} \frac{|C_m(x)|}{(x^{m/2})((m+1)^{\frac{1}{2}}/m!)} = 1. \tag{2}$$

Substituting in (2) the expression (1) yields

$$\lim_{x\to 0} (f(x)/x)^{m/2}(g(x))(\eta(m, x)) = 1 \tag{3}$$

which suggests that

$$\lim_{x\to 0} f(x) = x, \tag{4}$$

$$\lim_{x\to 0} g(x) = 1. \tag{5}$$

Next substitute the expression of (1) into equation (2.5.2A). This yields

$$\dot{f}(x) + [2f(x)/m]\,[\dot{\ln}\, g(x) + \dot{\ln}\, \eta(m, x)] = [F(m)/F(m+x)][\eta(m-2, x)/\eta(m, x)], \tag{6}$$

where dotting indicates differentiation. Assuming that $\dot{\ln}\, \eta(m, x)$ tends to zero with m large and that $[\eta(m-2, x)/\eta(m, x)] - 1$ is of order less than m^{-1}, and using the fact that $F(m)/F(m+x) = 1 - 2x/m + o(m^{-1})$, equate terms of order 1 and terms of order m^{-1} in (6), giving respectively

$$\dot{f}(x) = 1 \tag{7}$$

and

$$(f(x))\,(\dot{\ln}\, g(x)) = -x. \tag{8}$$

Equation (7), together with (4), yields

$$f(x) = x, \tag{9}$$

so that equation (8), together with (5), yields

$$g(x) = e^{-x}. \tag{10}$$

Finally, if (1), (9) and (10) are true, then substituting (9) and (10) in (1), and substituting the resulting expression for $|C_m(x)|$ in (2), leads to

$$\lim_{x\to 0, m \text{ fixed}} \eta(m, x) = 1.$$

In summary, assuming (4) and (5), plus the smoothness assumption discussed following (6) one is lead from (1) to

$$|C_m(x)| = (x^{m/2}(m+1)^{\frac{1}{2}}/e^x m!)(\eta(m, x)), \tag{11}$$

where $\eta(m, x)$ tends to unity for x fixed and m large, and also for m fixed and x small.

It seems of interest to point out the similarity of (11) and the upper bound in (1.7). (11) is proved in the next section; note that the asserted behavior of $\eta(m, x)$ for m fixed and x small needs no further verification, since it follows immediately from (1.7).

3.3. *The asymptotic measure of an equilateral spherical simplex.* Let $\lambda(m, x)$ be the function $x^{m/2}(m + 1)^{\frac{1}{2}}/e^x m!$ of (2.11). This section is devoted to showing that $|C_m(x)|$ is uniformly asymptotically equal to $\lambda(m, x)$ in the following sense. Let

$$\eta(m, x) = |C_m(x)|/\lambda(m, x), \tag{1}$$

and let ξ be arbitrary and fixed. Then

$$\lim_{m\to\infty} (\sup_{0 \leqq x \leqq \xi} |\eta(m, x) - 1|) = 0. \tag{2}$$

The proof of (2) is as follows. Relation (2.5.2a) states that

$$\partial|C_m(x)|/\partial x = [(m + 1)(m)(|C_{m-2}(x)|)]/[(2)(m - 1)(F(m + x))] \tag{3}$$

where $F(t) = t(t^2 - 1)^{\frac{1}{2}}$. In view of Lemma 2.4.1 and (2.5.1), $|C_m(0)| = 0$, so that integrating (3) yields

$$\eta(m, x) = x^{m/2}e^x \int_0^x (t^{m/2}e^{-t}) \left(\frac{mF(m)}{2tF(m + t)} \cdot \eta(m - 2, t)\right) dt. \tag{4}$$

Relation (4) is now investigated for m large. Below, $K(x)$ will denote a constant depending on x, and not necessarily the same constant each time that it appears; $o_i(m, t)$ will denote a function of m and t such that $o_i(m, t) < K(x) \cdot (t/m)^i$ for m large and $0 \leqq t \leqq x$; $w_i(m, t)$ will denote a function of m and t such that

$$w_i(m, t) < K(x) \cdot m^{-i} \quad \text{for } m \text{ large and } \quad 0 \leqq t \leqq x. \tag{5}$$

