ABSTRACTS OF PAPERS

(Abstract of a paper presented at the Central Regional Meeting, Manhattan, Kansas, May 7–9, 1964. Additional abstracts appeared in the June issue.)

22. On the Construction of a Class of Optimum Balanced Factorial Fractions by Linear Integer Programming. J. N. Srivastava, University of Nebraska.

In this paper the construction of a class of balanced fractions for $(2^n \times 3^m)$ $(n \geqq 0, m \geqq 0)$ factorials which would permit the estimation of all the main effects and two factor interactions has been considered. For any fixed m and n, different criteria for calling a fraction "optimum" could be laid down, e.g. $|\Sigma|$, trace Σ, or largest root of Σ $(\substack{\mathrm{ch}\ \Sigma \\ \mathrm{max}})$. The merits of these have been compared, and the last one is selected. Finally it has been shown how the problem of construction of a balanced fraction with a minimum number of assemblies, and for which $(\substack{\mathrm{ch}\ \Sigma \\ \mathrm{max}}) < c$ (a constant > 0), could be reduced to a problem in linear integer programming. This has been achieved by using the properties of the association schemes arising in factorial designs, as discussed by Bose and Srivastava in University of North Carolina, Institute of Statistics, mimeo series numbers 373 and 376.

(Abstracts of papers presented at the Annual Meeting, Amherst, Massachusetts, August 26–29, 1964. Additional abstracts appeared in earlier issues and others will appear in the December issue.)

8. Missing Values in Multiple Regression. A. Afifi and R. M. Elashoff, University of California, Berkeley.

We propose, survey and evaluate probabilistically several estimation methods for the unknown parameters in a multiple regression when some of the observations on the independent and dependent variables are missing. Point and interval estimation are considered, univariate and multivariate predictands are taken up, Bayesian and non-Bayesian techniques are used. Simple recommendations as to the preferred method of estimation in particular instances are given by tabular presentations. Our work quantifies and spells out in detail the widely held view of data analysts that classical least squares estimation should not be used when the number of vector observations with at least one component missing is large.

9. Conditional Expectation Given a σ-Lattice, and Applications. H. D. Brunk, University of Missouri. (Invited).

In the past ten years a number of writers have studied problems whose solutions may be expressed as conditional expectations given σ-lattices [Brunk, *Proc. Amer. Math. Soc.* **14** (1963) 298–304]. Many properties of conditional expectation given a σ-field carry over, including, for example, mean-square and almost sure convergence of (generalized) martingales. Applications discussed are of two kinds; in each the solution of a problem posed is the conditional expectation of a given function on a finite set, given a prescribed sub-σ-lattice of the class of all subsets. In problems of the first type, observed values are given of random variables whose joint distribution depends on an unknown function on a prescribed space, given to be measurable with respect to a prescribed σ-lattice. In problems of the second type, a measure space $(T, \mathcal{B}, \nu)$ is given, and observations are made on T

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according to a probability density f. The density f is to be estimated, subject to the condition that it is determined in a prescribed way by an unknown function which is, however, known to be measurable with respect to a prescribed sub-σ-lattice of $\mathcal{B}$.

10. Allocation in Multivariate and Analytical Surveys. SAMPRIT CHATTERJEE, Harvard University.

The problem of allocation in multivariate and analytical surveys has been engaging lately a considerable amount of attention. If the allocation problem is formulated such that the variances of the estimates meet certain prescribed tolerances, and the objective is to minimise cost (the cost function being linear) then the allocation problem reduces itself to a problem of minimisation of a convex function subject to a set of constraints. This formulation covers a number of sampling schemes such as stratified sampling, two stage sampling, and analytical surveys. An algorithm is presented for the solution of the above problem and a number of numerical solutions are obtained. An alternative method called the method of augmented sampling is also discussed. In this method there is a basic sample in which we measure both variates, this is augmented by another in which we measure the second variate alone; the samples being so chosen that the variances of the estimates meet their prescribed tolerances. Situations where this is likely to prove useful are discussed. An attempt is also made to formulate working rules regarding the suitability of variates in the inclusion of a single survey.

11. A Bayes Sequential Sampling Inspection Plan. HERMAN CHERNOFF and S. N. RAY, Stanford University.

Given a lot of size N whose items are obtained from a statistically controlled process with an unknown probability p of an item being defective, a rectifying sampling inspection problem arises when the loss involved in sending out the lot is proportional to the number of defectives in it while the cost of inspection is proportional to the number of items inspected. Assuming a beta prior distribution for p, the optimum (Bayes) procedure is computed by backward induction. The asymptotic behavior for large N of the optimum solution is found to be related to that of an associated Wiener process problem in the sense that they both give rise to the same free boundary problem. The latter problem is tackled with the help of the techniques developed in Chernoff (*Proc. Fourth Berkeley Symp. Math. Statist. Prob.*, *Ann. Math. Statist.* (1964), Stanford Technical Report 90) and then reinterpreted in terms of the sampling inspection problem. Lastly, the optimum procedure is compared with the best linear procedure in terms of the associated problem.

12. A Birth-Illness-Death Process. CHIN LONG CHIANG, University of California, Berkeley.

Human population is divisible into a number of groups according to illness. An individual is said to be in a particular "state of illness" when affected with the corresponding disease; he may leave a "state" either through recovery, death, or by contracting other diseases. The number of individuals in each "state" at time t depends on the birth, death, and illness taking place prior to t. For every t, $0 \leqq t < \infty$, let an R-dimensional stochastic process $\mathbf{N}(t) = (N_1(t), \cdots, N_R(t))'$ be the numbers of individuals living in R illness "states" $(s_1, \cdots, s_R)$ at time t; the sum $N(t) = N_1(t) + \cdots + N_R(t)$ is the total population size. For each time element $(t, t + h)$, let $\lambda_\alpha^*(t)h + o(h)$ be the probability of a birth occurring in "state" s_α; $\mu_{\alpha\beta}^*(t)h + o(h)$, a death occurring in s_α from risk d_β; and $\nu_{\alpha\beta}^*(t)h + o(h)$, one individual will leave s_α for s_β, for $\alpha, \beta = 1, \cdots, R$. A general system of differential equations for the probability distribution of $\mathbf{N}(t)$. When $\lambda_\alpha^*(t) = \lambda_\alpha$, and $\mu_{\alpha\beta}^*(t)$ and $\nu_{\alpha\beta}^*(t)$

are proportional to the number of individuals living in state s_α at time t but otherwise independent of t, an explicit function is obtained for the probability distribution of $\mathbf{N}(t)$. Various aspects of the process are discussed.

13. A Non-Linear Integral Equation and Its Application to Critical Branching Processes (Preliminary report). J. Chover and P. E. Ney, University of Wisconsin and Cornell University.

Let f be a (probability) generating function with $f'(1) = 1$, $f'''(1) < \infty$. Let G be a distribution function on the real line such that $G(0-) = 0$, $G(0) < 1$, $1 - G(t) = O(t^{-3})$. Then: *Theorem.* If F is a non-negative bounded function defined on the non-negative real line which satisfies the equation $F(t) = \int_0^t f[F(t-y)]\, dG(y)$, then $\lim_{t\to\infty} t[1 - F(t)] = b$; where $b = 2m/f''(1)$, $m = \int_0^\infty t\, dG(t)$. $F(t)$ represents the probability that a critical age-dependent branching process be extinct at time t.

14. Small Sample Tests for the Mean and Variance of the Weibull Distribution. Satya D. Dubey, Procter & Gamble Co. (By title)

Recognizing that the standard functions of the cumulants of the Weibull distribution depend upon only shape parameter and functions of its sample moments tend to normality, the Cornish-Fisher technique has been employed for determining percentiles of the relevant statistics for testing the hypotheses about the mean and the variance of the Weibull distribution. Three test statistics have been considered which are analogous to normal, chi-square and Student's t. Some sampling studies are planned to compare the accuracy of percentiles obtained by the Cornish-Fisher method. Since the mean of the Weibull distribution is a function of all its three parameters, it can be argued that testing the hypothesis about its mean is equivalent to simultaneous testing of all its three parameters. Likewise, the variance of the Weibull distribution is a function of its scale and shape parameters, so testing the hypothesis about the variance is equivalent to simultaneous testing of its scale and shape parameters. When the variance is known, the use of normal test will imply testing the hypothesis about its location parameter. The computation becomes quite involved in extending this approach to $k(\geqq 2)$-sample problems. The results of this paper are applicable in other non-normal cases too.