The coefficient of $\eta(m - 2, t)$ in (4) equals

$$\begin{aligned}
&[m/2t][1 + t/m]^{-2}[(1 - m^{-2})/(1 - m^{-2}(1 + t/m)^{-2})]^{\frac{1}{2}} \\
&\qquad = [m/2t][1 - 2t/m + o_2(m, t)][1 + o_1(m, t)/(m^2 - 1)]^{-\frac{1}{2}} \\
&\qquad = (m/2t - 1) + o_1(m, t) + w_2(m, t) = (m/2t - 1) + w_1(m, t).
\end{aligned}$$

Using this last expression and working with the new functions

$$\epsilon(n, s) = (\eta(n, s) - 1)/s, \tag{6}$$

equation (4) becomes

$$\begin{aligned}
1 + x\epsilon(m, x) = x^{-m/2}e^x \int_0^x t^{m/2}e^{-t}(m/2t - 1)(1 + t\epsilon(m - 2, t))\, dt \\
+ R(m, x).
\end{aligned} \tag{7}$$

Since $\int_0^x t^{m/2}e^{-t}(m/2t - 1)\,dt = 1$, the right hand side of (7) equals

$$(8) \quad 1 + x^{-m/2}e^x \int_0^x t^{m/2+1}e^{-t}(m/2t - 1)(\epsilon(m-2, t))\,dt + R(m, x) =$$

$$(9) \quad 1 + \left(\frac{m}{m+2}\right)\left(x^{-m/2}e^x \int_0^x t^{\frac{1}{2}(m+2)}\, e^{-t}\left(\frac{m+2}{2t} - 1\right)(\epsilon(m-2, t))\,dt\right) + S(m, x) + R(m, x),$$

or

$$(10) \quad \epsilon(m, x) = \left(\frac{m}{m+2}\right)\left(x^{-\frac{1}{2}(m+2)}e^x \int_0^x \epsilon(m-2, t)\, d(t^{\frac{1}{2}(m+2)}e^{-t})\right) + (S(m, x) + R(m, x))/x.$$

The next step is to bound $\sup_{x \leqq \xi} |[S + R]/x|$ for m large. To this end, define

$$(11) \qquad M(n, x) \equiv \sup_{0 \leqq t \leqq x} |\epsilon(n, t)|.$$

Comparing (4) and (7),

$$\sup_{x \leqq \xi} |R(m, x)/x| = \sup_{x \leqq \xi} \left| x^{-\frac{1}{2}(m+2)}e^x \int_0^x t^{m/2}e^{-t}(w_1(m, t))(1 + t\epsilon(m-2, t))\,dt \right|,$$

and, since $\int_0^x t^{n-1}e^{-t}\,dt \leqq x^n e^{-x}/(n - x)$, this last expression is no greater than

$$\begin{aligned} \sup_{x \leqq \xi} \{&[\sup_{0 \leqq t \leqq x} (w_1(m, t))]/(m/2 + 1 - x) \\ &+ [\sup_{0 \leqq t \leqq x} (w_1(m, t)\,\epsilon(m-2, t))][x/((m/2) + 2 - x)]\} \\ = {}&[\sup_{0 \leqq t \leqq \xi} (w_1(m, t))]/[(m/2) + 1 - \xi] \\ &+ [\sup_{0 \leqq t \leqq \xi} (w_1(m, t)\,\epsilon\,(m-2, t))][\xi/((m/2) + 2 - \xi)], \end{aligned}$$

which, by (5), is in turn no greater than $K(\xi)m^{-2} + K(\xi)M(m-2, \xi)m^{-2}$. Summarizing this derivation of a bound for $\sup_{x \leqq \xi} |R(m, x)/x|$,

$$(12) \qquad \sup_{x \leqq \xi} |R(m, x)/x| \leqq K(\xi)m^{-2}(1 + M(m-2, \xi)).$$

A similar argument, following the comparison of (8) and (9), yields

$$(13) \quad \sup_{x \leqq \xi} |S(m, x)/x| = \sup_{x \leqq \xi} \left| \frac{2}{m+2}\left(x^{-\frac{1}{2}(m+2)}e^x \int_0^x t^{\frac{1}{2}(m+2)}e^{-t}\varepsilon\,(m-2, t)\,dt\right)\right| \leqq K(\xi)m^{-2}M(m-2, \xi).$$

Relations (10), (12), and (13) now yield that, for m large,

$$(14) \quad M(m, \xi) \leqq [m/(m+2)](M(m-2, \xi)) + [(K(\xi)/m^2](M(m-2, \xi)) + [K(\xi)/m^2].$$

Let p be a fixed integer large enough so that (12) holds with $m - 2 = p$. I show next that

$$M(p, \xi) < K < \infty. \tag{15}$$

By (1), $|C_p(x)| = \eta(p, x)\lambda(p, x) = \eta(p, x)(x^{p/2}(p + 1)^{\frac{1}{2}}/e^x p!)$, so that (1.7) yields

$$e^x/(1 + x)^{p/2} \leqq \eta(p, x) \leqq e^x \bigg/ \left(1 + \frac{x}{p + 1}\right)^{p/2}$$

or

$$\frac{e^x - (1 + x)^{p/2}}{x(1 + x)^{p/2}} \leqq \epsilon(p, x) \leqq \frac{e^x - \{1 + [x/(p + 1)]\}^{p/2}}{x\{1 + [x/(p + 1)]\}^{p/2}}.$$

Hence, to demonstrate (15), it is sufficient to show that a function of type $[e^x - (1 + Ax)^B]/[x(1 + Ax)^B]$ does not become infinite in the interval $0 \leqq x \leqq \xi$, which obviously is true, since, in the questionable range, i.e., near zero, both e^x and $(1 + Ax)^B$ are of the form $1 + O(x)$.

Iterating (14) for $m, m + 2, m + 4, \cdots$, and using (15),

$$M(p + 2(q + 1), \xi) \leqq K(\xi) \cdot \left(\frac{p + 2}{p + 2(q + 2)}\right) + K(\xi) \cdot \left(\sum_{i=1}^{q+1} \left(\frac{1}{p + 2i}\right)^2\right)$$

which, letting $q \to \infty$, implies that, for m large, $M(m, \xi) = O(m^{-1})$, or, recalling (6) and (11),

$$\sup_{0 \leqq x \leqq \xi} |[\eta(m, x) - 1]/x| = O(m^{-1}). \tag{16}$$

Since $(\xi/x) \geqq 1$ for $0 \leqq x \leqq \xi$, multiplying through by (ξ/x) in (16) yields

$$\sup_{0 \leqq x \leqq \xi} |\eta(m, x) - 1| = O(m^{-1}) \tag{17}$$

which, of course, implies (2).

4. Applications of the geometry.

4.1. *The mean among the order statistics in normal samples of size four.* Let $P(4, k)$ be the probability that, in a normal random sample of size four, the sample mean lies between the k'th smallest and $(k + 1)$'st smallest observation. Since $P(4, 3) = P(4, 1)$ and $P(4, 2) = 1 - 2P(4, 1)$, only $P(4, 1)$ need be computed.

Clearly

$$P(4, 1) = \binom{4}{1} \Pr\{x_1 \leqq \bar{x} \leqq x_2, x_3, x_4\}, \tag{1}$$

where the x_i's are unordered. (1) can be rewritten

$$\begin{aligned} P(4, 1) = 4 \Pr\{&-3x_1 + x_2 + x_3 + x_4 \geqq 0; \\ &-x_1 + 3x_2 - x_3 - x_4 \geqq 0; \\ &-x_1 - x_2 + 3x_3 - x_4 \geqq 0; \\ &-x_1 - x_2 - x_3 + 3x_4 \geqq 0\}. \end{aligned} \tag{2}$$

Let μ be the population mean. Conditionally on $(x_1 - \mu)^2 + \cdots + (x_4 - \mu)^2 = R^2$, the point (x_1, x_2, x_3, x_4) is distributed uniformly on the 3-sphere with center at (μ, μ, μ, μ) and radius R. Hence, conditionally on $(x_1 - \mu)^2 + \cdots + (x_4 - \mu)^2 = R^2$, the probability of (2) is the spherical measure of the subset $\{x : x \cdot w_1, x \cdot w_2, x \cdot w_3, x \cdot w_4 \geqq 0\}$ of the 3-sphere with radius R, relative to the spherical measure of this 3-sphere, where w_1, w_2, w_3 and w_4 are the vectors $(-3, 1, 1, 1)$, $(-1, 3, -1, -1)$, $(-1, -1, 3, -1)$ and $(1, -1, -1, 3)$. But this relative measure does not depend on R; hence, setting $R = 1$ and recalling Definition 2.1.1, (2) can be rewritten