15. Some Test Functions for the Parameters of the Weibull Distributions. Satya D. Dubey, Procter & Gamble Co.

Appropriate test functions for testing several hypotheses about the location (threshold), scale and shape parameters of the Weibull distribution are proposed. A test function based on $(b - a + 1)$ ordered observations and to simple test functions based on ath and bth ordered observations $(1 \leqq a < b \leqq n)$ and rth ordered observation $(1 \leqq r \leqq n)$ in a random sample of size n are suggested for simultaneous testing of all its three parameters. The $(b - a + 1)$-observation statistic obeys the chi-square law with $2(b - a)$ degrees of freedom and the two simple statistics are reduced to equivalent statistics which obey F-distributions with $2(b - a)$ and $2(n - b + 1)$ degrees of freedom and $2r$ and $2(n - r + 1)$ degrees of freedom respectively. When the scale parameter is unknown, two new statistics, based on $(b - a + 1)$ ordered observations and ath and bth ordered observations are considered for testing the hypotheses about the other two parameters. The probability density functions of these statistics are derived and necessary formulas are established for computing the desired percentiles. The test functions, introduced in this paper, can enable one to test several other relevant hypotheses. Some extensions of the present approach to $k(\geqq 2)$-sample problems are also considered in the paper.

16. Extremal Processes II. Meyer Dwass, Northwestern University.

$Y(t)$, $t \geqq 0$, is called an extremal process (EP) if the c.d.f. of $Y(t)$ is $\exp - tQ(x) = F(t, x)$ and the joint law of $Y(t_1), Y(t_2), \cdots, Y(t_n)$ is that of U_1; $\max U_1, U_2$; $\cdots \max U_1, \cdots, U_n$, where the U_i's are independent with c.d.f.'s $F(t_1, x), F(t_2 - t_1, x), \cdots, F(t_n - t_{n-1}, x)$, $0 < t_1 < \cdots < t_n$. The processes in *Ann. Math. Statist.* (abstract) **34** (1963) 1625 are EP's. For another example, let $M(t)$ be the maximum discontinuity in an i.d. process on $(0, t)$, and h a continuous 1-1 function. Then $h(M(t))$ is EP. Also any EP can be represented in this way. EP's are Markovian, step functions (away from origin). The number of jumps in (u, v) is Poisson with parameter $\log v/u$ if and only if Q is continuous. A detailed structural analysis of the evolution of EP's is obtained. Generalizations are made to the multivariate case. (Replace the word "maximum" above by "first k largest.")

17. Sufficiency for Independent Variables. D. A. S. Fraser, University of Toronto.

For sampling from a probability distribution having a fixed carrier region of positive density, the Koopman-Darmois-Pitman problem is concerned with distribution forms admitting a sufficient statistic whose dimension remains fixed under increasing sample size. Most analyses in the literature assume a real variable, a real parameter, and analyticity of the density function, and they find that the distribution form must be exponential. For an open but possibly disconnected sample space, a generalized exponential model is defined. The likelihood-function statistic is used to describe the local form of the minimal sufficient statistic; this gives a characterization of the generalized exponential model. The Koopman-Darmois-Pitman result is then available on the assumption of an Euclidean variable, a general parameter, and sample-space differentiability of the density function for all θ. A Barankin-Maitra extension of the Koopman-Darmois-Pitman problem is also available under these assumptions.

18. Maximum Likelihood Estimation of the Parameters of the Beta Distribution From Smallest Order Statistics. R. Gnanadesikan, R. S. Pinkham and Mrs. L. P. Hughes, Bell Telephone Laboratories, Incorporated.

Numerical methods, that can be used with computers, are described for obtaining the maximum likelihood estimates of the two parameters of a beta distribution using the smallest M order statistics, $u_1 \leqq u_2 \leqq \cdots \leqq u_M$, in a random sample of size $K(\geqq M)$. For this formulation in terms of order statistics, the maximum likelihood estimates are functions only of the ratio, $R = K/M$, the Mth ordered observation, u_M, and the two statistics, $G_1 = [\prod_{i=1}^{M} u_i]^{1/M}$ and $G_2 = [\prod_{i=1}^{M} (1 - u_i)]^{1/M}$. Tabulation of the roots of the likelihood equations in general would therefore entail four-way tables and, hence, explicit tables of the maximum likelihood estimates are given only for the case of a complete sample (i.e., $R = 1$) for a grid of values of the corresponding G_1 and G_2. Some examples of the use of the procedures described are given.

19. On the Selection of the First Sample Size in Some Two-Stage Sampling Problems. Aaron Goldman, Los Alamos Scientific Laboratory, University of California.

The expected total sample size of the methods of Graybill, (F. Graybill, "Determining sample size for a specified width confidence interval," *Ann. Math. Statist.* **29** (1958) 282–287), Stein (C. Stein, "A two sample test for a linear hypothesis where power is independent of the variance," *Ann. Math. Statist.* **16** (1945) 243–258), and Birnbaum and Healy (A. Birnbaum and W. Healy, Jr., "Estimates with prescribed variance based on two-stage sampling," *Ann. Math. Statist.* **31** (1960) 710–720) are compared for setting a prescribed width confidence interval on the mean and variance of a normal population.

20. Estimation of Response Function in a Dynamic Regression Model. ZAKKULA GOVINDARAJULU and YUKIO SUZUKI, Case Institute of Technology. (By title)

This paper is concerned with the estimation of the impulse response function in a time-invariant, linear system for the input in a finite time interval and observation of output at discrete time points. The time-discrete and time continuous models are discussed. The criterion of optimality used is the minimum square error. For the time-discrete model, "dead time" between input and output, and replication of observations are also considered.

21. On the Convergence Rate in the Law of Large Numbers for Linear Combinations of Independent Random Variables. D. L. HANSON and L. H. KOOPMANS, University of Missouri and Sandia Laboratory.

Let $\{\xi_k\}$ be a sequence of independent random variables with zero means, and with moment generating functions which exist in a common interval about zero and whose derivatives converge uniformly to zero at zero. Define $S_n = \sum_k a_{n,k}\xi_k$ and suppose $A > 0$. For each $\epsilon > 0$ there exists $0 < \rho < 1$ such that if $\sum_k |a_{n,k}| \leqq A$ and $f(n) = \max_k |a_{n,k}|$ then $P\{|S_n| > \epsilon\} \leqq 2\rho^{1/f(n)}$. This result is used to obtain exponential convergence in the strong law of large numbers for all subsequences of a linear process of the form $\{X_n = \sum_k b_{n-k}\xi_k\}$ where $\sum_k |b_k| < \infty$. The result is also applied to summability theory yielding (among other results): if $\alpha > 0$ and S_n^α is the nth Toeplitz partial sum of $\{\xi_k\}$ for the Cesaro summability method of order α, then S_n^α converges to zero in probability.

22. Convergence Rates for Linear Combinations of Exchangeable Random Variables. D. L. HANSON and L. H. KOOPMANS, University of Missouri and Sandia Laboratory.

Let $\{\xi_k : k = 0, \pm 1, \pm 2, \cdots\}$ be a process of exchangeable random variables with zero means. Then the ξ_k's are conditionally independent and identically distributed relative to some sub sigma field $\mathcal{B}$ of the underlying sigma field. Let $\{a_{n,k} : k = 0, \pm 1, \pm 2, \cdots; n = 1, 2, \cdots\}$ be real constants satisfying (i) $\sum_k |a_{n,k}| \leqq A < \infty$ all n, and (ii) $f(n) = \sup_k |a_{n,k}| \to 0$ as $n \to \infty$. Then $S_n = \sum_k a_{n,k}\xi_k$ is defined a.s. and as a l.i.m. of order 1. The basic result is, *Theorem 1*. For $0 < \beta < 1$, let $T_\beta(\omega)$ be the largest value of $|t|$ such that the conditional ($\mathcal{B}$) moment generating function of ξ_k is no larger than $\exp \beta |t|$. Then the moment generating function, $g_\beta(t)$, of $T_\beta(\omega)$ exists for all $t \leqq 0$ and for every $\epsilon > 0$ there exists $\beta > 0$ and a negative number t^* such that $P[|S_n| \geqq \epsilon] \leqq 2g_\beta(t^*/f(n))$. This theorem is employed to obtain various rates of convergence to zero of the S_n's. For example, explicit sufficient conditions for exponential and algebraic convergence rates can be given in terms of the distribution of $T_\beta(\omega)$.

23. Expected Values of Exponential, Weibull, and Gamma Order Statistics. H. LEON HARTER, Aerospace Research Laboratories.

Five-decimal-place tables, accurate to within a unit in the last place, are given of the expected values of the Mth order statistics $[M = 1(1)N]$ of samples of size N from the exponential population [$N = 1(1)120$] and from the Weibull and Gamma populations [$N = 1(1)40$]. In each case, the values of the location and scale parameters are assumed to be 0 and 1, respectively. Results are tabulated for the Weibull population with shape parameter $K = 0.5(0.5)4(1)8$ and for the Gamma population with shape parameter $\alpha = 0.5(0.5)4$, as well as for the exponential population, which is a special case (shape parameter 1) of each of the other two populations. Also given is an eight-decimal-place table, accurate to within a unit in the last place, of the moments (mean, variance, skewness, and kurtosis) of the exponential population and of the Weibull and Gamma populations with the above-mentioned values of the shape parameters. The mathematical formulations are given, along with a description of the methods of computation and a discussion of uses of the tables.