$$P(4, 1) = 4|C_3\{w_1 ; w_2 ; w_3 ; w_4\}|/\Sigma_3 . \tag{3}$$

Now $w_1 = w_2 + w_3 + w_4$, so that $[x \cdot w_2, x \cdot w_3, x \cdot w_4 \geqq 0]$ implies $[x \cdot w_1 \geqq 0]$. Hence $C_3\{w_1 ; w_2 ; w_3 ; w_4\} = C_3\{w_2 ; w_3 ; w_4\}$, and (3) can be rewritten

$$P(4, 1) = 4|C_3\{w_2 ; w_3 ; w_4\}|/\Sigma_3 .$$

Since w_2, w_3 and w_4 are independent, $C_3\{w_2 ; w_3 ; w_4\}$ is a proper lune of type 1 (see Definition 2.1.4), and, by (2.7.1),[10]

$$P(4, 1) = 4|C_2\{w_2 ; w_3 ; w_4\}|/\Sigma_2 . \tag{4}$$

The three dihedral angles arcos $[-w_i \cdot w_j/|w_i| \cdot |w_j|]$ of $C_2\{w_2 ; w_3 ; w_4\}$ are equal, and equal to arcsec 3. Hence $C_2\{w_2 ; w_3 ; w_4\}$ is the equilateral spherical simplex C_2 [arcsec 3] with dihedral angle arcsec 3, and

$$P(4, 1) = 4|C_2 \text{ [arcsec 3]}|/\Sigma_2 . \tag{5}$$

Equation (2.6.3) for $\lambda = 3$ is the spherical excess formula for spherical triangles, which yields in this case $|C_2 \text{ [arcsec 3]}| = 3 \text{ arcsec } 3 - \pi$. Since $\Sigma_2 = 4\pi$, (5) therefore can be rewritten

$$P(4, 1) = [(3 \text{ arcsec } 3)/\pi] - 1. \tag{6}$$

4.2. *Normal samples of size five.* The definition of $P(5, k)$ is analogous to the definition of $P(4, k)$. Since $P(5, 4) = P(5, 1)$ and $P(5, 3) = P(5, 2) = (1 - 2P(5, 1))/2$, only $P(5, 1)$ need be computed.

An argument similar to that leading from (1.1) to (1.5) shows that

$$P(5, 1) = 5|C_3 \text{ [arcsec 4]}|/\Sigma_3 \tag{1}$$

and Schläfli's decomposition[11] of spherical tetrahedra into "orthoschemes" (see for example p. 155 of [5]) yields

$$|C_3 \text{ [arcsec 4]}| = 24|W|, \tag{2}$$

[10] H. Ruben has pointed out that R. A. Fisher used a dimension reduction in [11] that is similar to the step from (3) to (4); indeed, this type of argument appears already in Schläfli's work.

[11] The following derivation of the value of $P(5,1)$, suggested by the referee, replaces a more cumbersome earlier derivation.

where $|W|$ is the spherical measure of the spherical simplex W with dihedral angles $\theta_{12} = \theta_{23} = \pi/3$, $\theta_{34} = (\text{arcsec } 4)/2$, and $\theta_{13} = \theta_{14} = \theta_{24} = \pi/2$. But, as is pointed out by Plackett in [22] (eleventh line of the table on p. 359 of [22]), Schläfli's exact value for the Schläfli function $f(\alpha, \beta, \nu)$ at $(\alpha, \beta, \nu) = (\pi/3, \pi/3, (\text{arcsec } 4)/2)$ yields

$$(3) \qquad |W|/\Sigma_3 = (\text{arcsec } 4)/48\pi - \tfrac{1}{120},$$

so that (1), (2) and (3) yield

$$(4) \qquad P(5, 1) = (5 \text{ arcsec } 4)/2\pi - 1.$$

4.3. *Normal samples of arbitrary size.* Let $(x_1, \cdots, x_n)$ be an unordered normal sample, and set

$$R(n, k) = \Pr\{x_1, \cdots, x_k \leqq \bar{x} \leqq x_{k+1}, \cdots, x_n\}.$$

Define as well

$$(1) \qquad Q(n, k) = \Pr\{\bar{x} \leqq x_{k+1}, \cdots, x_n\}.$$

It is clear that $Q(n, k) = \sum_{i=0}^{k} \binom{k}{i} R(n, i)$, $0 \leqq k \leqq n$, or

$$(2) \qquad (Q(n, 0), \cdots, Q(n, n)) = (R(n, 0), \cdots, R(n, n))(M),$$

where M is a matrix with elements M_{ij} equal to $\binom{j}{i}$, so that inverting (2) yields

$$(3) \qquad R(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} Q(n, i), \quad 0 \leqq k \leqq n.$$