24. **Limit Theorems for Queues With Traffic Intensity One.** DONALD L. IGLEHART, Cornell University.

Consider the single server queueing process with Poisson arrivals and general service time. Assume that the customers are served in the order of their arrival and that the traffic intensity is equal to one. If the service distribution belongs to the domain of attraction of a stable law of index α $(1 < \alpha \leqq 2)$, then occupation time laws are obtained for many of the random variables of interest in queueing theory. For example, the occupation time laws are obtained for the virtual waiting time, the waiting time of the nth customer, and the number of customers in the system at the points of departure. Furthermore, the distributions of the number of customers served in a busy period and the length of a busy period belong to the domain of attraction of a stable law of index α $(0 < \alpha < 1)$. This last fact immediately gives limit distributions for the expired and remaining times in a cycle and the number of customers served to date in a busy period and the number of customers remaining to be served. Similar results are also obtained for the queues GI/M/1 and for certain queues with priorities and with batch service.

25. **A Note on Moments of Gamma Order Statistics.** P. R. KRISHNAIAH and M. HASEEB RIZVI, Aerospace Research Laboratories. (By title)

This note derives expressions for the moments of order statistics and for the moments of pairs of order statistics and give the modal equations of order statistics from a gamma distribution with any positive parameter. Earlier, Gupta (*Technometrics* **2** (1960) 243–262) has derived expressions restricting the gamma parameter to be a positive integer. Some recurrence relations between the expected values of a specified function of order statistics from any arbitrary distribution with continuous cumulative distribution function are also given. They are closely related to some results of Srikantan (these *Annals*, **33** (1962) 169–177).

26. **The Combination of Unbiased Estimators Having Variances in Unknown Ratios.** PAUL S. LEVY, Johns Hopkins University.

Suppose there are k independent normally distributed estimators X_i of a parameter μ, where each X_i has associated with it an estimator s_i^2/f_i of its variance σ_i^2/f_i. Suppose the s_i^2 are distributed as mean squares with n_i degrees of freedom independent of each other and of the X_i, and the f_i are known constants depending on the nature of the X_i. The problem treated here is that of combining the X_i into a single unbiased estimator of μ. The two most frequently used combinations of the X_i are $\bar{X}$ defined as $(\sum f_i X_i)/(\sum f_i)$ and $\bar{X}_w$ defined as $(\sum f_i X_i/s_i^2)/(\sum f_i/s_i^2)$. The approach taken here is to find an unbiased estimator $\bar{X}_0$ of μ, which is a linear combination of $\bar{X}$ and $\bar{X}_w$ and which has minimum variance over the class of all such linear combinations of $\bar{X}$ and $\bar{X}_w$. It is shown that $\bar{X}_0$ has the form $\alpha\bar{X} + (1 - \alpha)\bar{X}_w$ where α is defined as [Var $(\bar{X}_w)$ − Cov $(\bar{X}, \bar{X}_w)$]/[Var $(\bar{X})$ + Var $(\bar{X}_w)$ − 2 Cov $(\bar{X}, \bar{X}_w)$]. Methods of approximation similar to those used by Paul Meier (*Biometrics* **9** (1953) 59–73) are used to obtain an explicit formula for α. Because α depends on unknown parameters, the estimator $\bar{X}_0$ is in general unobtainable but can be approximated by the estimator $\dot{X}_0$ which is obtained from $\bar{X}_0$ by finding an estimator $\hat{\alpha}$ of α. Empirical sampling studies show that $\dot{X}_0$ is usually more efficient than either $\bar{X}$ or $\bar{X}_w$, especially when the number of X_i being combined is large.

27. **A Modification of Kendall's Rank Correlation Measure** (Preliminary Report). KAY K. MAZUY and E. C. VENEZIAN, Arthur D. Little, Inc., Cambridge, Massachusetts.

M. G. Kendall's rank correlation measure τ satisfies the conditions that it will be: (1) +1, if the rankings of two systems of interest are identical; (2) −1, if the ranking for one system is the reverse of the ranking for the second system; (3) in the open interval (−1, +1) for all other ranking arrangements. This paper proposes a modification of Kendall's τ which satisfies the above conditions, and, in addition, is applicable when the rankings of two systems are naturally truncated or when we wish to examine only a given subset of successive rankings. The modified measure takes into account such truncation when it occurs at either or both ends of the set of rankings. We discuss the true measure, its sample counterpart, and certain distribution properties of the sample counterpart, including its mean and variance.

28. **Clusters in a Poisson Process.** M. V. MENON, IBM Research Laboratory, San Jose, California.

Given a positive integer r, no clusters of size $r + 1$ are said to occur in an interval of time $(0, t)$ if no sub-interval of length b contains more than r incidents of a given Poisson process. Newell (*Time Series Analysis*, ed. M. Rosenblatt, John Wiley, 1962, pp. 89–103), derives asymptotic formulae for the probability $p_0(t)$ of no clusters for a process slightly more general than the Poisson process and also mentions various applications. In this paper, we attempt to compute exact (non-asymptotic) expressions for $p_0(t)$. If μ is the parameter of the process, then $p_0(t)$ equals $\exp(-\mu t)\sum_{m=0}^{N+1}\sum_{k=0}^{rm} S_{m,k}\{t - (m-1)b\}^k$, where $N = [t/b]$ and $S_{m,k} = \sum c_{m,i_1,\cdots,i_r}$, the summation extending over all the positive integers $i_1, \cdots, i_r$ whose sum is k. Recursion formulae are derived for the $c_{m,\cdots}$, which are explicitly solved when $r = 2$.

29. **A Class of Tests With Monotone Power Functions for Two Problems in Multivariate Statistical Analysis.** G. S. MUDHOLKAR, University of Rochester.

The problem of testing the general multivariate linear hypothesis and the problem of testing independence between two sets of multivariate random variables are the two problems considered in this paper. Two sufficient conditions for the power function of an invariant test of the linear hypothesis to be a monotonically increasing function of each of the noncentrality parameters were obtained by Das Gupta, Anderson and Mudholkar (*Ann. Math. Statist.* **34** (1963) 200–205). One of these conditions was extended to the invariant tests of the latter problem by Anderson and Das Gupta (*Ann. Math. Statist.* **34** (1963) 206–208). The two conditions are, respectively, in terms of the convexity and symmetry of certain sections of the acceptance regions of the tests. Therefore, their verification is, in general, nontrivial. In this paper it is shown that the invariant tests based on statistics "generated" by the symmetric gauge functions of the maximal invariants have the relevant monotonicity properties. In this process useful extensions of some results on the symmetric gauge functions and the convexity in the matrix theory are obtained.

30. Consistent Estimation of Sum of Squares of Jumps for a Spectrum and a Probability Distribution. V. K. Murthy and P. M. Gluckman, Douglas Aircraft Company, Inc.

Let $F(x)$ be a probability distribution function. $F(x)$ can be decomposed into $F(x) = F_1(x) + F_2(x)$ where $F_1(x)$ is an everywhere continuous function and $F_2(x)$ is a pure step function with jumps of magnitude S_ν at the saltuses $x = x_\nu$, $\nu = 1, 2, \cdots \infty$ and both $F_1(x)$ and $F_2(x)$ are non-decreasing and are uniquely determined. In this paper the problem of estimating $\sum_{\nu=1}^{\infty} S_\nu^2$ is considered. Based on a random sample of size $2N$ consistent and asymptotically normal class of estimators is obtained for $\sum_{\nu=1}^{\infty} S_\nu^2$. Also considered is the problem of estimating the sum of squares of the spectral jumps. Based on a realization of size N (T in the continuous case) from a stationary stochastic process of normal varieties, a consistent class of estimators for estimating the sum of squares of spectral jumps is obtained and it is further shown that this class is asymptotically normal.

31. Spectral Analysis of a Class of Non-Stationary Stochastic Processes. V. K. Murthy and P. M. Gluckman, Douglas Aircraft Company, Inc.

One of the most important properties of the class of stationary (wide-sense) processes is that they admit a spectral representation by means of a stochastic integral of Fourier type. A more general class of processes (non-stationary) called harmonizable processes by Loéve are considered in this paper as they also possess spectral representations of a similar kind. The covariance function $R(m, n)$ of a harmonizable process $\{x_n\}$ has the representation $R(m, n) = E(x_m \bar{x}_n) = \int_0^{2\pi} \int_0^{2\pi} e^{i(mu-nv)}\, dF(u, v)$, where the function $F(u, v)$ is called the spectral function which is a distribution of a complex valued mass such that the density at the element $du\, dv$ is $f(u, v)$ where $dF(u, v) = f(u, v)\, du\, dv$. The function $f(u, v)$ is called the spectral density of the processes $\{x_n\}$. In this paper based on a realization of size $2N$ ($2T$ in the continuous case) from the process $\{x_n\}$, a consistent class of estimators is obtained for estimating $f(u, v)$ which is also shown to be asymptotically normal assuming the process $\{x_n\}$ is also of normal variates.

32. The Distribution of the Size of the Maximum Cluster of Points Drawn From a Non-Uniform Distribution. Joseph I. Naus, The Franklin Institute.

N points are independently drawn from the distribution on $(0, 1)$ with c.d.f., $F(x)$. Let $n_{y,p}$ denote the number of points in $(y, y + p)$. Let M_p denote $\max_y\{n_{y,p}\}$ for $0 \leqq y \leqq 1 - p$. For $p = 1/L$, L an integer, let m_p denote $\max_i\{n_{ip,p}\}$ for $i = 0, 1, 2, \cdots, L - 1$. We find limiting forms for $\Pr(M_p \geqq k)$, for $k > N/2$, p small, and $F(x)$ satisfying certain regularity conditions, and use these results to show that $\lim_{p\to 0} [\Pr(M_p \geqq k)/\Pr(m_p \geqq k)] = k$.

33. Existence of Optimal Estimators Relative to Convex Loss. M. M. Rao, Carnegie Institute of Technology.

Let $W(\cdot)$ be a (nonnegative) symmetric convex function on the real line R such that $W(0) = 0$. If T is an estimator of a parameter θ of a probability distribution P_θ, $\theta \varepsilon A$ where

A is a subset of R, then T is optimal among all estimators of θ if $E_\theta[W(T - \theta)]$ is a minimum. The problem of existence of optimal estimators, for $W(x) = |x|^p$, $p \geqq 1$, was treated by E. W. Barankin (these *Annals* **20** (1949) 477–501), whose main result was generalized by the author (*Math. Annalen* **143** (1961) Theorem 9) if $W(2x) \leqq CW(x)$. However, this condition on $W(\cdot)$ has no particular statistical significance and was used because of certain mathematical limitations. The case that $W(\cdot)$ is discontinuous is seen to be covered by Barankin's work. In the present paper, the above results are extended for continuous $W(\cdot)$ without any other conditions whatever, and it is pointed out that there are no further generalizations in this direction. (This paper is included as a part of my "Linear functionals on Orlicz spaces," which will appear in *Nieuw Arch. Wiskunde*, (1964), Netherlands.)

34. **Bounds on the Maximum Sample Size of a Bayes Sequential Procedure.** S. N. Ray, Stanford University.

Bounds on the maximum sample size of a Bayes sequential procedure are derived using a class of results the simplest one of which may be stated roughly as follows. A Bayes procedure whose risk can be approximated arbitrarily closely by that of a Bayes truncated sequential procedure must stop at a stage when the expected reduction in the stopping risk derivable from taking one more observation cannot, at any future time, be compensated for by the expected cost of that additional observation. If such a situation must arise by stage n irrespective of the data, then n gives an upper bound on the maximum sample size of the Bayes procedure. Sufficient conditions are given for these bounds to be exact. These results are then applied to certain special decision problems.

35. **Exact Distributions of Wilks' Criterion** (Preliminary Report). Martin Schatzoff, IBM and Harvard University.

The likelihood ratio statistic for testing the general linear hypothesis, derived by Wilks (*Biometrika* **24** (1932) 471–494), is defined by $W = \det(E)/\det(E + H)$, where E and H are the sample covariance matrices for error and hypothesis respectively. Under the null hypothesis, W has the distribution of $\prod_1^p X_i$, where the X_i are independently distributed as beta random variables with parameters $(\frac{1}{2}(n - i + 1), \frac{1}{2}q)$, p = number of variates, q = degrees of freedom for hypothesis, and n = degrees of freedom for error. In this paper, we derive exact closed form expressions for the density and distribution functions of W under the null hypothesis for cases in which p or q is an even integer. The approach consists of making the transformation $Z = -\log W$, observing that Z is distributed as a sum of independent random variables, and performing successive convolutions. It is shown that when p or q is even, the distribution function for W is always of the form $F(w) = \sum_1^M c_i[2/(n - l_i)]^{r_i} w^{\frac{1}{2}(n-l_i)} (-\log w)^{s_i}$, and a computational algorithm is given for deriving the constants M, c_i, r_i, l_i and s_i, given the parameters p, q and n. Tables of percentage points are provided for $p = 3(1)10$, $q = 4(2)10$, as well as comparisons with the Rao approximations (*Biometrika* **35** (1948) 58–79).

36. **Spectral Analysis With Randomly Missed Observations: The Binomial Case.** Perry A. Scheinok, Hahnemann Medical College.

Estimation of the spectral density of a discrete Gaussian stationary time series is considered under the assumption that the device being used to observe the time series misses an occasional observation due to a binomially distributed random failure. A modified version of the periodogram is introduced under the condition that the estimate of the spectral density be asymptotically unbiased. Using weighting functions similar to those introduced by Parzen (*Ann. Math. Statist.* **28** 329–348), the asymptotic variance of such an estimate is calculated.

37. **Admissible-Bayes Character of T^2, R^2, and Other Classical Multivariate Tests.** R. SCHWARTZ and J. KIEFER, Cornell University and General Electric Co. (Invited)

For many classical multivariate normal testing problems, certain fully invariant procedures are shown to be admissible (proper) Bayes procedures. For example, suppose that the columns of $X(p \times r)$, $Y(p \times n)$, $Z(p \times m)$ are independent and normal with common nonsingular covariance matrix Σ, and with $EX = \xi$, $EY = 0$, $EZ = \nu$. When $n \geqq p$, for testing $H_0: \xi = 0$, various tests (all reducing to T^2 when $r = 1$) are shown to be admissible Bayes procedures, for example, the critical region $\text{tr}(XX' + YY')^{-1} XX' \geqq$ const. Previous admissibility proofs (M. N. Ghosh, abstract, *Ann. Math. Statist.* **34** (1963) 1131) for certain other tests; more generally, R. Schwartz in abstract 38 used the Birnbaum-Stein exponential family approach which can be applied to more tests but (even in the T^2 case of Stein) does not imply good power performance except at a great distance from H_0; for any $\epsilon > 0$, the *a priori* distribution here can be supported on a set where $\text{tr}\, \Sigma + \text{tr}\, \Sigma^{-1} \xi\xi' < \epsilon$. Moreover, the Birnbaum-Stein approach fails in such problems as the following: H_0: the matrices Y_i $(i = 1, \cdots, k)$ are independent, where Y has been decomposed into matrices $Y_i(p_i \times n)$ with $\sum_1^k p_i = p$. In this case, for example, when $n > p$, the likelihood ratio critical region $\prod_i \det(Y_i Y_i')/\det YY' \geqq$ const. is admissible Bayes. When $p_1 = 1$ this is the R^2 test, for which admissibility had been known previously only when $k = 2$, $p_2 = 1$. In these and other problems the tests considered are Bayes with respect to an infinite-dimensional set of *a priori* distributions.

38. **Admissible Invariant Tests in MANOVA.** RICHARD SCHWARTZ, Cornell University and General Electric Co. (Invited)

In the p-dimensional Euclidean space of points $t = (t_1, t_2, \cdots, t_p)$ let $A = \{t: \min_i t_i \geqq 0\}$ and let $B = \{t: 0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_p\}$. For σ in the symmetric group S_p, write $\sigma t = (t_{\sigma(1)}, \cdots, t_{\sigma(p)})$. Let $\mathfrak{C}$ consist of every subset C of B with the property that $\bigcup_{\sigma \varepsilon S_p} \sigma C$ is a convex monotone subset of A. In the setting of the preceding abstract it is shown that, for testing $H_0: \xi = 0$ vs. $H_1: \xi \neq 0$, every (invariant) acceptance region in the space B of latent roots of $XX'(YY' + XX')^{-1}$, that is in $\mathfrak{C}$, is admissible. The proof follows Stein, "Admissibility of Hotelling's T^2-test", *Ann. Math. Statist.* **27** (1956) 616. Special cases had previously been given by Ghosh (abstract, *Ann. Math. Statist.* **34** (1963) 1131) and Schwartz (abstract, *Ann. Math. Statist.* **35** (1964) 939).

39. **Asymptotic Properties of an Age Dependent Branching Process.** HOWARD WEINER, University of California, Berkeley. (By title)

Let $Z(t)$ denote the number of cells at time t which are progeny of a single cell born at $t = 0$, $G(t)$ with $G(0) = 0$ be the lifetime distribution function of each cell, and

$$h(s) = \sum_{r=0}^{\infty} p_r s^r$$

with $p_r \geqq 0$, $\sum_{r=0}^{\infty} p_r = 1$ be the generating function of the number of progeny which replace each cell on completion of its life. Cells develop and proliferate independently of each other. For $h^{(1)}(1) = 1$, $0 < h^{(2)}(1) < \infty$, $h^{(n)}(1) < \infty$, $n = 3, 4, \cdots$, and $\int_0^\infty u \, dG(u)$

$\equiv m_G < \infty$, and by means of an integral equation satisfied by $D(s, t) \equiv 1 - E[\exp - (sZ(t))]$, it is shown that, for $n = 1, 2, \cdots$, $\lim_{t\to\infty} E[Z(t)]^n/t^{n-1} = n!/b^{n-1}$ where $b = 2m_G/h^{(2)}(1)$. Under the assumptions that $1 - G(t) = O(t^{-3})$ and $h^{(3)}(1) < \infty$, Chover and Ney have shown that $\lim_{t\to\infty} tP[Z(t) > 0] = b$, and this result along with that on moments above yield, by moment methods, that for $u > 0$, $\lim_{t\to\infty} P[bZ(t)/t > u \mid Z(t) > 0] = \exp(-u)$.