Further, geometric interpretation of (1) and dimension reduction based on (2.7.1) give

$$(4) \qquad Q(n, i) = |C_{n-i-1}[\text{arcsec } (n-1)]|/\Sigma_{n-i-1}.$$

Finally, defining $P(n, k)$ analogously to $P(4, k)$ and $P(5, k)$, it is clear that

$$(5) \qquad P(n, k) = \binom{n}{k} R(n, k),$$

so that (3), (4) and (5) yield

$$(6) \qquad P(n, k) = \binom{n}{k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} |C_{n-i-1}[\text{arcsec } (n-1)]|/\Sigma_{n-i-1}.$$

A form of (6) more convenient for computation is obtained by applying Lemma 2.4.1 and (2.5.1), and by changing the summation index from i to $j = k - i$. This yields

$$(7) \qquad P(n, k) = \binom{n}{k} \sum_{j=0}^{k-1} (-1)^{j} D(n, k, j),$$

where

$$D(n, k, j) = \binom{k}{j} |C_{n-k+j-1}(k - j)|/\Sigma_{n-k+j-1}. \tag{8}$$

Given (7), bounds for $P(n, k)$ can be computed using (3.1.7) and the fact that ([3], p. 303)

$$\Sigma_m = 2\pi^{\frac{1}{2}(m+1)}/\Gamma(\tfrac{1}{2}(m + 1)). \tag{9}$$

Thus, for example, $P(n, 1)$, which equals $n|C_{n-2}(1)|/\Sigma_{n-2}$ by (7), is bounded from above by

$$\left(\frac{n-1}{n}\right)^{\frac{1}{2}(n-2)} \cdot \frac{(n)(n-1)^{\frac{1}{2}}\Gamma((n-1)/2)}{2(n-2)!\pi^{(n-1)/2}}$$

and from below by

$$\left(\frac{1}{2}\right)^{\frac{1}{2}(n-2)} \cdot \frac{(n)(n-1)^{\frac{1}{2}}\Gamma((n-1)/2)}{2(n-2)!\pi^{(n-1)/2}}.$$

4.4. *The mean among the extreme order statistics.* The asymptotic probability that the mean falls among the extreme order statistics is computed in a series of lemmas.

LEMMA 1. *Let $D(n, k, j)$ be defined as in* (3.8); *then*

$$\lim_{n\to\infty} \{D(n, k, j)/D(n, k, j + 1)\} = \infty, \quad 0 \leqq j \leqq k - 2. \tag{1}$$

PROOF. For $0 \leqq j \leqq k - 1 \leqq n - 2$,

$$\begin{aligned} D(n, k, j) = {} & \left(\binom{k}{j}(k - j)^{\frac{1}{2}(n-k+j-1)}\right) \\ & \cdot ((n - k + j)^{\frac{1}{2}})(\Gamma(\tfrac{1}{2}(n - k + j)))(\eta(n - k + j - 1, k - j))/ \\ & (2(\pi)^{\frac{1}{2}(n-k+j)})(e^{k-j})((n - k + j - 1)!) \end{aligned} \tag{2}$$

by (3.8), since Σ_m is given by (3.9) and $|C_m|$ is given by (3.3.1).