(Abstracts to be presented at the European Regional Meeting, Bern, Switzerland, September 14–16, 1964. Additional abstracts appeared in the June issue and others will appear in the December issue.)

3. **Confidence Region Tests.** JOHN AITCHISON, University of Liverpool.

To test a hypothesis that a parameter θ has value θ^* we usually ask whether an observation falls in a critical region of the outcome space. It is well known that, by a suitable choice of confidence region, an equivalent test is to ask whether θ^* lies outside the confidence region. In this paper the concept of such a confidence region test is extended to composite hypotheses in directions which are relevant to the study of data-snooping and multiple-hypothesis testing. The method consists of first constructing a confidence region $C(x)$, based on the observation x and selected with the experimental setting in mind; any hypothesis ω suggested for testing is then rejected if and only if $\omega \cap C(x) = \phi$. This procedure is investigated for linear models (the relation to the S-method being indicated) and, in the asymptotic case, for more general parametric models. Investigation of the properties of the equivalent critical region $A(\omega) = \{x:\omega \cap C(x) = \phi\}$ reveals that the method is easy to apply, involving only the usual standard computations, and provides an illuminating comparison with "fixed significance level" testing by way of an "equivalent significance level".

4. **Optimal One-Sample Distribution-Free Tests and Their Two-Sample Extensions.** C. B. BELL and KJELL DOKSUM, Mathematisch Centrum, Amsterdam, and University of California, Berkeley.

For the one-sample goodness-of-fit test of $H_0:F = F_0$ vs. $H_1:F = G_\theta$, G_θ can be written $h_\theta(F_0)$; h_θ is a cpf on the unit interval, and all cpfs are appropriately regular. Using the Neyman-Pearson lemma one obtains strongly distribution-free tests which are most powerful (MP) for any K_0 vs. K_θ satisfying $K_\theta K_0^{-1} = h_\theta$, and whose statistics are of the form $T = \sum h_\theta'(F_0(x_i))$. For $\{h_\theta\}$ with $h_{\theta_0}(u) \equiv u$, tests based on the statistic

$$T' = \sum (\partial/\partial\theta) \ln h_\theta'(F_0(x_i)) \,|_{\theta=\theta_0}$$

are *almost locally* MP. Further, the classes of alternatives for which T'-tests are almost locally MP and *asymptotically efficient* are *contaminated Koopman-Pitman* classes containing $\{h_\theta\}$. For the 2-sample problem randomized statistics of the form $S = m^{-1} \sum J(U(R(x_i))) - n^{-1} \sum J(U(R(y_j)))$ where $J = \ln h_\theta'$ or $J = (\partial/\partial\theta) \ln h_\theta'|_{\theta=\theta_0}$, and $U(r)$ is the rth order statistic of a uniform random sample independent of $x_1, \cdots, x_m; y_1, \cdots, y_n$ have *continuous* H_0- and H_1-distributions; and are *asymptotically locally* MP and *asymptotically efficient* for $\{h_\theta\}$ and for certain *contaminated Koopman-Pitman classes* containing $\{h_\theta\}$.

5. **Random Graphs and Digraphs.** T. N. BHARGAVA and S. D. CHATTERJI, Kent State University and University of Heidelberg. (Invited)

We have extended the study of random (undirected) graphs as initiated by Erdös and Rényi (See in particular their paper "On the Evolution of Random Graphs" *Publ. Math.*

Inst. Hungar. Acad. Sci. **5** *Ser. A.* 1960) to the study of directed graphs (called digraphs). Various models of randomness are possible and we discuss each one correlating our models with the ones used in the undirected theory (e.g. by Gilbert; Austin, Fagen, Penney, and Riordan). We have concentrated on obtaining only asymptotic results instead of closed form formulae since (as in Erdös and Rényi) the asymptotic results offer a good qualitative picture of the evolution of graphs.

6. **Exact Probability Distribution Functions of Some Rényi Type Statistics.** MIKLÓS CSÖRGŐ, Princeton University.

Let $F(x)$ be the continuous distribution function of a random variable X, and $F_n(x)$ be the empirical distribution function of a random sample $X_1, \cdots, X_n$, taken on X. Using the method of Z. W. Birnbaum and Fred H. Tingey, *Ann. Math. Statist.* **22** (1951) 592–596 and a combinatorial lemma whose proof is based on some results of N. H. Abel, *Oevres Completes*, Christiania, C. Groendahl, 1839, **1**, p. 102 and Z. W. Birnbaum and Ronald Pyke, *Ann. Math. Statist.* **29** (1958) 179–187, we derive exact probability distribution functions for the random variables $\sup_{F(x)\leqq b}\{F(x) - F_n(x)\}/\{1 - F(x)\}$, $\sup_{a\leqq F(x)}\{F_n(x) - F(x)\}/F(x)$, $\sup_{F_n(x)\leqq b}\{F(x) - F_n(x)\}/\{1 - F_n(x)\}$ and $\sup_{a\leqq F_n(x)}\{F_n(x) - F(x)\}/F_n(x)$ where $0 < a < 1, 0 < b < 1$, and also for these random variables such that respective sups in order are taken over all x's so that $F(x) < 1, 0 < F(x), F_n(x) < 1$ and $0 < F_n(x)$.

7. **Some Rényi Type Limit Theorems for Empirical Distribution Functions.** MIKLÓS CSÖRGŐ, Princeton University. (By title)

Let $F(x)$ be the continuous distribution function of a random variable X, and $F_n(x)$ be the empirical distribution function of a random sample $X_1, \cdots, X_n$, taken on X. Using some results of Alfréd Rényi, *Acta Math. Acad. Sci. Hungar.* **4** (1953) 191–231, and an extended form of a convergence theorem of Harald Cramér, *Mathematical Methods of Statistics*, (1946), p. 254, the limiting distribution of the supremum of the random variables $\{F_n(x) - F(x)\}/F_n(x)$, $|F_n(x) - F(x)|/F_n(x)$, $\{F_n(x) - F(x)\}/(1 - F(x))$, $|F_n(x) - F(x)|/(1 - F(x))$, $\{F_n(x) - F(x)\}/(1 - F_n(x))$, $|F_n(x) - F(x)|/(1 - F_n(x))$ is derived where sup is taken over suitable ranges of x respectively. Relevant tests and some combination of them to provide for symmetrical asymptotic statistical tests are also discussed.

8. **Simultaneous Test Procedures.** KUNO RUBEN GABRIEL, Hebrew University.

Simultaneous Test Procedures (STPs) are defined as tests for each component hypothesis of a set, all having the same critical value and allowing for a fixed probability of rejecting at least one of the hypotheses if they are all true. Where the over-all hypothesis is rejected, the STP will show which of the component hypotheses must also be rejected. Under fairly general conditions, STPs have desirable properties of transitive decisions, and probabilities of error which cannot exceed those obtaining when all the hypotheses are true. STPs and the corresponding type of confidence sets are derived for the univariate and multivariate general linear hypothesis in a normal model, including T^2 type tests. Tukey's multiple comparisons method is shown to be an STP. STPs are closely related to union-intersection tests and simultaneous confidence bounds. For their distinct purpose, STPs are at least as good as the above bounds. A result is given where the STP is superior. Significance levels for component hypotheses are defined and their relation to the over-all level pointed out. The importance of the significance levels for hypotheses of various ranks is discussed and their calculation illustrated.

9. A System of Inequalities for the Incomplete Gamma Function and the Normal Integral. S. S. GUPTA and M. N. WAKNIS, Purdue University. (By title)

Two different sets of bounds (which form monotone sequences) for the functions $\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt$ and $\Gamma(a, x) = \int_0^\infty e^{-t}t^{a-1} dt$, $a > 0$, $x > 0$, have been derived by using the continued fraction expansions of these integrals. The first set of inequalities is given by

$$\sum_{k=0}^{n-1} x^k/(k + a)_{k+1} < e^x x^{-a} \gamma(a, x) < \sum_{k=0}^{n-1} x^k/(k + a)_{k+1} + x^n(n + 1 + a)/(n + a)_{n+1}(n + 1 + a - x)$$

where $a > 0$, $x > 0$ and $n = 1, 2, 3, \cdots$ and where the right hand side of the above inequalities is valid only for $x < n + 1 + a$. The second set of inequalities for $\Gamma(a, x)$ is valid for all positive numbers a and x. Comparison between the two systems of inequalities shows that the first set is better than similar inequalities obtained from the second, for $\gamma(a, x)$, when x is small compared to a. For large x, the second set of inequalities for $\Gamma(a, x)$ appears to be better than the first set. From the above two systems of inequalities, bounds are obtained for the Mill's ratio for the normal distribution. Applications include derivation of bounds on the probability of correct selection for some problems of ranking and selection of parameters of the gamma and normal distributions. Also, these bounds have applications to evaluation of the probability integrals of $F_{\max}$ and $F_{\min}$ distributions.

10. On the Block Structure of Singular Group Divisible Partially Balanced Incomplete Block (SGDPBIB) Designs. C. H. KAPADIA, Southern Methodist University.

Shah [*Ann. Math. Statist.* **35** (1964) 398–407] obtained an upper bound for the number of disjoint blocks in certain PBIB designs. The purpose of this note is to prove the following similar results for the SGD designs. (1) For a SGD design, k is divisible by n. If $k = cn$, then the number of groups in each block must be c. (2) A given block of the SGD design cannot have more than $b - 1 - k(\lambda_1 - 1)^2/[(\lambda_1 - 1)n + (k - n)(\lambda_2 - 1)]$ disjoint blocks with it and if some block has that many disjoint blocks, then $n + [(k - n)(\lambda_2 - 1)]/(\lambda_1 - 1)$ is an integer and each non-disjoint block has $n + [(k - n)(\lambda_2 - 1)]/(\lambda_1 - 1)$ treatments common with that given block. (3) The necessary and sufficient condition that a block of SGD design has the same number of treatments common with each of the remaining blocks is that (i) $b = r + m - k/n$ and (ii) $kn(\lambda_1 - 1)/[n(\lambda_1 - 1) - (v - k)]$ is an integer. (4) If a block of the SGD design with parameters $v = mn = tk$, $b = tr$ (t an integer greater than 1) has $(t - 1)$ blocks disjoint with it then the necessary and sufficient conditions that it has the same number of treatments common with each of the non-disjoint blocks is that (i) $b = r + m - 1$ and (ii) k/t is an integer. Using this, one can derive results on resolvable and affine resolvable SGD designs.

11. Distribution of the Studentized Smallest Chi-Square, With Tables and Applications. P. R. KRISHNAIAH and J. V. ARMITAGE, Aerospace Research Laboratories. (By title)

Let $x_1, \cdots, x_k$ be k independently distributed chi-square variates each with n degrees of freedom and let x_0 be another chi-square variate with m degrees of freedom distributed independently of the other x_i's. Then, the distribution of $V = m \min(x_1, \cdots, x_k)/nx_0$ is known to be the studentized smallest chi-square distribution. In the present paper, the evaluation of the probability integrals associated with V is considered. Also, lower 10%, 5%, 2.5% and 1% points of V are tabulated when $n = 1(1)20$, $m = 5(1)45$ and $k = 1(1)12$. These tables overlap slightly with the tables of Gupta and Sobel (*Biometrika* **49** (1962) 509–523) and of Ramachandran (*J. Amer. Statist. Assoc.* **53** (1958) 868–872). Various applications of these tables are also discussed.

12. **Dynamic Programming for Countable State Systems.** ASHOK MAITRA, University of Copenhagen.

We consider a system with a countable state space S, which we take to be the set of positive integers. The states are labeled by s or s'. Once a day, we observe the current state s of the system, and then choose an action a from a finite set A of actions. As a result of this: (1) we receive an immediate income $i(s, a)$ and (2) the system moves to a new state s' with probability $q(s' \mid s, a)$. We assume that the incomes are bounded, that is, there exists $M > 0$ such that $|i(s, a)| \leqq M$ for all $s \varepsilon S$ and $a \varepsilon A$. Further, there is specified a discount factor β, $0 \leqq \beta < 1$, so that the value of unit income n days in the future is β^n. The problem is to choose a policy which maximises our total expected income. It is proved that for $\beta < 1$, there is an optimal policy which is stationary. For $\beta = 1$, which is studied as a limiting case of $\beta < 1$, it is shown that an optimal policy will not always exist.

13. **On the Estimation of Missing Elements of a Markov Chain.** ROBERT H. RIFFENBURGH, University of Connecticut.

For k a finite non-negative integer, let us denote by $N^{(k)}$ a $(l \times r)$ vector of quantities $n_l^{(k)}$, $l = 1, \cdots, r$, representing states of the system after the kth discrete pass of occurrence through the system. Let us denote by $A^{(k)}$ an $(r \times r)$ matrix of quantities $a_{ij}^{(k)}$, $i, j = 1, \cdots, r$, $a_{ij}^{(k)} \subset [0, 1]$, all i, j, in which $a_{ij}^{(k)}$ represents the probability that a member of the system in state i will alter to lie in state j on the kth discrete pass of occurrence through the system. The problem considered here involves a non-stationary chain in which at least some of the elements of $N^{(m)}$, $A^{(m)}$ are known or estimated, $m = 0, 1, \cdots, k$, but for which no quantitative information at all is available for the missing elements. Aspects of the problem studied are: (1) determination of configurations of unknown elements for which a solution exists, (2) derivation of estimates for the missing elements from the relationships with other elements implicit in a Markov chain, (3) derivation of the small-sample probability density functions of such estimates for estimated $N^{(m)}$, $A^{(m)}$, and (4) obtaining of expressions for multivariate confidence bounds on the single estimates of the unknown elements when $N^{(m)}$, $A^{(m)}$ are estimated.

14. **The Application of Bayes Theorem to the Adaptive Control Processes.** J. TORRENS-IBERN, Escuela Superior de Ingenieros Industriales and the Escuela de Administración de Empresas, Barcelona.

Dynamic programming is applied with success to the selection of optimal policies in the adaptive control processes, both in deterministic and in stochastic cases. As to the latter, Howard has solved the problem where the transition probabilities are known. The problem remains unsolved where the transition probabilities are unknown and we must collect data while decisions are made up, hoping for the best ones. In such circumstances Bayes theorem can be employed to estimate the unknown transition probabilities. The Laplace-Bayes formula yields the best estimations on average, that is to say, the unbiased estimations, but it does not give us the estimations of maximal likelihood. The conditions of utilisation appropriate to the adaptive control recommend the last ones, which have still further practical advantages, without there being any coincidences between them.

(Abstracts not connected with any meeting of the Institute.)

1. **On Estimating the Frequency of a Sinusoid in the Presence of Noise.** ARTHUR ALBERT, ARCON Corporation.

Let $Z^*(t) = A^*\sin(\lambda t + \varphi^*) + N^*(t)$, where $N^*(t)$ is a second order weakly stationary Gaussian process whose power density spectrum, $f(\omega)$, is known. A^*, λ, and φ^* are not known. λ is to be estimated. Let $\mathcal{H}$ be any time invariant linear filter whose unit impulse response is $h(t)$ and whose transfer function is $H(\omega)$. *Theorem*: If $\int_{-\infty}^{\infty} \omega^2|H(\omega)|^2 f(\omega)\, d\omega < \infty$ and $Z(t) = \int_0^t h(t - s)Z^*(s)\, ds$ is the result of passing Z^* through $\mathcal{H}$, then $Z(t)$ is everywhere mean square differentiable and $\lambda_T^2 = \{T^{-1}\int_0^T Z^2(t)\, dt + \alpha\}/\{T^{-1}\int^T Z^2(t)\, dt - \beta\}^2$ converges to λ^2 in probability as $T \to \infty$, where $\alpha = \int_{-\infty}^{\infty}\omega^2|H(\omega)|^2 f(\omega)\, d\omega$ and $\beta = \int_{-\infty}^{\infty}|H(\omega)|f(\omega)\, d\omega$.

2. **On Some Robust Estimates of Location** (Preliminary report). PETER J. BICKEL, University of California, Berkeley.

The asymptotic normality and variances of the trimmed and Winsorized means proposed by Tukey, as estimates of location in the presence of contamination, are established by a method different from that of Tukey and Harris. Some alternative methods of trimming and Winsorizing (not equivalent to those of Tukey) which generalize to higher dimensions are also explored. The minimum efficiency with respect to the families of all symmetric and symmetric unimodal distributions of the Winsorized and trimmed means with respect to the mean are established. The lower bound found for the trimmed means (for small trimming proportions) in the unimodal case compare well with that found by Hodges and Lehmann for the median of averages of pairs. The Winsorized mean has minimum efficiency $\frac{1}{3}$ for unimodal distributions, and 0 in general. The median of averages of pairs is also compared directly with the trimmed mean and found to be at least 90% as efficient for moderate trimming proportions. It can be arbitrarily more efficient.

3. **Main Effect Analysis of the General Non-Orthogonal Layout With Any Number of Factors.** DAN BRADU, Hebrew University. (Introduced by Joseph Putter)

In the analysis of a non-orthogonal layout the main difficulty is the size of the matrix which has to be inverted to obtain a direct solution of the normal equations. This paper presents a method of stepwise solution of the normal equations for the main effect model without interactions, in which at each stage the parameters of the model are transformed into new ones. The number of stages is equal to the number of factors, and each stage requires the inversion of a matrix whose order is equal to the number of levels of one of the factors. Thus, provided the number of levels of each factor does not exceed the maximal order of the matrices which can be handled by computing facilities available, the method can be easily applied to a non-orthogonal layout with any number of factors.