Hence, for $0 \leqq j \leqq k - 2 \leqq n - 3$, $D(n, k, j)/D(n, k, j + 1)$ equals

$$\begin{aligned} & \left(\frac{\binom{k}{j}}{\binom{k}{j+1}}\right)\left(\frac{(k - j)^{\frac{1}{2}(n-k+j-1)}}{(k - j - 1)^{\frac{1}{2}(n-k+j)}}\right)\left(\frac{(n - k + j)^{\frac{1}{2}}}{(n - k + j + 1)^{\frac{1}{2}}}\right) \\ & \left(\frac{\Gamma(\frac{1}{2}(n - k + j))}{\Gamma(\frac{1}{2}(n - k + j + 1))}\right)\left(\frac{\eta(n - k + j - 1, k - j)}{\eta(n - k + j, k - j - 1)}\right) \\ & \left(\frac{(\pi)^{\frac{1}{2}(n-k+j+1)}}{(\pi)^{\frac{1}{2}(n-k+j)}}\right)\left(\frac{e^{k-j-1}}{e^{k-j}}\right)\left(\frac{(n - k + j)!}{(n - k + j - 1)!}\right) \end{aligned} \tag{3}$$

For n, k and j in the range $0 \leqq j \leqq k - 2 \leqq n - 3$, the first factor of (3) is a constant; the second factor of (3) equals $1/(1 - (k - j)^{-1})^{n-k+j/2}(k - j)^{\frac{1}{2}} \geqq 1/(1 - k^{-1})^{(n-k)/2}(k)^{\frac{1}{2}}$; the third factor of (3) is no smaller than

$$(n - k)^{\frac{1}{2}}/(n - k + 1)^{\frac{1}{2}};$$

the fourth factor of (3) is no smaller than $n - 1$; by (3.3.2), the fifth factor of (3) tends to 1 with n large; the sixth and seventh factors of (3) are constants; and the eighth factor is no smaller than $(n - k)$. Hence, for n large, the second factor of (3) exceeds any power of n, whereas all other factors of (3) exceed n^{-1}, which verifies (1).

LEMMA 2. *Let $P(n, k)$ be defined as in* (3.5). *Then*

$$\lim_{n\to\infty} \left\{P(n, k) \Big/ \left[\binom{n}{k} D(n, k, 0)\right]\right\} = 1. \tag{4}$$

PROOF. For n large, the D's of (3.7) decrease and the second is negligible with respect to the first. Hence, in accordance with a fundamental property of sums of oscillating terms, $P(n, k)$ is approximated by the leading term

$$\binom{n}{k} D(n, k, 0)$$

of (3.7).

LEMMA 3.

$$\lim_{n\to\infty} \left\{\binom{n}{k} D(n, k, 0)/f(n, k)\right\} = 1, \tag{5}$$

where

$$f(n, k) = [k^{n-k-1}e^{n-2k}/2(k!)^2(n)^{n-3k-1}(2\pi)^{n-k}]^{\frac{1}{2}}. \tag{6}$$

PROOF. Consider the expression for $D(n, k, 0)/\eta(n - k - 1, k)$ obtained from (2) by setting $j = 0$ and dividing by $\eta(n - k - 1, k)$. Easy simplifications using Sterling's formula show that $\binom{n}{k} D(n, k, 0)/\eta(n - k - 1, k)$ is asymptotically equal to $f(n, k)$. Hence (5) is proved if it can be shown that

$$\lim_{n\to\infty} \eta(n - k - 1, k) = 1. \tag{7}$$

But (7) is an immediate consequence of (3.3.2).

THEOREM 1. *Let $P(n, k)$ be defined as in* (3.5), *and $f(n, k)$ by* (6). *Then* $\lim_{n\to\infty} \{P(n, k)/f(n, k)\} = 1$.

PROOF. The proof follows from Lemmas 2 and 3.

4.5. *Calculations.* It may be of interest to show $P(n, k)$ for several values of n and k, computed exactly when possible, computed by the asymptotic $f(n, k)$ of (4.6), and computed also by a National Bureau of Standards Monte Carlo run whose results kindly were made available to me by J. Youden. Note that Monte Carlo sample size was 7000 for $n = 4$, and 1000 for all other n; also, letting $\hat{P}(n, k)$ denote the Monte Carlo estimate of $P(n, k)$, the Monte Carlo estimates reported here are not $\hat{P}(n, k)$, but rather the more stable averages $(\hat{P}(n, k) + \hat{P}(n, n - k))/2$.

	Exact Value	$f(n, k)$	Monte Carlo
$P(4, 1)$	.175	.122	.174
$P(5, 1)$	.049	.036	.049
$P(6, 1)$	.011*	.0088	.011
$P(7, 1)$		.0019	.0025
$P(7, 2)$		.064	.094
$P(8, 1)$		.0004	.0015
$P(8, 2)$		.002	.033

* Obtained by numerical integration of the expression verified in Lemma 2.4.2.

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