4. **Polyspectra.** DAVID R. BRILLINGER, Princeton University.

Consider a stationary multivariate time series $\{X_1(t), \cdots, X_k(t)\}$ in which a number of the components are possibly identical. The kth order polyspectrum of this series is defined to be the Fourier transform of the kth order cumulant (which is effectively of order $k - 1$). The polyspectrum may be estimated by at least three distinct techniques; (i) Fourier transforming an estimated cumulant (with the use of convergence factors), (ii) forming an appropriate combination of band-passed versions of the individual series, (iii) forming an appropriate combination of the results of complex demodulating the individual series. Under regularity conditions, these estimates may be seen to be consistent and asymptotically normal. Polyspectra would appear to be of use in the following two problems; (i) does a particular frequency component in one of the series under consideration appear to be linearly related to the product of certain frequency components in the other series, (ii) is there some function of a series $X(t)$, say $H(X(t))$, admitting a simpler harmonic analysis. These two questions lead to the definition of a number of polyspectral coefficients which may be estimated from observed series.

5. **A Result on Estimation of Mixtures.** David B. Cooper, Raytheon Company and Columbia University. (Introduced by Paul W. Cooper)

A problem arising in signal detection is testing the hypothesis that a sample has the cumulative distribution function (c.d.f.) $F(x + \alpha)$ against the alternative that it has the c.d.f. $F(x - \alpha)$ when the a priori probabilities of the two events are each $\frac{1}{2}$. As our following theorems show, if $F(x)$ and α are unknown they can often be estimated though the specific associations with $F(x + \alpha)$ and $F(x - \alpha)$ of the samples are not known. Specifically, let $\{z_i\}$ be a sequence of statistically independent samples of a population having c.d.f. $(1/2)[F(x + \alpha) + F(x - \alpha)]$. *Theorem* 1. Let α be known and denote the empirical distribution function determined by $z_1, z_2, \cdots, z_m$ by $G_m(x)$. Then

$$P\{\lim_{m\to\infty} \sup_{-\infty<x<\infty} |F(x) - \textstyle\sum_{k=1}^{\infty}(-1)^{k+1}G_m[x - (2k - 1)\alpha]| = 0\} = 1.$$

Denote the characteristic function of $F(x)$ by $\psi(s)$. *Theorem* 2. For any bounded interval $[-S, S]$ of the real line, $(1/m) \sum_{k=1}^{m} \exp(iz_k s) \to \cos(\alpha s)\psi(s)$ uniformly in $s(-S \leqq s \leqq S)$ w.p.1. Theorem 2 is often effective in the determination of the zeroes of $\cos(\alpha s)\psi(s)$ and consequently may be effective in the estimation of an unknown α. A more comprehensive theorem though only an existence statement is: *Theorem* 3. Let $\mathfrak{F}$ be a class of continuous c.d.f.'s such that if $\bar{F}(x)$, $F(x) \varepsilon \mathfrak{F}$, then $\bar{F}(x + \bar{\alpha}) + \bar{F}(x - \bar{\alpha}) = F(x + \alpha) + F(x - \alpha)$ implies that $\bar{\alpha} = \alpha$. Then a random function $F_m(x)$ and a r.v. α_m can be obtained from $z_1, z_2, \cdots, z_m$ such that $\alpha_m \to \alpha$ w.p.1 and $F_m(x) \to F(x)$ uniformly in x $(-\infty < x < \infty)$ w.p.1.

6. **On the Problem of Matching Documents With Missing and Inaccurately Recorded Items** (Preliminary report). N. S. D'Andrea Du Bois, Jr., University of California, Los Angeles.

Let $\Omega = \{(X_iY_k):i = 1, \cdots, M; k = 1, \cdots, N\}$ be the set of all pairs of L-item documents (X_iY_k) in $X = \{X_i:i = 1, \cdots, M\}$ and $Y = \{Y_k:k = 1, \cdots, N\}$, where M and N are finite numbers. Given that some of the $(x_{ij}y_{kj})$ on (X_iY_k) are missing or inaccurately recorded or both, a statistical procedure is developed to identify all "identifiable" matching pairs of documents in Ω representing the same individuals such that the set of false matches representing different individuals is minimized. The theory includes classification criterions based on the similarity statistic, the test statistic for the likelihood ratio test and Bayes procedure. Among these criterions, we select the one which maximizes the set $A_{ik} = [X_i \equiv Y_k] \subset \Omega$ representing the same individuals while minimizing the set $B_{ik} = [X_i \equiv Y_k] \subset \Omega$ which represents different individuals. The joint distribution of the number of true and false matching documents in a sample of size (m, n) from (X, Y) is derived and moments are computed; a uniformly most powerful test for testing the hypothesis with respect to the proportion of documents common to (X, Y) is given; and the asymptotic forms of the joint distribution of the number of true and false matches in large samples are given. Finally the theoretical results are compared with the known results of the 1960 Death Clearance Project of the California State Department of Public Health.

7. Markovian Sequential Control Processes—Denumerable State Space (Preliminary report). CYRUS DERMAN, Columbia University.

A dynamic system is observed periodically and classified into one of a denumerable number of states. After each observation one of a possible number of decisions is made. The decisions determine the chance laws governing the behavior of the system. Let (Y_t) and (Δ_t), $t = 0, 1, \cdots$ denote respectively the sequences of observed states and decisions. The class C of *all* rules for "controlling" the system is as defined in Derman (*Ann. Math. Statist.* **35** 341–349). Let $P_t(j, k \mid i, R)$ denote the probability that $Y_t = j$, $\Delta_t = k$, given $Y_0 = i$ and that the rule R is used to control the process. Attention is focused on the expression $\psi(i, \alpha, R) = \sum_{t=0}^{\infty} \alpha^t \sum_{j,k} P_t(j, k \mid i, R) w_{jk}$ where α (a discount factor satisfying $0 < \alpha < 1$) and $\{w_{jk}\}$ (costs of being in state j and making decision k) are given. It is shown that if the number of available decisions at each state is finite and the w_{jk}'s are bounded, then there exists a non-randomized stationary rule R_0 such that $\psi(i, \alpha, R_0) = \min_{R \varepsilon C} \psi(i, \alpha, R)$ for all initial states i. The proof of existence of a minimizing rule follows along the lines suggested by Karlin (*Nav. Res. Log. Quart.* **2** 285–294) and that it can be taken to be non-randomized stationary makes use of a device employed by Blackwell (*Ann. Math. Statist.* **33** 719–726).

8. Multivariate Two Sample Problems With Discrete and Continuous Data. R. M. ELASHOFF, University of California, Berkeley.

The work takes up techniques to analyze multivariate data with discrete and continuous variable and is motivated by Fisher's treatment of the 2×2 table. Several probability models are proposed for this type of data, estimators for the parameter are obtained, and different test statistics are proposed, including the use of modified Kullback-Liebler information numbers. For certain models and test statistics the distributional problems are solved; for others, approximations are obtained. A numerical example is worked out based on the data analysis of a clinical trial of psychotropic agents.

9. Admissibility of Linear Estimates for Estimating the Mean of Distributions of the Exponential Type With Square Error Loss. MILAN KUMAR GUPTA, University of California, Berkeley. (Introduced by E. L. Lehmann)

Professor Karlin (*Ann. Math. Statist.* **29** 406–436) gave sufficient conditions for the admissibility of ax as estimator for $E[X]$ for distributions of the type

$$dF(X) = \beta(\alpha)\exp(\alpha X)\, d\mu(X).$$

The present paper extends the results of Karlin to estimators of the type $ax + b$; more specifically we show that if $\beta^{-\lambda}(\alpha)\exp(-\nu\alpha)$ is not integrable in the neighbourhood of both the boundaries of Ω—the natural range of α— then $[1/1 + \lambda]x + [\nu/1 + \lambda]$ is admissibile for $E[X]$ with respect to square error loss. Certain necessary conditions for admissibility of $ax + b$ are also given. A number of examples e.g. Normal, Binomial, Poisson, Negative Binomial and X etc. are considered. In all these cases it turns out that the conditions of admissibility are also necessary.

10. Some Applications of Sverdrup's Lemma to Large Sample Treatment of Categorical Data. D. G. KABE, Wayne State University.

Bhapkar (*Ann. Math. Statist.* **32** (1961) 72), Mitra, Diamond, and others (Cf. Diamond, *Ann. Math. Statist.* **34** (1963) 1432) use large sample minimum modified Chi-square methods to treat multinomially distributed categorical sample data. They point out the analogy between the large sample modified Chi-square methods used for treatment of categorical data and the usual methods of the univariate normal analysis of variance theory with fixed effects model. In this paper we use Sverdrup's lemma (Sverdrup Erling, *Skand. Aktuarietidskrift* **30** (1947) 151) to give an alternative derivation of some of the results of the above authors.

11. A Generalization of a Property of the Mean Deviation for a Class of Discrete Distributions. A. R. KAMAT, Gokhale Institute of Politics and Economics.

It was observed by N. L. Johnson that in the case of some well known discrete distributions the mean deviation δ_1 equals the product of twice the variance and the frequency function at the mean μ, when it is an integer. It is proved in this note that for a certain class of discrete distributions this property can be generalized to show that $\delta_1 = (A_1(m - \mu)^2 + A_2(m - \mu) + A_3)f_m$, where $m = [\mu]$, $f_m =$ the frequency function at m, and A_1, A_2, A_3 can be expressed in terms of the cumulants K_2, K_3, K_4.

12. A Property of the Mean Deviation of a Class of Continuous Distributions. A. R. KAMAT, Gokhale Institute of Politics and Economics.

The mean deviation of some distributions can be expressed as the product of the variance and twice the frequency function at the mean. It is shown in this note that this property always holds for a subclass of continuous distributions of the linear exponential type.

13. On the Behaviour of a Standard Markov Transition Function Near $t = 0$. D. G. KENDALL, Cambridge University.

K. L. Chung asked in his book whether the standard Markov transition function p_{ii} is monotone near $t = 0$, when the state i is instantaneous; G. Smith (*Trans. Amer. Math. Soc.* **110** (1964) 185–195) has now given a very drastic counter-example for which

$$\limsup_{t\to 0} p_{ii}'(t) = +\infty.$$

This raises a new question: how non-monotone can p_{ii} be, near $t = 0$? Let $W_i^+(t)$ and $W_i^-(t)$ be the positive and negative variations of p_{ii} on $[0, t]$; in the present paper it is shown that W_i^+ is at most of the order of the square of W_i^- near $t = 0$, so that $W_i^+(t)/W_i^-(t) \to 0$ as $t \to 0$. In this sense, therefore, p_{ii} is "nearly" decreasing in the neighbourhood of $t = 0$.

14. The Superposition of Point Processes. P. M. LEE, Churchill College, Cambridge. (Introduced by J. F. C. Kingman)

A point process, or random sequence of events, is a random variable ω taking its values in the set Ω of all non-decreasing sequences of real numbers which have only a finite number of members in each finite time interval. A point process which may be regarded as the superposition of an indefinitely large number of independent point processes, no one of which represents a significant fraction of the whole, is termed infinitely divisible. A representation theorem is derived for all infinitely divisible point processes in terms of σ-finite measures Λ on Ω; formally, this can be summarized by the statement that any infinitely divisible point process has the form $\int_\Omega \omega\pi[\Lambda(d\omega)]$ where $\pi[\lambda]$ denotes a Poisson variable of mean λ, and all such variables are supposed independent. From this it follows that such processes are generalizations of the Poisson clustering processes considered by M. S. Bartlett and others. A number of special cases, of which the most important is the Poisson process, are considered, and characterizations of these are given. Finally, a full investigation is made of such topics as the conditions under which these processes are stationary, and the nature and extent of the after-effects present.

15. **A Wide Class of Distributions in Geometrical Probability.** R. E. Miles, Emmanuel College, Cambridge.

For an oval (a bounded closed convex set) C in R^n, $V_s(C)$ denotes: for $s = 1, 2, \cdots, n - 1$ the mean s-content of its orthogonal projection onto a random uniformly oriented s-flat, and for $s = n$ its n-content. $\mathfrak{H}_1$ is a stochastically homogeneous and isotropic system of independent or "Poisson" r-flats in R^n $(0 \leqq r \leqq n - 1)$ of density ρ, so that the number m intersecting an arbitrary oval C has a Poisson distribution with mean $\rho V_{n-r}(C)$; $\mathfrak{H}_k$ denotes the class of k-subsets of $\mathfrak{H}_1$ (k an integer $\geqq (n + 1)/(n - r)$). Δ is a rule associating with each member H_k of $\mathfrak{H}_k$ an oval $C\{H_k\}$ (possibly ϕ), the association being invariant under all affine transformations of the type $\mathbf{x} \to a\mathbf{x} + \mathbf{b}$ ($a > 0$, $\mathbf{b} \in R^n$), and otherwise arbitrary. $\mathfrak{C}^{(k,l)}$ denotes the class of $C\{H_k\}$ ($H_k \in \mathfrak{H}_k$) such that (i) $V_{n-r}(C\{H_k\}) > 0$; and (ii) $C\{H_k\}$ is intersected by exactly l r-flats of $\mathfrak{H}_1$ other than the k r-flats of H_k (whose intersections with $C\{H_k\}$ are determined by Δ). Ergodic theory is used to demonstrate the existence, with probability 1, of a probability distribution of V_{n-r} for $\mathfrak{C}^{(k,l)}$, each member having equal weight. This distribution is gamma, with f.f.

$$f^{(k,l)}(V_{n-r}) = \rho(\rho V_{n-r})^{k+l-1-n/(n-r)} \exp\{-\rho V_{n-r}\}/\Gamma(k + l - n/(n - r)) \qquad (V_{n-r} > 0).$$

Differing choices of Δ and $(k, l; n, r)$ yield a variety of distributions in Geometrical Probability which, loosely, constitute inversions $P[V_{n-r} \mid k + l]$ of the Poisson, regarded as a conditional, distribution $P[m \mid V_{n-r}]$. Generalisations exist in a number of directions.

16. **Positive Unbiased Estimators of a Positive Mean of a Poisson Distribution** (Preliminary report). Martin Sandelius, Skövde Handelsgymnasium, Box 70, Sweden.

From a population divided into two strata, I and II, where I is known to have a relative size > 0 and, if finite, an absolute size $\geqq A$ (>2), individuals are drawn at random until A individuals have been obtained from I. On the n individuals thus sampled, except the last one, a numerical characteristic is observed. Then standard formulas for estimators of the population mean and variance and of the variance of the former estimator, with $n - 1$ instead of n, yield unbiased estimators. (For a finite population cf. Sandelius, *Kungl. Lantbrukshögskolans Ann.* **19** 1952 (1953), 113-126.) Applied to a Poisson distribution, where II corresponds to the zero class, this procedure yields a mean estimator which is always positive. For another application let sample values be drawn from a Poisson distribution until their sum attains or exceeds a given positive integer S. For this sampling procedure unbiased estimators of the mean and its reciprocal are constructed and also unbiased estimators of the variances of these estimators. If now S is chosen so small that the mean estimator has a non-negligible probability of equalling zero repeated sampling until A positive mean estimates have been obtained will yield a positive mean estimator.

17. Quotients of Exponentially Distributed Random Variables. P. Scheinok, Hahnemann Medical College.

Let X and Y be independent and identically distributed random variables defined on $[0, \infty)$, and possessing a common density function $f(x)$ with finite moments of all orders. Let $Z = X/Y$ have the density function $G_Z(u)$, where $g_Z(u) = 1/(1 + u)^2$ for $u \geqq 0$ and $g_Z(u)$ equals zero otherwise. This is identical to the density function of the quotient of two independent and identically exponentially distributed random variables. Let $\Phi(z) = E[\exp(iz \ln x)]$ have no zeroes in its region of analyticity (z complex), then $f(x)$ must be exponential. The result is completely analogous to the one obtained for the Normal distribution by R. G. Laha [On the laws of Cauchy and Gauss, *Ann. Math. Statist.* **30** (1959) 1165–1174]. One can use the example in Laha's paper to show that here the condition of no zeroes for $\Phi(z)$ is also essential, for $f(x)$ to be exponential.

18. On the Evaluation of AOQL's for a Large Class of Continuous Sampling Inspection Plans (Preliminary report). Leon S. White, Columbia University. (Introduced by Cyrus Derman)

A large class S_c of "Dodge-type" continuous sampling plans is defined and methods are given for the evaluation of any plan in this class. S_c includes CSP-1, 2, 3, 4, and 5, MLP-1, r, and T, and H-106 type plans—among others—as sub-classes. The evaluation of any plan $S \varepsilon S_c$ is in terms of its average outgoing quality limit (AOQL). The AOQL of S may be defined as an upper bound to the long run proportion of defectives that remain in the output after inspection, given certain assumptions about Nature's (the processes') ability to control process quality. Specific methods of evaluation are given in the cases where Nature is unrestricted in her ability to produce defectives and where Nature is partially restricted. The unrestricted AOQL is sometimes referred to as the AOQL for a process "out of control". The derivation of these methods relies heavily on the recent work of Derman (*Ann. Math. Statist.* **35** (1964) 341–49) in Markovian Sequential Control Processes. In both cases, AOQL's are actually computed by linear programming techniques. As an example, the unrestricted AOQL's for the class of H-106 plans have been computed and are included in the form of five graphs. An average cost model for a principal sub-class of S_c is also formulated and shown to be amenable to a linear programming solution.